Scattering of H-polarized Wave by a Periodic Array of Thick-Walled Parallel Plate Waveguides

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Abstract—Electromagnetic wave scattering by a periodic array of semi-infinite thick-walled parallel plate waveguides is studied in this paper. The scattered field in the free-space region above the waveguides is sought in the form of a series of spatial harmonics in accordance with Floquet's theorem, whereas in the waveguide regions it is sought in the form of parallel plate waveguide modes. To satisfy the boundary and edge conditions in the free-space region, Fourier series expansion with corresponding coefficients being properly chosen Legendre functions is used. As a result, the amplitudes of spatial harmonics are given in the form of an infinite series of Legendre functions with unknown coefficients being the solution of certain doubly infinite system of linear equations. The exact solution of this system can not be constructed and numerical calculations were performed to find the expansion coefficients approximately.

Index Terms—Electromagnetic scattering by periodic structures, Parallel plate waveguides, Waveguide arrays, Floquet expansions, Maxwell equations.

I. INTRODUCTION

The scattering and radiation of an electromagnetic wave by an infinite periodic system of parallel plate waveguides is a classical problem of the diffraction theory. It has been studied by numerous authors and is still a problem of current significance to many engineering applications. From the theoretical point of view it gives a perfect example for the study of periodic structures, whereas in the practical aspect it simulates a phased array of waveguides which is widely used in today’s communication and radar systems. Exact closed-form solutions have been obtained only in few cases of waveguides with infinitely thin walls [1]-[3]. In the case of finite waveguide wall thickness an approximate solution can only be obtained [4]-[7]. The purpose of this paper is to present a new approach to the analysis of electromagnetic scattering by a periodic array of thick-walled parallel plate waveguides. The scattered field is sought in the form of spatial harmonics in the space above the array and in the form of parallel plate waveguide modes in the waveguide regions. The key point of this approach is the application of Fourier series expansion with coefficients being properly chosen Legendre functions for field representation in the free-space region above the waveguides which assures that the boundary and edge conditions are satisfied. A similar expansion was used in the analysis of electromagnetic waves in structures containing periodic strips of infinitesimal thickness [8]. The problem is reduced to solving the doubly infinite system of linear equations for unknown expansion coefficients. The exact solution cannot be obtained, however, and numerical calculations are required to solve this system approximately.

II. FIELD SOLUTIONS

Let us consider an infinite system of perfectly conducting thick-walled parallel plate waveguides as shown in Fig. 1. The period of the structure is \( \Lambda \) and the waveguide aperture is \( d \). The system is homogeneous in the \( z \)-direction and periodic in the \( x \)-direction. Waveguides occupy the lower half-plane \( y < 0 \). An incident plane harmonic wave of the angular frequency \( \omega \) impinges on the system at the angle \( \theta \) counted from the \( y \)-axis. The wave is assumed to be H-polarized in the \( z \)-direction. In what follows, the term \( \exp(j\omega t) \) will be omitted. For the case of H-polarization in this paper, the field components resulting from the Maxwell equations are

\[
H_z(x, y), \quad E_z(x, y) = \frac{1}{j\omega \varepsilon_0} \frac{\partial H_z}{\partial y}, \quad E_y(x, y) = \frac{j}{\omega \varepsilon_0} \frac{\partial H_z}{\partial x}.
\]

The total field can be represented in the following form:

\[
H_z = H_z^+ + H_z^I, \quad E_i = E_i^+ + E_i^I, \quad y > 0,
\]

\[
H_z = H_z^-, \quad E_i = E_i^-, \quad y < 0,
\]

where the subscript \( i = x, y \) and the superscripts \( +, - \) denote the scattered field in the free-space and in the waveguide regions respectively. The superscript \( I \) denotes the given field of incident wave:

\[
H_z^I = e^{-j(k_x x + k_y y)}, \quad E_z^I = k_y \omega \varepsilon_0 H_z^I, \quad E_y^I = k_x \omega \varepsilon_0 H_z^I,
\]

where \( k_x \) and \( k_y \) are the components of incident wave vector \( k = k \sin \theta \), \( k_x = k \cos \theta \), and \( k = 2\pi/\lambda \), \( \lambda \) is the wavelength of the incident wave. The problem is to find the scattered field.
Let us consider first the region $y > 0$. Due to the system periodicity the scattered field can be represented by a series of spatial harmonics according to Floquet’s theorem yielding

$$H_z^+ = \sum_n A_n e^{-j(k_{zx}^+ x + k_{zy}^+ y)},$$

$$E_x^+ = \frac{1}{\omega_0} \sum_n k_{zx}^+ A_n e^{-j(k_{zx}^+ x + k_{zy}^+ y)},$$  

$$E_y^+ = \frac{1}{\omega_0} \sum_n k_{zy}^+ A_n e^{-j(k_{zx}^+ x + k_{zy}^+ y)},$$

