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On two motions of a particle driven by equivalent ergodic and chaotic reflection laws

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IN THE PAPER we analyse dynamical systems describing the motion of a free particle in a domain on a plane (a square). We show that topologically equivalent reflection laws (each of them ergodic and chaotic) governing particle's motion at the moment of reflection can lead to two dynamical systems with entirely different qualitative properties. We also indicate a general problem of transferring such properties like chaos and ergodicity from a subsystem to the extended one.

1. Introduction

THE MOTION OF A FREE PARTICLE in a bounded domain is inherently determined by the shape of the boundary and the reflection law at this boundary. The reflection law is responsible for the global behaviour of the velocity of the particle during its contact with the boundary of the domain. In such dynamical systems (in the idealised theoretical model), the fundamental physical laws like the conservation of linear momentum and the conservation of energy are assumed to be satisfied what leads to extensively studied classical billiards. This means that the incidence angle is equal to the reflection one. In general, analysing the transformation of the angles of the moving particle at the moment of reflection one can observe that the reflection law itself is a dynamical system. This has created a temptation to consider the reflection law as an independent dynamical system.

The theory of the non-classical reflection laws found its place in the literature [1–5]. Up to now there are only hypotheses on what happens when the particle reaches the boundary, more or less confirmed by experiment. Reflection law models are an intermediate case between the deterministic systems first considered by SCHNUTE and SHINBROT [2] and systems with random reflection laws [6]. Namely, we admit a system with a strictly deterministic reflection laws that are not one-to-one maps. Thus, in this case it can happen that two different initial configurations in the phase space lead to the same final configuration what is impossible in the Schnute and Shinbrot model. There is a number of maps playing the role of the reflection law. The authors investigate the properties of the reflection laws finding that they can lead to such phenomena like: non-slip reflection on the boundary, non-increasing entropy, chaos, ergodicity (mixing property) of systems describing behaviour of the particle.

The reflection laws describe the global behaviour of the velocity of a freely moving particle during its contact with the boundary of the domain. From this point of view, non-classical reflection laws do not satisfy such a fundamental physical law as the conservation of linear momentum. However, one can find some situations where such laws can describe realistic physical phenomena. Consider for example the container, the wall of which has some microstructure (Fig. 1). We assume that the mass of the reflected particle is negligible in comparison to the mass of the container. Then the reflection process, observed as non-classical, can in fact be the effect of few classical elastic reflections where, for every microreflection, the conservation of linear momentum is satisfied. In this model, due to the small scale of the microreflection, we identify the outgoing positions with the incoming point.

