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Wave pulses in two-dimensional randomly stratified elastic media

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IN THE PAPER the propagation of the planar wave pulses in a two-dimensional randomly stratified elastic medium is considered. Two cases: the plane and anti-plane deformations are studied. The problem is described by means of the transition matrix method. In both the cases the transition matrices are obtained and the equations for the wave fields reflected from and transmitted through the randomly stratified elastic slab are derived. Finally, the law of large numbers for the product of random matrices is applied to obtain the effective material constants needed for the description of the elastic wave pulse in the homogenized medium.

1. Introduction

ELASTIC WAVE propagation in stratified media has been widely studied in the literature [4, 10, 11] in the context of mechanical and geotechnical applications. Also some elements of structures are segmented in such a way, that they can be considered as stratified waveguides (see e.g. [5]). Among various methods of the analysis of the waves in the stratified media, the transition matrix method is one of the most effective ones. The method, introduced for the investigation of the harmonic surface waves in deterministic stratified media [17, 21, 24] has been applied in the cases of planar volume harmonic waves in elastic media [6, 7], harmonic elastic waves generated by space-distributed sources [8] and waves in stochastic stratified media [2, 25]. The transition matrix method has been also adopted for the investigation of the propagation of wave pulses in segmented elements of structures, both deterministic [1, 18] and stochastic [15, 16].

In this paper we consider the wave pulses in a two-dimensional elastic stratified medium. The results obtained are a generalization of the results obtained in paper [15], where a one-dimensional medium was considered. On the other hand, this paper extends the model of two-dimensional harmonic waves, considered in [12, 13, 14], on the non-stationary phenomenon of wave pulses.

The schedule of the paper is the following. In Sec. 2 the fundamental equations and notation used through the paper are introduced. In the following sections we give the elastic wave equation for the planar elastic wave pulse and derive the expressions for the transition matrices for the anti-plane (Sec. 3) and plane (Sec. 4) state of deformation. Section 5 contains the wave equation in a layered medium written in the transfer matrix language. The main result of the paper is contained in Sec. 6, where, by applying the law of large numbers for the product of random matrices, we obtain the effective material parameters for the homogenized elastic medium. Section 7 summarizes the results of the paper.

2. The governing equations in homogeneous medium

Consider a non-harmonic linear elastic wave propagating in the homogeneous isotropic medium. In such a case the equations of motion of the medium constitute the following system of partial differential equations (cf. [19]):

$$\varrho \frac{\partial^2}{\partial t^2} u_i = \sigma_{ij,j}\,, \tag{2.1}$$

$i = 1, 2, 3$, where σ_{ij} is the stress tensor, defined as

$$\sigma_{ij} = \mu\left(u_{i,j} + u_{j,i}\right) + \lambda u_{k,k}\delta_{ij} \tag{2.2}$$

(double indices denote summation from 1 to 3). In the above equations λ and μ are the elastic Lamé constants and ϱ is density of the medium.

Let us assume that the elastic medium has a discontinuity surface (plane). We introduce such a system of independent variables that this plane is perpendicular to the x-axis (x_1-axis) of coordinates. At the discontinuity plane (being the interface between two homogeneous and isotropic materials) the wave field must satisfy two following continuity conditions (see [9]): continuity of the displacement vector $\mathbf{u}$ and continuity of the traction vector $\mathbf{t}$.

3. The anti-plane state of deformation

Consider the simplest two-dimensional problem of elastic wave propagation of the transversal, horizontally polarized plane wave. We assume that the displacement of the medium has the following form:

$$\mathbf{u} = \Big(0, 0, u(x_1, x_2, t)\Big)^T, \tag{3.1}$$

that is, it is perpendicular to the plane x_1, x_2. In such a case the elements of the stress tensor are:

$$\begin{aligned} \sigma_{11} &= \sigma_{12} = \sigma_{21} = \sigma_{22} = \sigma_{33} = 0, \\ \sigma_{13} &= \sigma_{31} = \mu \frac{\partial u}{\partial x_1}\,, \\ \sigma_{23} &= \sigma_{32} = \mu \frac{\partial u}{\partial x_2}\,. \end{aligned} \tag{3.2}$$

Substituting these particular stress tensor components into the system of equations (2.1) we obtain a single non-trivial governing equation:

$$\mu \frac{\partial^2 u}{\partial x_1^2} + \mu \frac{\partial^2 u}{\partial x_2^2} = \varrho \frac{\partial^2 u}{\partial t^2}\,. \tag{3.3}$$

Using the given form of the stress tensor (3.2), we obtain the coordinates of the traction vector in the following form:

$$\tau_1 = \tau_2 = 0, \tag{3.4}$$

$$\tau = \tau_3 = \sigma_{13} = \mu \frac{\partial u}{\partial x_1}. \tag{3.5}$$

From the formulae (3.2) and (3.5) we can obtain the system of equations for two mechanical fields u and τ, remaining continuous at a discontinuity plane:

$$\frac{\partial u}{\partial x_1} = \frac{1}{\mu}\tau, \tag{3.6}$$

$$\frac{\partial \tau}{\partial x_1} = \varrho \frac{\partial^2 u}{\partial t^2} + \mu \frac{\partial^2 u}{\partial x_2^2}. \tag{3.7}$$