where $A_n$ are the unknown amplitudes; $n \in \mathbb{Z}$ throughout the paper, unless otherwise stated. The wavenumbers $k_{zx}^+$ and $k_{zy}^+$ are defined as follows:

$$k_{zx}^+ = k_x + nK, \quad K = 2\pi/\Lambda,$$  

and

$$k_{zy}^+ = \begin{cases} (k^2 - (k_{zx}^+)^2)^{1/2} & \text{for real } k_{zx}^+, \\ -j \left( (k_{zx}^+)^2 - k^2 \right)^{1/2} & \text{for imaginary } k_{zy}^+. \end{cases}$$

The field in the waveguide regions is represented by a series of parallel plate waveguide modes as follows (for one period $-\Lambda/2 < x < \Lambda/2$):

$$H_z^- = \sum_p B_p \cos \left[ \frac{\pi}{d} \left( x + \frac{d}{2} \right) \right] e^{j\kappa_{zp}^- y},$$

$$E_x^- = \frac{1}{\omega_0} \sum_p k_{zp}^- B_p \cos \left[ \frac{\pi}{d} \left( x + \frac{d}{2} \right) \right] e^{j\kappa_{zp}^- y},$$

$$E_y^- = -\frac{j}{\omega_0} \sum_p B_p \sin \left[ \frac{\pi}{d} \left( x + \frac{d}{2} \right) \right] e^{j\kappa_{zp}^- y},$$

where $B_p$ are the unknown mode amplitudes; $p \in \mathbb{N} \cup \{0\}$ throughout the paper, unless otherwise stated. It is quite obvious that it is sufficient to consider the field distribution in one period only. The mode amplitudes in different waveguides are related to those in (7) by simple relation:

$$B_p^m = B_p e^{-j\kappa_m x \Lambda}, \quad m \in \mathbb{Z}.$$  

The propagation constant $\kappa_{zp}$ is defined as:

$$\kappa_{zp}^- = \begin{cases} (k^2 - (p\pi/d)^2)^{1/2} & \text{for real } \kappa_{zp}^-, \\ -j \left( (p\pi/d)^2 - k^2 \right)^{1/2} & \text{for imaginary } \kappa_{zp}^- \end{cases}$$  

### III. Boundary Conditions

The tangential component of the total electric field vector must vanish on the perfectly conducting walls:

$$E_x = 0, \quad d/2 < |x| < \Lambda/2, \quad y = \pm 0,$$  

$$E_y = 0, \quad x = \pm d/2, \quad y < 0.$$  

Besides, the electric field exhibits singular behavior near the plate edges:

$$E_x = O \left( \rho^{-\frac{1}{2}} \right), \quad E_y = O \left( \rho^{-\frac{1}{2}} \right), \quad \rho \to 0,$$  

$$\rho = ((x \pm d/2)^2 + y^2)^{1/2}.$$  

The second condition in (10) is satisfied directly by (7). To obey the boundary condition for the $E_x$-component in the free-space region, we shall make use of the following expansion [9]:

$$\sum_n P_n^\mu(\Delta)e^{-jn\kappa K x} = \begin{cases} C e^{\kappa K x/2} / (\cos(K x) - \Delta)^{\mu+\frac{1}{2}}, & |x| < d/2, \\ 0, & d/2 < |x| < \Lambda/2, \end{cases}$$

where the constant terms

$$C = (\pi/2)^\frac{1}{2} (\sin(\pi d/\Lambda))^\mu/\Gamma(\frac{1}{2} - \mu)$$

and $\Delta = \cos(\pi d/\Lambda)$ are introduced to shorten notation; $P_n^\mu$ denotes the Legendre functions, $\Gamma$ is the gamma-function. Later on the argument of the Legendre functions $\Delta$ will be omitted. Multiplying (12) by $\exp(-j\lambda K x)$, where $\lambda$ is some integer, and taking linear combination of the resulting equations after straightforward algebraic manipulations we obtain

$$\sum_{m,n} \alpha_m P_m^{\mu} e^{jn\kappa K x} = \begin{cases} C \sum_{m,n} \alpha_m e^{-j(m\pi/2)} e^{-jn\kappa K x}, & |x| < d/2, \\ 0, & d/2 < |x| < \Lambda/2, \end{cases}$$