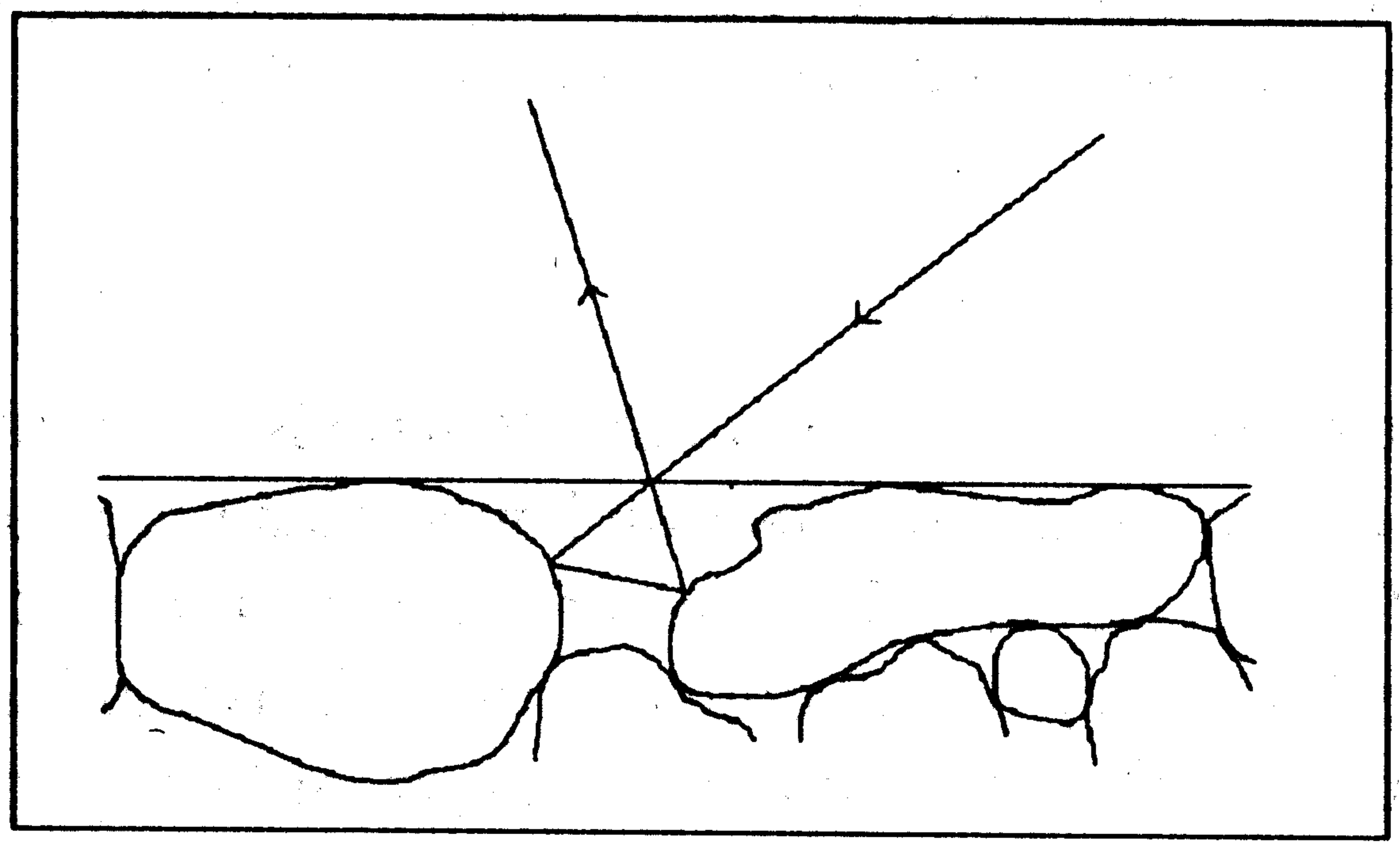

FIG. 1. Effect of the boundary microstructure on the reflection law.

After the reflection law was extracted from the extended dynamical system describing the motion of freely moving particle and then independently considered, one can ask the following questions: What are the properties of the extended system if we use non-classical reflection law? What is the effect of the specific properties of the reflection law (like chaos or ergodicity) on the behaviour of the particle? Is the particle motion chaotic or ergodic? Let us remark that this is a different problem than the chaotic or ergodic motion of the particle observed in classical billiard systems (connected with a specific shape of the domain's boundary). In this paper we just try to answer the question of transferring the specific properties from a non-classical reflection law to the dynamical system of a moving particle. We perform our considerations in two dimensions, where qualitative results we are interested in can be observed. Extensions of the results to more-dimensional spaces lead to some technical problems, what can be also observed in the case of the widely studied classical billiards theory. However, the results in two dimensions can give some suggestions concerning the behaviour of more-dimensional systems.

Problems of transferring of imposed properties from a dynamical system to its extension appear in various situations [4, 5, 7, 8] and seem to be interesting both from the theoretical and practical point of view. They naturally arise from the problems of physics, engineering dynamics, mathematical economy and many others. In general, by an extended dynamical system we understand a system with state space of dimension greater than the original one and functionally dependent on it. Such a system can be a simple extension of the given dynamical system obtained by adding more co-ordinates without changing the form of the primary ones, or it can be some higher-dimensional dynamical system driven by the lower-dimensional one. In this paper we consider the transfer problems in the case of a free particle motion inside a bounded plane domain. We assume the reflection law as a primary dynamical system and the motion of the reflecting particle as an extended system.

To establish a reflection law model one must select a domain with a certain shape of the boundary and define the reflection law. Usually, the boundary is assumed to be a closed, sufficiently smooth curve. The reflection law can be quite general; in our considerations we assume that the particle moves with a constant velocity, changing the direction at the moment of reflection. In the particular case of the reflection law conserving the angle of incidence (the angle of incidence is equal to the angle of reflection), one obtains the class of dynamical systems called billiards. This conservative reflection law (as a map) is neither ergodic nor chaotic (see formula (*) in the next Section). However, it is well known that in appropriate domains it can lead to ergodic or chaotic motion of a particle. Thus, to obtain ergodic [9] and chaotic properties [8, 10–11] of a reflection law, one must assume another map relating the incident and outcoming angles. Such models have been studied in [1–5].

Applying various reflection laws, we face some natural questions when describing the motion of particles:

• Fix a reflection law. Do the ergodic and chaotic properties of the law transfer to the same properties of particles' motion for some typically used shapes of the domain?