To solve the system of Eqs. (3.6)–(3.7) in a non-stationary case, we calculate its Fourier transform with respect to time t and spatial variable x_2 (the corresponding transformation variables are ω and k, respectively). We obtain the system of equations for the transformed functions $\widehat{u}$ and $\widehat{\tau}$ in the following form (we replaced x_1 with x):

$$\frac{d\widehat{u}}{dx} = \frac{1}{\mu}\widehat{\tau}, \tag{3.8}$$

$$\frac{d\widehat{\tau}}{dx} = \left(\mu k^2 - \varrho\omega^2\right)\widehat{u}. \tag{3.9}$$

In the matrix form the system of Eqs. (3.8)–(3.9) can be written as

$$\frac{d}{dx}\widehat{\mathbf{u}} = \mathbf{M}\,\widehat{\mathbf{u}}, \tag{3.10}$$

where, by definition,

$$\widehat{\mathbf{u}} = \begin{bmatrix} \widehat{u} \\ \widehat{\tau} \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \mathbf{M}(\omega) = \begin{bmatrix} 0 & \frac{1}{\mu} \\ \mu k^2 - \varrho\omega^2 & 0 \end{bmatrix}. \tag{3.11}$$

To solve the wave problem described by Eq. (3.10), we complete it with the following boundary condition:

$$\widehat{\mathbf{u}}(0,k,\omega) = \widehat{\mathbf{u}}_0(k,\omega) = \begin{bmatrix} \widehat{u}_0(k,\omega) \\ \widehat{\tau}_0(k,\omega) \end{bmatrix}, \tag{3.12}$$

representing jointly the incident wave pulse reaching the plane $x = 0$ and the pulse reflected from it. Then the value at $x = L$ of the solution of the wave equation (3.10), satisfying boundary condition (3.12), can be represented as

$$\widehat{\mathbf{u}}(L,k,\omega) = \mathbf{T}(L)\widehat{\mathbf{u}}_0(k,\omega), \tag{3.13}$$

where

$$\mathbf{T}(L) = \exp\{\mathbf{M}(\omega)L\} \tag{3.14}$$

is the transition matrix through the layer of thickness L, for the elastic wave in anti-plane state of deformation. Construction of the transition matrix requires the knowledge of the eigenvalues of the system matrix $\mathbf{M}$ of the wave equation (3.10). Solving the characteristic equation

$$\det\{\mathbf{M} - p\mathbf{Id}\} = \det\begin{bmatrix} -p & \dfrac{1}{\mu} \\ \mu k^2 - \varrho\omega^2 & -p \end{bmatrix} = p^2 - \left(k^2 - \varrho\omega^2/\mu\right) = 0, \tag{3.15}$$

we obtain the following eigenvalues of the system matrix:

$$p_1 = p = \sqrt{k^2 - \varrho\omega^2/\mu}\,, \qquad p_2 = -p = -\sqrt{k^2 - \varrho\omega^2/\mu}\,. \tag{3.16}$$

According to the following Lagrange interpolation formula (see [23]):

$$\exp\{\mathbf{M}L\} = \frac{(\mathbf{M} - p_2\mathbf{Id})}{p_1 - p_2}\exp\{p_1 L\} + \frac{(\mathbf{M} - p_1\mathbf{Id})}{p_2 - p_1}\exp\{p_2 L\}, \tag{3.17}$$

we obtain the explicit expression for the transition matrix $\exp\{\mathbf{M}(\omega)L\}$ in the following form:

$$\mathbf{T}(L) = \exp\{\mathbf{M}L\} = \begin{bmatrix} \operatorname{ch} pL & \dfrac{\operatorname{sh} pL}{\mu p} \\ \mu p \operatorname{sh} pL & \operatorname{ch} pL \end{bmatrix}, \tag{3.18}$$

where ch and sh are, respectively, the hyperbolic cosine and sine functions.