A similar expansion was used for modeling an infinite array of infinitely thin strips in the work of Danicki et al. [8]. Expansion (13) represents the Fourier series of certain $\Lambda$-periodic function as required by Floquet’s theorem, vanishing in certain domains as required by the boundary condition for the $E_x(x, +0)$-component of electric field vector (see (10)) and having singular behavior at the bounds of the above domains (at the edges of the waveguide apertures) in accordance with the edge conditions for $E_x(x, +0)$ (see (11), if $\mu = -1/6$ is applied). Noting the similar behavior of the above expansion and $E_x(x, +0)$ we can rewrite the expression for the tangential component of the total electric field vector in the free-space region in the form satisfying conditions (10), (11) as follows:

$$E_x(x, 0) = e^{-j\kappa K x} \sum_{m,n} \alpha_m P_m^{\mu} e^{-jn\kappa K x},$$

where $\mu = -1/6$ hereinafter. In a general case the summation domain over $m$ is infinite, but it can always be appropriately truncated for the purpose of numerical evaluation without significant loss of accuracy [10]. Comparing (14) with (2) and (4) we obtain the following relation between corresponding amplitudes of spatial harmonics $A_n$ and coefficients $\alpha_m$:

$$A_n = \delta_{n0} - \frac{1}{k_{yz}^m} \sum_m \alpha_m P_m^{\mu} e^{-jn\kappa K x},$$

where $\delta_{n0}$ is Kronecker delta.

### IV. Evaluation of Expansion Coefficients

To evaluate the unknown coefficients $\alpha_m$ we use the continuity conditions of the tangential field components:

$$H_z^+ + H_z^- = H_z^-, \quad E_x^+ + E_x^- = E_x^-,$$

$$y = 0, \quad x \in (-d/2, d/2).$$
We first substitute (3), (4) and (7) into (16). Next, we multiply the resulting equations by \( \cos [\pi (x/d + 1/2)] \) and integrate with respect to \( x \) from \(-d/2\) to \(d/2\) to obtain

\[
B_p = \frac{4e^{i\pi/2}}{d(1 + \delta_{p0})} \sum_{n} k_{xn}^+ \sin \frac{1}{2} (k_{xn}^+ d - \pi \gamma) (\delta_{n0} + A_n),
\]

\[
B_p = \frac{4e^{i\pi/2}}{d(1 + \delta_{p0})} \sum_{n} k_{yn}^+ k_{yn}^+ \sin \frac{1}{2} (k_{yn}^+ d - \pi \gamma) (\delta_{n0} - A_n). \tag{17}
\]

Subtracting the second equation from the first in (17), after rearrangement of terms we get

\[
\sum_{n} k_{yn}^+ k_{yn}^+ \sin \frac{1}{2} (k_{yn}^+ d - \pi \gamma) \left( \frac{1}{k_{yp}} + \frac{1}{k_{yp}} \right) A_n = \left( 1 - \frac{k_y}{k_{yp}} \right) k_x k_y \sin \frac{1}{2} \left( k_x d - \pi \gamma \right) \frac{1}{k_x^2 - (\pi \gamma)^2}. \tag{18}
\]

Taking into account (15), after simple algebraic manipulations we obtain the following system of linear equations for the unknown \( \alpha_m \):

\[
\sum_{m,n} \alpha_m P_{n-m}^\mu k_{xn}^+ \sin \frac{1}{2} (k_{xn}^+ d - \pi \gamma) \left( \frac{1}{k_{yp}} + \frac{1}{k_{yp}} \right) = 2k_x \sin \frac{1}{2} \left( k_x d - \pi \gamma \right) \frac{1}{k_x^2 - (\pi \gamma)^2}. \tag{19}
\]

The above doubly infinite system of linear equations can only be solved numerically (see Sec.5). If the coefficients \( \alpha_m \) are known, the scattered field in both regions can be evaluated. Namely, the amplitudes \( A_n \) can be found from (15) whereas \( B_p \) can be determined from the following relation resulting from (17):

\[
B_p = \frac{4e^{i\pi/2}}{d(1 + \delta_{p0})} \sum_{m,n} \alpha_m P_{n-m}^\mu k_{yn}^+ k_{yn}^+ \sin \frac{1}{2} (k_{yn}^+ d - \pi \gamma) \left( \frac{1}{k_{yp}} + \frac{1}{k_{yp}} \right). \tag{20}
\]

V. NUMERICAL RESULTS

A detailed discussion concerning the uniqueness and existence of the solution of infinite system of linear equations analogous to (18) is presented in [10]. In order to find an approximate solution of the system (18) we reduce a number of equations taking into account only the modes transporting energy from the plane \( y = 0 \) and a finite number of lower order evanescent modes. The higher order ones are ignored in the solution for the scattered field. To find the unknown coefficients \( \alpha_m \) and consequently the partial wave amplitudes \( A_n \) and \( B_p \) of the scattered field in the upper and lower half-planes respectively, we consider the truncated system of linear equations (19):

\[
\sum_{m=-M}^{M} G_{pm} \alpha_m = g_p, \; p = 0, 1, 2, \ldots, 2M, \tag{21}
\]

where the corresponding coefficients are

\[
G_{pm} = \sum_{n=-N}^{N} P_{n-m}^\mu x(k_{xn}^+ d - \pi \gamma) \left( \frac{1}{k_{yn}^+} + \frac{1}{k_{yp}} \right),
\]

\[
g_p = \frac{2k_x \sin \frac{1}{2} (k_x d - \pi \gamma)}{k_x^2 - (\pi \gamma)^2}. \tag{22}
\]