• Fix a shape of the domain. Do topologically conjugate ergodic and chaotic reflection laws generate equivalent motion of the particle?

Some insight into the first problem was given in [5]. It was shown that for two simple domains, the ergodic and chaotic properties of the same reflection law can transfer in a quite different manner. In this paper we deal with the second question.

2. Formulation

Now we specify the model. We assume that the domain of a moving particle is a square. In the domain, the particle moves along straight lines with a constant velocity; when it encounters a wall it "reflects", that is, its velocity instantaneously changes (according to some reflection law) to another "reflected" value to make the particle remain inside the domain. The motion of the particle is described by two co-ordinates (Fig. 2):

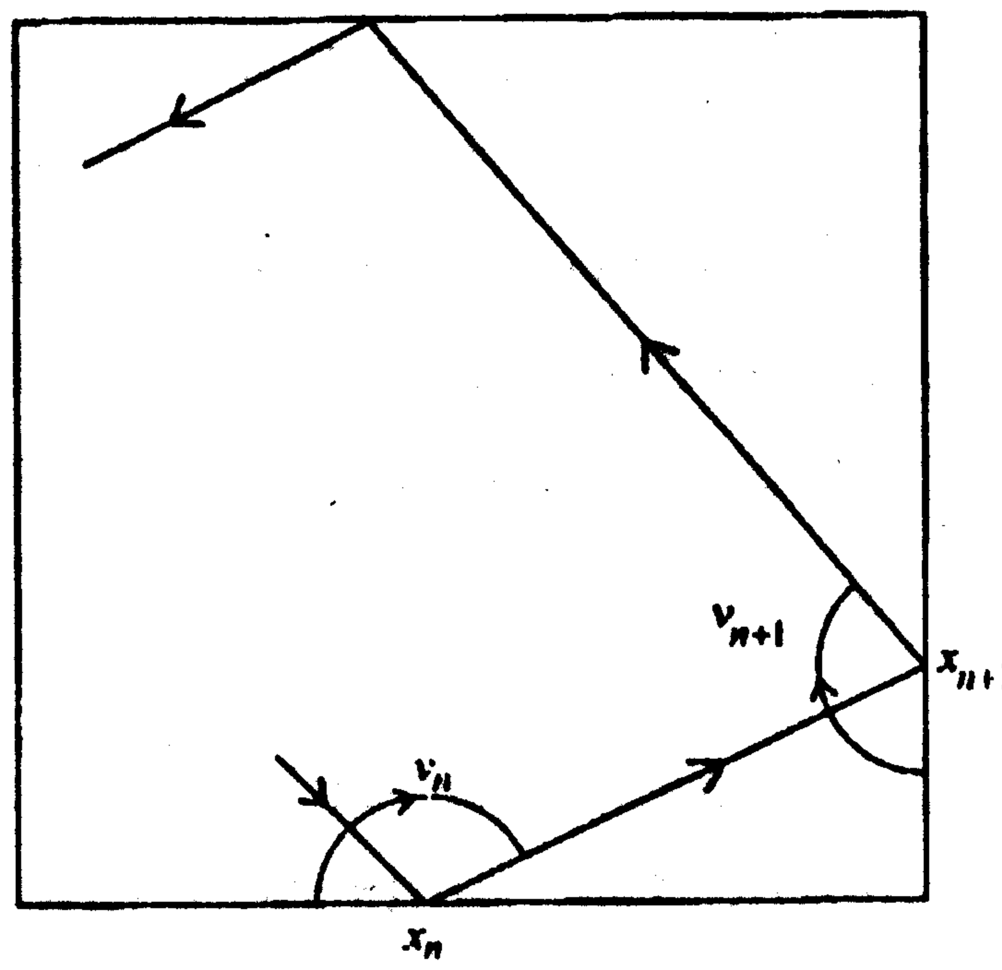

FIG. 2. The co-ordinate system used to describe the motion of a particle in a square.

- the position x_n at the square's boundary at the moment of the n-th reflection (measured counterclockwise from the fixed vertex of the square);
- the angle ν_n measured from the tangent to the boundary to the velocity vector of the point after reflection (clockwise).