4. The plane state of deformation

In the second possible form of planar wave the displacement vector is

$$\mathbf{u} = \left(u_1(x_1, x_2, t),\ u_2(x_1, x_2, t),\ 0\right)^T. \tag{4.1}$$

Then the stress tensor has the following elements:

$$\sigma_{13} = \sigma_{23} = \sigma_{31} = \sigma_{32} = 0, \tag{4.2}$$

$$\sigma_{11} = (\lambda + 2\mu)\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2}, \tag{4.3}$$

$$\sigma_{12} = \sigma_{21} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \tag{4.4}$$

$$\sigma_{22} = (\lambda + 2\mu)\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial u_1}{\partial x_1}, \tag{4.5}$$

$$\sigma_{33} = \lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right), \tag{4.6}$$

and the traction vector has the following coordinates:

$$\tau_1 = \sigma_{11} = (\lambda + 2\mu)\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2}, \tag{4.7}$$

$$\tau_2 = \sigma_{21} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \tag{4.8}$$

$$\tau_3 = 0. \tag{4.9}$$

In this particular case, the non-trivial governing equations (2.1) can be written as:

$$\frac{\partial \tau_1}{\partial x_1} = \varrho\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial \tau_2}{\partial x_2}, \tag{4.10}$$

$$\frac{\partial \tau_2}{\partial x_1} = \varrho\frac{\partial^2 u_2}{\partial t^2} - \frac{\lambda}{(\lambda + 2\mu)}\frac{\partial \tau_1}{\partial x_2} - \frac{4\mu(\lambda + \mu)}{(\lambda + 2\mu)}\frac{\partial^2 u_2}{\partial x_2^2}. \tag{4.11}$$

From Eqs. (4.7), (4.8) for the traction vector we obtain the pair of equations connecting stresses and displacements,

$$\frac{\partial u_1}{\partial x_1} = \frac{1}{(\lambda + 2\mu)}\tau_1 - \frac{\lambda}{(\lambda + 2\mu)}\frac{\partial u_2}{\partial x_2}, \tag{4.12}$$

$$\frac{\partial u_2}{\partial x_1} = \frac{1}{\mu}\tau_2 - \frac{\partial u_1}{\partial x_2}. \tag{4.13}$$

Equations (4.10)–(4.13) describe completely the wave pulse in elastic media in the plane state of deformation.

After the Fourier transformation with respect to time t and spatial variable x_2, the system of equations (4.10)–(4.13) becomes the following system of ordinary differential equations (also in this case $x_1 \equiv x$):

$$\frac{d\widehat{u}_1}{dx} = \frac{1}{(\lambda + 2\mu)}\widehat{\tau}_1 - ik\frac{\lambda}{(\lambda + 2\mu)}\widehat{u}_2, \tag{4.14}$$

$$\frac{d\widehat{u}_2}{dx} = \frac{1}{\mu}\widehat{\tau}_2 - ik\widehat{u}_2, \tag{4.15}$$

$$\frac{d\widehat{\tau}_1}{dx} = -\omega^2\varrho\widehat{u}_1 - ik\widehat{\tau}_2, \tag{4.16}$$

$$\frac{d\widehat{\tau}_2}{dx} = -ik\frac{\lambda}{(\lambda + 2\mu)}\widehat{\tau}_1 + \left(k^2\frac{4\mu(\lambda + \mu)}{(\lambda + 2\mu)} - \omega^2\varrho\right)\widehat{u}_2. \tag{4.17}$$

It is seen that the wave process depends on the following five material parameters (similarly to the stationary harmonic case – see [14]):

$$\alpha = \frac{\lambda}{(\lambda + 2\mu)}, \tag{4.18}$$

$$\beta = \frac{4\mu(\lambda + \mu)}{(\lambda + 2\mu)}, \tag{4.19}$$

$$\kappa = \frac{1}{(\lambda + 2\mu)}, \tag{4.20}$$

$$\eta = \frac{1}{\mu}, \tag{4.21}$$

and density ϱ.

Using the above symbols, we can rewrite the system (4.14)–(4.17) in the abstract matrix form analogous to (3.10) where, by definition,

$$\hat{\mathbf{u}} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{\tau}_1 \\ \hat{\tau}_2 \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} 0 & -ik\alpha & \kappa & 0 \\ -ik & 0 & 0 & \eta \\ -\omega^2\varrho & 0 & 0 & -ik \\ 0 & 4k^2\beta - \omega^2\varrho & -ik\alpha & 0 \end{bmatrix}. \tag{4.22}$$

To find the transition operator (matrix) for the system of equations we must know the eigenvalues of the matrix $\mathbf{M}$. Solving the characteristic equation:

$$\det(\mathbf{M} - p\mathbf{Id}) = p^4 + p^2\left(\omega^2\varrho(\kappa + \eta) + 2k^2(\alpha - 2\beta\eta)\right) \\ + \omega^4\kappa\eta\varrho^2 - \omega^2k^2\varrho\left(\alpha^2\eta + 4\beta\kappa\eta + \kappa\right) + k^4\left(\alpha^2 + 4\beta\kappa\right) = 0 \tag{4.23}$$

and substituting the definitions of the parameters, we obtain

$$p_{1,2,3,4} = \pm\sqrt{2k^2A_1 - \omega^2A_2 \pm \omega^2A_3}\,/\sqrt{2}\,, \tag{4.24}$$

where the parameters A_1, A_2 and A_3 are defined by

$$A_1 = 2\beta\eta - \alpha = 1, \tag{4.25}$$

$$A_2 = \varrho(\kappa + \eta) = \varrho\frac{\lambda + 3\mu}{\mu(\lambda + 2\mu)}, \tag{4.26}$$

$$A_3 = \varrho(\kappa - \eta) = \varrho\frac{\lambda + \mu}{\mu(\lambda + 2\mu)}. \tag{4.27}$$