It should be noted that the infinite series in (19) has been truncated at some \( n = N \) (large enough to assure convergence) in order to evaluate approximate values of the matrix coefficients \( G_{pm} \) in (22). Generally, the series in (22) is fast converging and it is sufficient to take \( N \) about one order of magnitude higher than \( 2M \) to assure an acceptable accuracy of numerical calculations. To estimate the accuracy of the numerical calculation we consider the relative error in the power relation resulting from the complex Poynting’s theorem. Without going into details we shall give here the final expression for the propagating modes derived for one period of the structure:

\[
\sum_{n,k_{yn}^+ \text{real}} |A_n|^2 k_{yn}^+ + \frac{d}{2\lambda} \sum_{p,k_{yp}^+ \text{real}} |B_p|^2 k_{yp}^+(1 + \delta_{p0}) = \delta P. \tag{23}
\]

This condition for the squares of amplitudes of propagating modes is equivalent to the energy conservation law for the scattered field. It should be satisfied by the field components in both regions. The relative error in the power relation is defined as follows:

\[
\delta P = \left| \left( P^I - P^+ - P^- \right) / P^I \right|,
\]

where \( P^\pm \) denotes the power of the scattered field in the upper (reflected field) and lower (transmitted field) half-planes marked by superscripts + and respectively; \( P^I \) is the power of the incident wave field. Table I shows the relative error in the power relation (23) for the following case: \( \Lambda/d = 2, \Lambda/\lambda = 3.8, \theta = 10^6 \), and \( N = 1000 \) in (22). It is worth noting that using (20) and (15) we can evaluate

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
M & 5 & 10 & 15 & 25 & 50 \\
\hline
\delta P(\%) & 3.49 & 2.37 & 1.71 & 1.22 & 0.78 \\
\hline
\end{array}
\]

TABLE I

RELATIVE ERROR \( \delta P \) FOR DIFFERENT NUMBER OF EQUATIONS IN (22)

In Table II the relative error in the transmitted field power versus the amplitudes of spatial harmonics \( A_n \) and waveguide modes \( B_p \) for \( |n| > M \) and \( p > 2M \) respectively, having solved the truncated system (22) for unknown \( \alpha_m \), \( |m| \leq M \). This allows us to improve the numerical accuracy of the scattered field evaluation. In Table II the relative error in the power relation (23) is shown for different numbers \( N_{\text{harm}} \) of spatial harmonics \( A_n \) and waveguide modes \( B_p \) in both half-planes:

\[
-N_{\text{harm}} \leq n \leq N_{\text{harm}}, 0 \leq p \leq 2N_{\text{harm}}.
\]

For fixed \( M = 50, N = 1000 \) in (22) and \( \Lambda/d = 2, \Lambda/\lambda = 3.8, \theta = 10^6 \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
N_{\text{harm}} & 50 & 100 & 500 & 1000 \\
\hline
\delta P(\%) & 0.78 & 0.46 & 0.14 & 0.075 \\
\hline
\end{array}
\]

TABLE II

RELATIVE ERROR \( \delta P \) FOR DIFFERENT NUMBERS OF SPACE HARMONICS
the normalized period is shown. It is defined as the real part of the complex Poynting flux (the second sum in (23)). A similar dependence of the normalized net reactive energy stored in the magnetic and electric fields in one period of the structure is shown in Fig. 3. It is related to the imaginary part of the complex Poynting flux (see (22)) being fast converging series, and can be easily implemented for an approximate evaluation of the scattered field. The presented approach can be generalized and applied to the case of oblique incidence as well as to a number of practically important problems like the scattering by the system of periodic conducting strips of finite thickness.

VI. CONCLUSION

In this paper we have presented a method for analyzing the electromagnetic wave scattering by an infinite array of thick-walled parallel plate waveguides. Its key point is the application of Fourier expansion with coefficients being properly chosen Legendre functions for field representation in the free-space region above the array to satisfy the boundary and edge conditions. It should be noted that, in contrast to [4] and [5], our method does not work for normal incidence, but it works well for an incidence close to normal, up to \( \theta \sim 10^{-10} \). The presented method is quite straightforward due to the simple form of matrix elements (see (22)) being fast converging series, and can be easily implemented for an approximate evaluation of the scattered field. The presented approach can be generalized and applied to the case of oblique incidence as well as to a number of practically important problems like the scattering by the system of periodic conducting strips of finite thickness.

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