To complete the definition of the system we assume some reflection law $T : (0,\pi) \to (0,\pi)$, $T(\nu_{\text{inc}}) = \nu_{\text{ref}}$ (Fig. 3). For example, in this formalism, the conservative reflection law is given by the map

$$\nu_{\text{ref}} = T(\nu_{\text{inc}}) = \pi - \nu_{\text{inc}}.$$

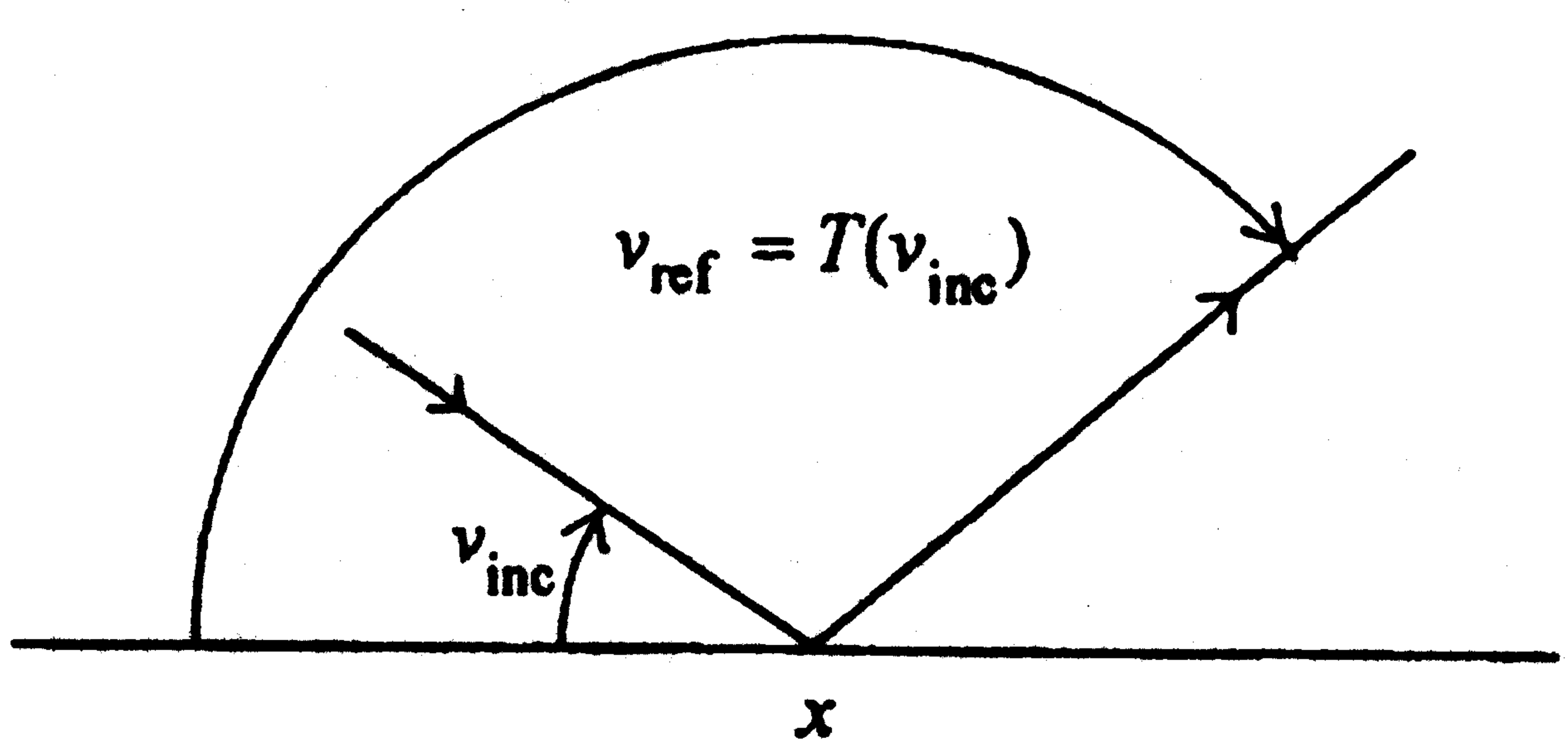

FIG. 3. The reflection law in local co-ordinates.

Thus, the motion is described by the two-dimensional map

$$F_T : [0,L) \times (0,\pi) \to [0,L) \times (0,\pi), \qquad (2.1)$$
$$F_T(x_n,\nu_n) = (x_{n+1},\nu_{n+1}),$$

where the subscript in F_T denotes the dependence of the function on the reflection law T, and L is the length of the boundary of the square.