In an explicit form, the eigenvalues are

$$p_{1,2} = \pm\frac{\sqrt{k^2\mu - \omega^2\varrho}}{\sqrt{\mu}}, \tag{4.28}$$

$$p_{3,4} = \pm\frac{\sqrt{k^2(\lambda + 2\mu) - \omega^2\varrho}}{\sqrt{\lambda + 2\mu}}. \tag{4.29}$$

The transition matrix for the wave in the plane state of deformation can be calculated according to the Lagrange interpolation formula, analogous to (3.17), (see [23]). The elements of the transition matrix have the following form:

$$\mathbf{T}_{11}(L) = \frac{2\mu k^2}{\omega^2\varrho}\operatorname{ch} p_1 L + \left(1 - \frac{2\mu k^2}{\omega^2\varrho}\right)\operatorname{ch} p_3 L, \tag{4.30}$$

$$\mathbf{T}_{12}(L) = -\frac{i(\omega^2\varrho - 2\mu k^2)k \operatorname{sh} p_1 L}{\sqrt{k^2 - \omega^2\varrho/\mu}\,\omega^2\varrho} - \frac{2i\sqrt{k^2 - \omega^2\varrho/(\lambda + 2\mu)}\,\mu k \operatorname{sh} p_3 L}{\omega^2\varrho}, \tag{4.31}$$

$$\mathbf{T}_{13}(L) = \frac{k^2 \operatorname{sh} p_1 L}{\sqrt{k^2 - \omega^2\varrho/\mu}\,\omega^2\varrho} - \frac{\sqrt{k^2 - \omega^2\varrho/(\lambda + 2\mu)}\operatorname{sh} p_3 L}{\omega^2\varrho}, \tag{4.32}$$

$$\mathbf{T}_{14}(L) = \frac{ik}{\omega^2\varrho}\left(\operatorname{ch} p_1 L - \operatorname{ch} p_3 L\right), \tag{4.33}$$

$$\mathbf{T}_{21}(L) = \frac{2i\sqrt{k^2 - \omega^2\varrho/\mu}\,\mu k \operatorname{sh} p_1 L}{\omega^2\varrho} + \frac{i(\omega^2\varrho - 2\mu k^2)k \operatorname{sh} p_3 L}{\sqrt{k^2 - \omega^2\varrho/(\lambda + 2\mu)}\,\omega^2\varrho}, \tag{4.34}$$

$$\mathbf{T}_{22}(L) = \left(1 - \frac{2\mu k^2}{\omega^2\varrho}\right)\operatorname{ch} p_1 L + \frac{2\mu k^2}{\omega^2\varrho}\operatorname{ch} p_3 L, \tag{4.35}$$

$$\mathbf{T}_{23}(L) = \frac{ik}{\omega^2\varrho}\left(\operatorname{ch} p_1 L - \operatorname{ch} p_3 L\right), \tag{4.36}$$

$$\mathbf{T}_{24}(L) = -\frac{\sqrt{k^2 - \omega^2\varrho/\mu}\operatorname{sh} p_1 L}{\omega^2\varrho} + \frac{k^2 \operatorname{sh} p_3 L}{\sqrt{k^2 - \omega^2\varrho/(\lambda + 2\mu)}\,\omega^2\varrho}, \tag{4.37}$$

$$\mathbf{T}_{31}(L) = \frac{4\sqrt{k^2 - \omega^2\varrho/\mu}\,\mu^2 k^2 \operatorname{sh} p_1 L}{\omega^2\varrho} - \frac{(\omega^2\varrho - 2\mu k^2)^2 \operatorname{sh} p_3 L}{\sqrt{k^2 - \omega^2\varrho/(\lambda + 2\mu)}\,\omega^2\varrho}, \tag{4.38}$$

$$\mathbf{T}_{32}(L) = 2i\left(1 - \frac{2\mu k^2}{\omega^2\varrho}\right)k\mu(\operatorname{ch} p_3 L - \operatorname{ch} p_1 L), \tag{4.39}$$

$$\mathbf{T}_{33}(L) = \frac{2\mu k^2}{\omega^2\varrho}\operatorname{ch} p_1 L + \left(1 - \frac{2\mu k^2}{\omega^2\varrho}\right)\operatorname{ch} p_3 L, \tag{4.40}$$

$$\mathbf{T}_{34}(L) = \frac{2i\sqrt{k^2 - \omega^2\varrho/\mu}\,\mu k \operatorname{sh} p_1 L}{\omega^2\varrho} + \frac{i(\omega^2\varrho - 2\mu k^2)k \operatorname{sh} p_3 L}{\sqrt{k^2 - \omega^2\varrho/(\lambda + 2\mu)}\,\omega^2\varrho}, \tag{4.41}$$

$$\mathbf{T}_{41}(L) = 2i\left(1 - \frac{2\mu k^2}{\omega^2\varrho}\right)k\mu(\operatorname{ch} p_3 L - \operatorname{ch} p_1 L), \tag{4.42}$$

$$(4.43)\qquad \mathbf{T}_{42}(L) = -\frac{(\omega^2\varrho - 2\mu k^2)^2 \operatorname{sh} p_1 L}{\sqrt{k^2 - \omega^2\varrho/\mu}\,\omega^2\varrho} + \frac{4\sqrt{k^2 - \omega^2\varrho/(\lambda + 2\mu)}\,\mu^2 k^2 \operatorname{sh} p_3 L}{\omega^2\varrho},$$

$$(4.44)\qquad \mathbf{T}_{43}(L) = -\frac{i(\omega^2\varrho - 2\mu k^2)k \operatorname{sh} p_1 L}{\sqrt{k^2 - \omega^2\varrho/\mu}\,\omega^2\varrho} - \frac{2i\sqrt{k^2 - \omega^2\varrho/(\lambda + 2\mu)}\,\mu k \operatorname{sh} p_3 L}{\omega^2\varrho},$$

$$(4.45)\qquad \mathbf{T}_{44}(L) = \left(1 - \frac{2\mu k^2}{\omega^2\varrho}\right) \operatorname{ch} p_1 L + \frac{2\mu k^2}{\omega^2\varrho} \operatorname{ch} p_3 L.$$

5. Elastic waves in layered media

The transition matrices obtained in Sec. 4 enable us to describe the transition of the two-dimensional elastic wave through a multi-layered medium. In such a case, knowing the transition matrices through individual layers, we can obtain the transition matrix through the whole stratified medium as a product of the matrices.

The transition matrix $\mathbf{T}(\cdot)$ enables us to express the wave field $\hat{\mathbf{u}}$,

$$(5.1)\qquad \hat{\mathbf{u}} = \begin{bmatrix} \hat{u} \\ \hat{\tau} \end{bmatrix} \qquad \text{or} \qquad \hat{\mathbf{u}} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{\tau}_1 \\ \hat{\tau}_2 \end{bmatrix},$$

at any point $x = L \in \mathbb{R}^+$ in a homogeneous medium, provided the boundary condition $\hat{\mathbf{u}}_0 = \hat{\mathbf{u}}(0)$ at $x = 0$ is known in the form (3.13).

Consider at the moment the multi-layered medium (slab) built of N layers of elastic materials, with thicknesses Δ_j, $j = 1, 2, \ldots, N$. Assume that the stratified medium is surrounded by the homogeneous elastic environment, at $x < 0$ and $x > L = \sum_{j=1}^{N} \Delta_j$. Since the wave field $\hat{\mathbf{u}}$ must be continuous at the interfaces of the layers in the stratified medium, we can express the wave, generated by some boundary conditions $\hat{\mathbf{u}}_0$ at $x = 0$, after it reaches the point L, in the form

$$(5.2)\qquad \hat{\mathbf{u}}(L) = \mathbf{T}_N(\Delta_N)\mathbf{T}_{N-1}(\Delta_{N-1}) \ldots \mathbf{T}_2(\Delta_2)\mathbf{T}_1(\Delta_1)\hat{\mathbf{u}}_0,$$

or, in a more convex form, by

$$(5.3)\qquad \hat{\mathbf{u}}(L) = \prod_{j=1}^{N} \mathbf{T}_j(\Delta_j)\hat{\mathbf{u}}_0,$$

where $\hat{\mathbf{u}}_0$ is the vector describing the incident and reflected wave, $\hat{\mathbf{u}}(L)$ is the vector of the transmitted wave, $\mathbf{T}_j(\cdot)$ is the transition matrix through j-th layer, for $j = 1, 2, \ldots, N$, depending on the material parameters of the layer.

In the above equation all the material properties of the multi-layered medium are completely described by the 4×4 matrix $\mathcal{T}$, being the product of the transition matrices through the individual layers and interpreted as a transition matrix through the slab built of N layers of homogeneous elastic materials,

$$\mathcal{T} = \prod_{j=1}^{N} \mathbf{T}_j(\Delta_j). \tag{5.4}$$

Let us remark that vector $\hat{\mathbf{u}}_0$ describes jointly the (Fourier transforms of) incident wave pulse (going to the right), and all the reflected pulses leaving the slab (going to the left), generated by all the reflections at the interfaces of the layers, measured at the plane $x \equiv 0$. Similarly, $\hat{\mathbf{u}}(L)$ represents the transmitted pulses, generated by all the reflections and transmissions at the internal interfaces of the layers, measured at the plane $x \equiv L$.