We consider the following two reflection laws:

$$\begin{aligned} &T_1 : (0,\pi) \to (0,\pi), \\ &\nu_{\text{ref}} = T_1(\nu_{\text{inc}}) = \frac{4}{\pi}\nu_{\text{inc}}(\pi - \nu_{\text{inc}}), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &T_2 : (0,\pi) \to (0,\pi), \\ &\nu_{\text{ref}} = T_2(\nu_{\text{inc}}) = \begin{cases} 2\nu_{\text{inc}} & \text{for} \quad \nu_{\text{inc}} \in (0,\pi/2), \\ 2(\pi - \nu_{\text{inc}}) & \text{for} \quad \nu_{\text{inc}} \in [\pi/2,\pi). \end{cases} \end{aligned} \tag{2.3}$$

T_1 is a unimodal map which is ergodic and chaotic [12]. T_2 is the so-called tent map, also ergodic and chaotic [13].

These maps are topologically conjugate [14]; the equivalence is given by the homeomorphism

$$g(\nu) = 2 \arcsin \sqrt{\frac{\nu}{\pi}}, \tag{2.4}$$

i.e. the following diagram is commutative:

$$\begin{array}{ccc} (0,\pi) & \xrightarrow{T_1} & (0,\pi) \\ \downarrow g & & \downarrow g \\ (0,\pi) & \xrightarrow{T_2} & (0,\pi) \end{array} \tag{2.5}$$

This diagram yields the following implications:

I. If $\nu_k \to \tilde{\nu}$ (so $T_1(\nu_k) \to T_1(\tilde{\nu})$) then the g-corresponding sequences satisfy: $g(\nu_k) \to g(\tilde{\nu})$ and $T_2(g(\nu_k)) \to g(T_1(\tilde{\nu}))$.

II. If the orbit $\{T_1^n(\nu_0), n = 0,1,2,...\}$ has some properties like periodicity, asymptotic periodicity or density, then the g-corresponding $\{T_2^n(g(\nu_0)), n = 0,1,2,...\}$ orbit has the same properties.

3. Results

Consider the motion of the particle in a square. In the models presented, the velocity of the particle inside the square is constant and the reflection law at the boundary is given by either T_1 or T_2. It was proved in [5] that if the reflection law is defined by T_1 then the motion F_{T_1} of the particle is asymptotically periodic, i.e. for almost all initial points (x_0, ν_0), after sufficiently many reflections, the particle moves closer and closer to the edges of the square. More precisely, the angle ν_n tends to π and so the motion of the particle converges to the periodic changes of the positions x_n from vertex to vertex.

Now assume that the reflection law is defined by T_2. We show that the motion F_{T_2} differs qualitatively from F_{T_1}. To study the behaviour of the system we observe the second co-ordinate ν of motion of the particle. First notice that due to the geometry of the square (see Fig. 4), the velocity ν_n changes in the following way:

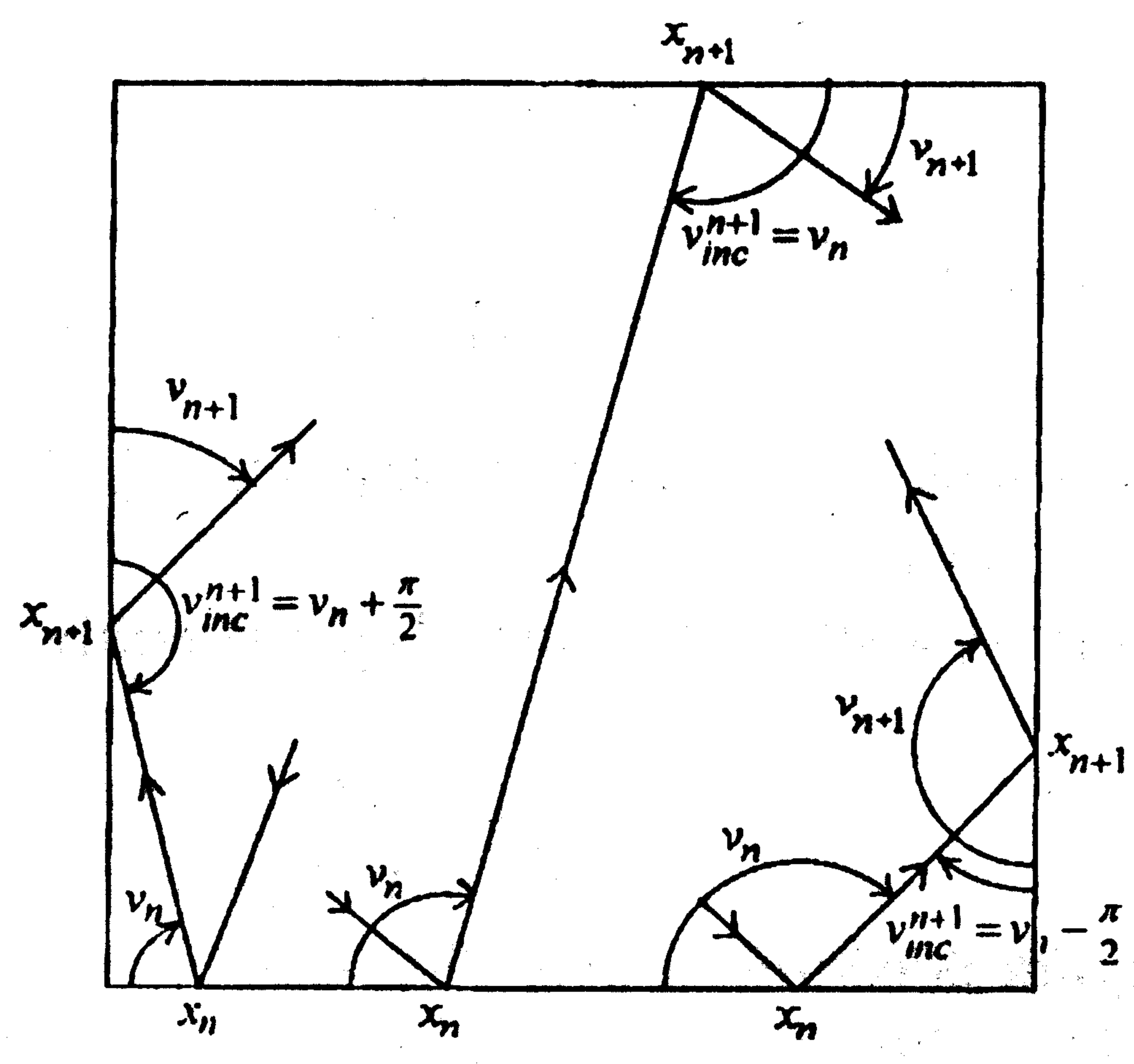

FIG. 4. Types of reflections in a square.

a) $\nu_{n+1} = T_2(\nu_n) = \begin{cases} 2\nu_n & \text{for} \quad \nu_n \in (\pi/4, \pi/2) \\ 2(\pi - \nu_n) & \text{for} \quad \nu_n \in [\pi/2, 3\pi/4) \end{cases}$ if the particle moves from one side to the opposite one. Notice that this is possible only when $\pi/4 < \nu_n < 3\pi/4$, which restricts the domain of the velocity in (2.3).

b) $\nu_{n+1} = 2\left(\frac{\pi}{2} - \nu_n\right)$ if the particle goes from one side to the clockwise adjacent side; this is possible only when $0 < \nu_n < \pi/2$.

c) $\nu_{n+1} = 2\left(\nu_n - \frac{\pi}{2}\right)$ if the particle goes from one side to the counterclockwise adjacent side; this is possible only when $\pi/2 < \nu_n < \pi$.

From the above we see that our two-dimensional system F_{T_2} is not a simple extension of the one-dimensional law T_2: due to the geometry of the square, the second co-ordinate is modified in comparison to the simple reflection law. Moreover, as we shall see below, the function describing the evolution of the second co-ordinate is multi-valued over the interval $(\pi/4, 3\pi/4)$ – see Fig. 5 (the choice of the value from two possibilities depends of the first co-ordinate, i.e. the position of the particle).