6. The limiting case – homogenization

Assume that the slab is built of $2K$ layers with thicknesses $l_1(\gamma), l_2(\gamma), \ldots, l_{2K}(\gamma)$, where $l_i(\gamma)$, $i = 1, 2, \ldots, 2K$ are random variables. In the above $\gamma \in \Gamma$ is an elementary event and $(\Gamma, \mathcal{F}, \mathcal{P})$ is the complete probabilistic space. Assume additionally that the material parameters of the layers and their thicknesses $(\varrho_{2j-1}(\gamma),\ \lambda_{2j-1}(\gamma),\ \mu_{2j-1}(\gamma),\ l_{2j-1}(\gamma),\ \varrho_{2j}(\gamma),\ \lambda_{2j}(\gamma),\ \mu_{2j}(\gamma),\ l_{2j}(\gamma))$ are, as the vector random variables, independent and identically distributed for $j = 1, 2, \ldots, K$. Moreover, we assume that the thicknesses of the layers have the following particular property:

$$(l_{2j-1}(\gamma), l_{2j}(\gamma)) = \left(\frac{L_{2j-1}(\gamma)}{2K} \ \frac{L_{2j}(\gamma)}{2K} \right), \tag{6.1}$$

for $j = 1, 2, \ldots, K$, are independent, identically distributed two-dimensional random variables with the given mean values:

$$E\{L_{2j-1}(\gamma)\} = L^1, \qquad E\{L_{2j}(\gamma)\} = L^2. \tag{6.2}$$

In this particular case the periodically repeated segments of the bar are built of the couples of the elements with lengths $l_{2j-1}(\gamma), l_{2j}(\gamma)$, $j = 1, 2, \ldots, K$. For such segments the transition matrices $\mathbf{M}_j(\gamma)$ are the products of the pairs of the transition matrices through the individual layers,

$$\mathbf{M}_j(\gamma) = \mathbf{T}_{2j-1}(l_{2j-1}(\gamma))\mathbf{T}_{2j}(l_{2j}(\gamma)), \qquad j = 1, 2, \ldots, K, \tag{6.3}$$

and the Eq. (5.3) for the Fourier transform of the amplitudes takes the following form ($2K = N$):

$$\hat{\mathbf{u}}(L) = \prod_{j=1}^{K} \mathbf{M}_j(\gamma)\, \hat{\mathbf{u}}_0\,, \tag{6.4}$$

where $L = L(\gamma) = \sum_{j=1}^{N} l_j(\gamma)$.

To study the asymptotic behaviour of the randomized equation for the amplitudes of the waves, we apply the law of large numbers for the products of random matrices obtained in [3]. This theorem can be written in the following form.

Consider the sequence of the products of real random matrices

$$\mathbf{P}_K(\gamma) = \prod_{j=1}^{K} \mathbf{M}_{j,K}(\gamma). \tag{6.5}$$

It is assumed that for K tending to infinity the matrices $\mathbf{M}_{j,K}$ can be represented by

$$\mathbf{M}_{j,K}(\gamma) = \mathbf{Id} + \frac{1}{K}\mathbf{B}_{j,K}(\gamma) + \mathbf{R}_j(K,\gamma), \tag{6.6}$$

where $\mathbf{B}_{j,K}(\gamma)$ for $j = 1, 2, \ldots, K$ are independent, identically distributed random matrices, integrable with respect to probability measure $\mathcal{P}$ and $|\mathbf{R}_j(K,\gamma)| = o(K^{-1})$ for large K. Under these conditions, the law of large numbers holds true and

$$\lim_{K\to\infty} \mathbf{P}_K(\gamma) = \exp\left(E\{\mathbf{B}_{j,K}(\gamma)\}\right), \tag{6.7}$$

in the sense of convergence in distribution of all the vectors obtained by multiplication of the random matrix by an arbitrary deterministic vector.

The presented method makes it possible to obtain the effective transition matrices in both cases of the anti-plane (Sec. 3) and the plane (Sec. 4) state of deformation. Let us begin the considerations from the more complicated, second example.

To analyze the limiting case of Eq. (6.4) when K tends to infinity, we expand, at the beginning, the transition matrix defined in (4.30)–(4.45) under the assumption (6.1) on the thickness of the layers, with respect to the powers of $1/K$:

$$\mathbf{T}_j\left(\frac{L_j}{K}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \frac{L_j}{K}\begin{bmatrix} 0 & -ik\alpha_j & \kappa_j & 0 \\ -ik & 0 & 0 & \eta_j \\ -\omega^2\varrho_j & 0 & 0 & -ik \\ 0 & 4k^2\beta_j - \omega^2\varrho_j & -ik\alpha_j & 0 \end{bmatrix} + o\left(\frac{L_j}{K}\right). \tag{6.8}$$

Multiplying the matrix $\mathbf{T}_1(L_1)$, corresponding to the transition matrices with odd indices by $\mathbf{T}_2(L_2)$ – with even indices, we obtain, that the matrices $\mathbf{B}_j$ required in formula (6.7) are defined as (we have changed the numeration of the random variables being the material parameters and the thicknesses of the layers according to the following rule: $b_{2j-1} = b_j^1$, $b_{2j} = b_j^2$ for any parameter (b is $\alpha, \beta, \kappa, \eta, \varrho$ or L) and $j = 1, 2, \ldots, K$, so the parameters with identical superscripts – 1 or 2 – have identical distributions):

$$(6.9) \qquad \mathbf{B}_j = \begin{bmatrix} 0 & -ik\left(\alpha_j^1 L_j^1 + \alpha_j^2 L_j^2\right) & \kappa_j^1 L_j^1 + \kappa_j^2 L_j^2 & 0 \\ -ik\left(L_j^1 + L_j^2\right) & 0 & 0 & \eta_j^1 L_j^1 + \eta_j^2 L_j^2 \\ -\omega^2\left(\varrho_j^1 L_j^1 + \varrho_j^2 L_j^2\right) & 0 & 0 & -ik\left(L_j^1 + L_j^2\right) \\ 0 & 4k^2\left(\beta_j^1 L_j^1 + \beta_j^2 L_j^2\right) - \omega^2\left(\varrho_j^1 L_j^1 + \varrho_j^2 L_j^2\right) & -ik\left(\alpha_j^1 L_j^1 + \alpha_j^2 L_j^2\right) & 0 \end{bmatrix}.$$

The common average value of the matrices $\mathbf{B}_j$ is

$$(6.10) \qquad E\{\mathbf{B}_j\} = \begin{bmatrix} 0 & -ik\,E(\alpha_j^1 L_j^1 + \alpha_j^2 L_j^2) & E(\kappa_j^1 L_j^1 + \kappa_j^2 L_j^2) & 0 \\ -ik\,E(L_j^1 + L_j^2) & 0 & 0 & E(\eta_j^1 L_j^1 + \eta_j^2 L_j^2) \\ -\omega^2 E(\varrho_j^1 L_j^1 + \varrho_j^2 L_j^2) & 0 & 0 & -ikE(L_j^1 + L_j^2) \\ 0 & 4k^2 E(\beta_j^1 L_j^1 + \beta_j^2 L_j^2) - \omega^2 E(\varrho_j^1 L_j^1 + \varrho_j^2 L_j^2) & -ikE(\alpha_j^1 L_j^1 + \alpha_j^2 L_j^2) & 0 \end{bmatrix},$$

where in the above formulae the parameters and the thicknesses are random variables with distributions identical for all couples of layers.

The matrix $e^{E\{\mathbf{B}_j\}}$ is of the form analogous to (4.30)–(4.45) where, instead of the parameters $\alpha(\gamma), \beta(\gamma), \kappa(\gamma), \varrho(\gamma), \eta(\gamma), p_1(\gamma), p_3(\gamma)$, being random variables, one has the effective constant parameters $\alpha^{\text{eff}}, \beta^{\text{eff}}, \kappa^{\text{eff}}, \varrho^{\text{eff}}, \eta^{\text{eff}}, p_1^{\text{eff}}, p_3^{\text{eff}}$, defined as

$$\alpha^{\text{eff}} = \frac{E\left\{\alpha^1(\gamma)L^1(\gamma) + \alpha^2(\gamma)L^2(\gamma)\right\}}{L}, \tag{6.11}$$

$$\beta^{\text{eff}} = \frac{E\left\{\beta^1(\gamma)L^1(\gamma) + \beta^2(\gamma)L^2(\gamma)\right\}}{L}, \tag{6.12}$$

$$\kappa^{\text{eff}} = \frac{E\left\{\kappa^1(\gamma)L^1(\gamma) + \kappa^2(\gamma)L^2(\gamma)\right\}}{L}, \tag{6.13}$$

$$\eta^{\text{eff}} = \frac{E\left\{\eta^1(\gamma)L^1(\gamma) + \eta^2(\gamma)L^2(\gamma)\right\}}{L}, \tag{6.14}$$

$$\varrho^{\text{eff}} = \frac{E\left\{\varrho^1(\gamma)L^1(\gamma) + \varrho^2(\gamma)L^2(\gamma)\right\}}{L}, \tag{6.15}$$

where

$$\alpha^1 = \frac{\lambda^1}{(\lambda^1 + 2\mu^1)}, \qquad \alpha_2 = \frac{\lambda^2}{(\lambda^2 + 2\mu^2)}, \tag{6.16}$$

$$\beta^1 = \frac{4\mu^1(\lambda^1 + \mu^1)}{(\lambda^1 + 2\mu^1)}, \qquad \beta^2 = \frac{4\mu^2(\lambda^2 + \mu^2)}{(\lambda^2 + 2\mu^2)}, \tag{6.17}$$

$$\kappa^1 = \frac{1}{(\lambda^1 + 2\mu^1)}, \qquad \kappa^2 = \frac{1}{(\lambda^2 + 2\mu^2)}, \tag{6.18}$$