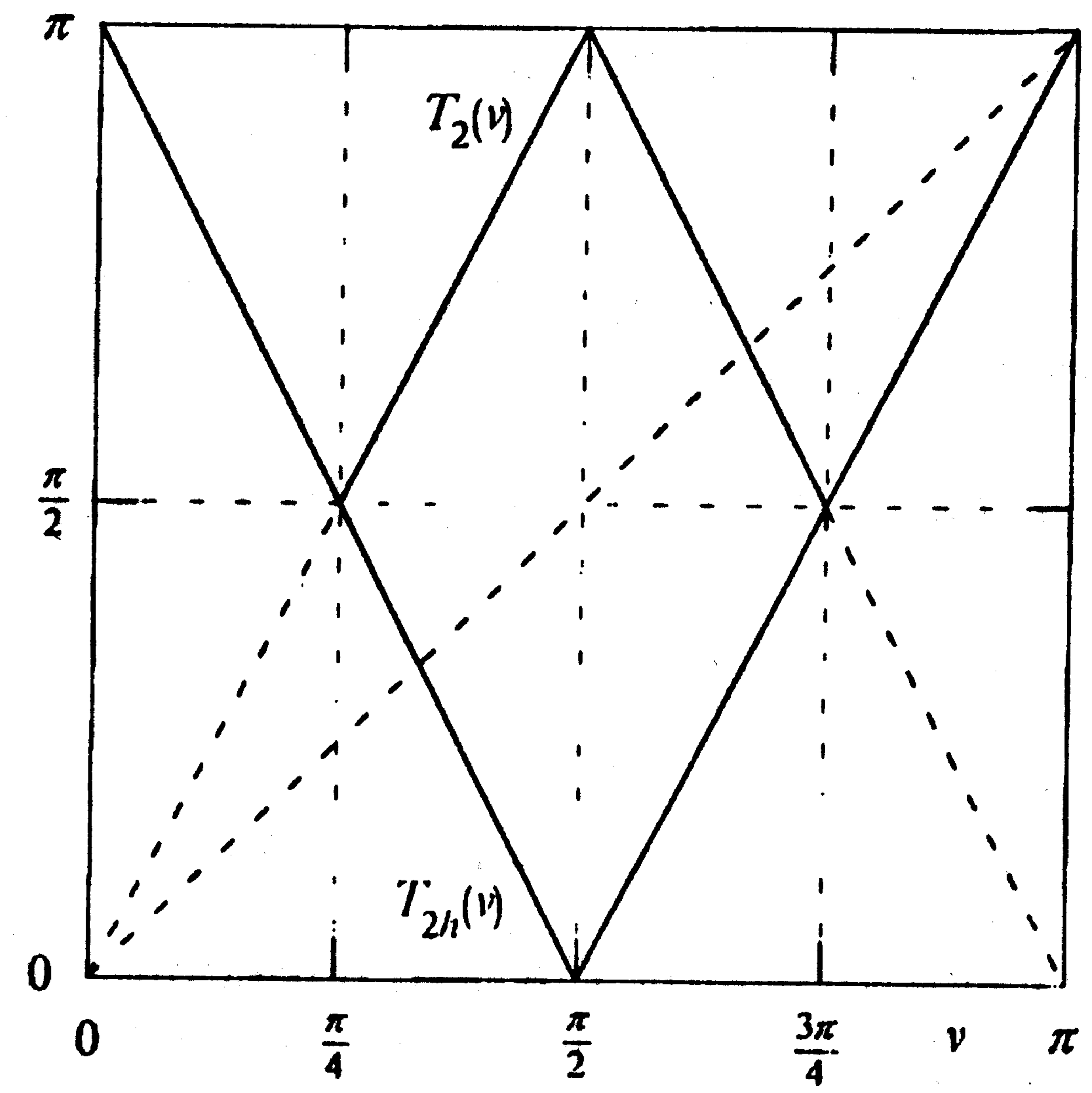

FIG. 5. The plot of the multi-valued map governed by the reflection law T_2.

Let us introduce a new function, based on the properties b) and c) of the reflection law:

$$T_{2h}(\nu) = \begin{cases} 2(\pi/2 - \nu) & \text{for} \quad 0 < \nu < \pi/2, \\ 2(\nu - \pi/2) & \text{for} \quad \pi/2 \leq \nu < \pi. \end{cases} \tag{3.1}$$

This function will be used for the study of the evolution of the second co-ordinate of F_{T_2}.

Observe that

$$T_{2h} = T_2 \circ h, \tag{3.2}$$

where h is a universal function, inherently connected with the shape of the square:

$$h(\nu) = \begin{cases} \nu + \pi/2 & \text{for} \quad 0 < \nu < \pi/2, \\ \nu - \pi/2 & \text{for} \quad \pi/2 \leq \nu < \pi. \end{cases} \tag{3.3}$$

One can see that after n reflections, the velocity of the particle, in the system of co-ordinates, is of the following form:

$$\nu_n = T_{\alpha_n} \circ T_{\alpha_{n-1}} \circ ... \circ T_{\alpha_1}(\nu_0), \tag{3.4}$$

where the subscripts are $\alpha_i = 2$ or $2h$ for $i = 1, 2, ..., n$. The sequence $(\alpha_i)_{i=1}^n$ is determined by the initial point (x_0, ν_0).

Notice that the reflection law T_2 has the following property:

$$T_2(\nu) = T_2(\pi - \nu). \tag{3.5}$$

Moreover, the function T_{2h} satisfies the condition:

$$T_{2h}(\nu) = \pi - T_2(\nu). \tag{3.6}$$

Both the above properties are satisfied for every $\nu \in (0, \pi)$.

From (3.5) and (3.6) we have

$$T_{2h}^2(\nu_0) = T_{2h}(T_{2h}(\nu_0)) = T_{2h}(\pi - T_2(\nu_0)) = \pi - T_2(\pi - T_2(\nu_0)) = \pi - T_2^2(\nu_0), \tag{3.7}$$

and generally, by induction,

$$\nu_n = T_2^n(\nu_0) \quad \text{or} \quad \nu_n = \pi - T_2^n(\nu_0). \tag{3.8}$$

We come to the conclusion that after the n-th reflection, the second co-ordinate of $F_{T_2}^n(x_0, \nu_0)$ is either $T_2^n(\nu_0)$ or the point symmetrical to $T_2^n(\nu_0)$ with respect to $\pi/2$. Now, because T_2 is ergodic (with an invariant measure equivalent to the Lebesgue measure), [13], we conclude that for almost all initial points ν_0 the set $\{\tilde{\nu}_n = T_2^n(\nu_0), n = 0, 1, 2, ...\}$ is dense in $(0, \pi)$ [9]. Thus, for almost all initial points (x_0, ν_0), the set of velocities $\{\nu_n, n = 1, 2, ...\}$ corresponding to each of them is dense in a set of Lebesgue measure of at least $\pi/2$. We see that the motion F_{T_2} is completely different from the motion F_{T_1}, where the sequence of velocities ν_n converged to the constant value π, independently of the initial position x_0 and the starting velocity ν_0.

Observe that an analogous result can be obtained for rectangles.

To end this section, we point out an interesting property of the relation (3.2). Consider the following chaotic and mixing reflection law:

$$T_3(\nu) = 2\nu \pmod{\pi}. \tag{3.9}$$

For this law applied to the motion of the particle in the square, the formula (3.2) becomes

$$T_{3h} = T_3 \circ h = T_3. \tag{3.10}$$

This is an example of a law invariant with respect to the function h. This class of reflection laws has an unusual property that the evolution of the second co-ordinate ν of particle's motion F_{T_3} is independent of the position x (the first co-ordinate of F_{T_3}).

4. Final remarks and conclusions

The problems studied in this paper were inspired by previous investigations connected with description of a single particle motion. The particle's motion with a non-classical reflection law arises in a number of practical physical phenomena.

The models of this kind can be observed in very rarefied gases, the so-called Knudsen gases [1, 4]. The investigation of the reflection law models allows us to predict, under some additional mathematical assumptions, the qualitative properties of the one-particle distribution function of the gas (e.g. the analyticity).

Another problem, directly related to the reflection law models, is the motion of a particle in accelerators [15]. Moreover, in this case the particle's motion can be described by the so-called "standard maps" which turned out to be the Poincaré maps generated by the moving particle [11, 16–17]. These maps are topologically conjugate to some dynamical systems obtained in the study of reflection law models [5].

The transfer of properties from smaller to extended dynamical systems can also be analysed in the motion of the particle in a viscous medium under the influence of a kick force. This phenomenon was modelled and investigated in [18].

Among many applications of chaos one can find also the recent utilisation of chaotic dynamical systems to construct secure communication (see e.g. [19–20]). In [21–22] we proposed the method of extending dynamical systems to construct safe cryptosystems. The results obtained in the above give some suggestions how such extensions can be performed. In the case of the block cryptosystems, the encryption and decryption is based on multiple inverse iterations and forward iterations. The secret key is introduced into the reflection law (the velocity of the particle) and the message is considered as the position of the particle [23]. Under the appropriate way of transferring the properties of the reflection law, the initial position of the particle cannot be reconstructed from the final position without the knowledge of the initial particle velocity (our secret key).

The considerations of this paper point out the interesting problem of constructing a chaotic and ergodic reflection law which would guarantee the transfer of these properties to certain extended dynamical systems, like the motion of a particle in a wide class of typical containers or some secure cryptosystems.

Our models show that there are no simple relations between the properties of a reflection law and the properties of the motion of the particle. Even for the same class of the reflection laws (in topological sense) with very strong properties like ergodicity and chaos, the qualitative properties of the motion of the particle (in commonly used containers) can be essentially different. It is an interesting open problem to find additional assumptions on the reflection law which would ensure the transfer of the above properties. It seems that such type of reflections could be interesting from the physical point of view.

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