$$\eta^1 = \frac{1}{\mu^1}, \qquad \eta^2 = \frac{1}{\mu^2}, \tag{6.19}$$

$$L = L^1 + L^2 \tag{6.20}$$

and

$$p_{1,3}^{\text{eff}} = \sqrt{2k^2 - \omega^2 A_2^{\text{eff}} \pm \omega^2 A_3^{\text{eff}}}/\sqrt{2}, \tag{6.21}$$

with the parameters A_2^{eff} and A_3^{eff} which are defined as:

$$A_2^{\text{eff}} = \varrho^{\text{eff}}(\kappa^{\text{eff}} + \eta^{\text{eff}}) = \varrho^{\text{eff}} \frac{\lambda^{\text{eff}} + 3\mu^{\text{eff}}}{\mu^{\text{eff}}(\lambda^{\text{eff}} + 2\mu^{\text{eff}})}, \tag{6.22}$$

$$A_3^{\text{eff}} = \varrho^{\text{eff}}(\kappa^{\text{eff}} - \eta^{\text{eff}}) = \varrho^{\text{eff}} \frac{\lambda^{\text{eff}} + \mu^{\text{eff}}}{\mu^{\text{eff}}(\lambda^{\text{eff}} + 2\mu^{\text{eff}})}. \tag{6.23}$$

Similarly in the case of anti-plane deformation, the transition matrix through the individual layer (3.18) can be written in the form (6.6),

$$\mathbf{T}_j\left(\frac{L_j}{K}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{L_j}{K} \begin{bmatrix} 0 & \dfrac{1}{\mu} \\ \mu k^2 - \varrho\omega^2 & 0 \end{bmatrix} + o\left(\frac{L_j}{K}\right). \tag{6.24}$$

The matrix $\mathbf{B}_j$ is, in the case of the anti-plane wave pulses, of the form

$$\mathbf{B}_j = \begin{bmatrix} 0 & \frac{L^1(\gamma)}{\mu^1(\gamma)} + \frac{L^2(\gamma)}{\mu^2(\gamma)} \\ \left(\mu^1(\gamma)L^1(\gamma)+\mu^2(\gamma)L^2(\gamma)\right) k^2 - \left(\varrho^1(\gamma)L^1(\gamma)+\varrho^2(\gamma)L^2(\gamma)\right)\omega^2 & 0 \end{bmatrix}. \tag{6.25}$$

By averaging the matrix $\mathbf{B}_j$ we can see that the effective transition matrix depends now on three parameters: κ^{eff} and ϱ^{eff}, defined in (6.14), (6.15), and on the additional parameter μ^{eff}, defined as:

$$\mu^{\text{eff}} = \frac{E\left\{\mu^1(\gamma)L^1(\gamma) + \mu^2(\gamma)L^2(\gamma)\right\}}{L}. \tag{6.26}$$

The transition matrix has the form analogous to (3.18) with suitable effective parameters, where

$$p^{\text{eff}} = \sqrt{k^2 - \varrho^{\text{eff}}\omega^2/\mu^{\text{eff}}}\,. \tag{6.27}$$

As a conclusion of the above considerations we can say that the elastic homogenized medium (obtained by the homogenization procedure from a randomly stratified medium) conducting wave pulses is completely described by six parameters: $\alpha^{\text{eff}}, \beta^{\text{eff}}, \kappa^{\text{eff}}, \eta^{\text{eff}}, \mu^{\text{eff}}$ and ϱ^{eff}, defined in (6.11)–(6.15) and (6.26).

7. Closing remarks

In the paper we have considered the model of stratified medium, that is the slab built of a number of isotropic, homogeneous elastic layers. Such a medium, globally, is both anisotropic and nonhomogeneous. After the presented averaging procedure, it becomes homogeneous but remains anisotropic (locally and globally transversally isotropic). The elastic properties of such a medium are described by a tensor, whose 5 elements are independent (see [22]). However, as we have seen from the considerations of the previous sections, to describe the elastic waves in the case of the plane state of deformation we need four elastic constants, while in the anti-plane state only two elastic constants are necessary (one of them – different than in the plane state). This statement remains valid both in the dynamic nonstationary case, studied in this paper, and in the stationary one (see [14]).

The above considerations were performed analytically. Solution of the equations, that is calculation of the resulting waves (reflected and transmitted) generated by some incident pulse needs numerical calculations. The most effective way of doing this is calculating the inverse Fourier transform using the Fast Fourier Transform algorithm (see [20]). Effective results in this field require precise specification of the incident pulse (that is, its shape, caused by the form of the source generating the disturbance).

Let us finally remark that the model of stratified medium considered in this paper is an idealization of a real physical medium. Therefore it neglects many effects observed in nature (like dissipation of energy or dispersion of waves in layered media) and needs some modifications. However, the improvement of the model of two-dimensional stratified medium is connected with the growth of the dimension of the system of the corresponding partial differential equations and is connected with numerical difficulties.

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