ELECTROMAGNETIC SCATTERING BY PERIODIC GRATING OF PEC BARS

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Abstract—Electromagnetic wave scattering by a periodic array of perfectly conducting bars is studied in this paper. In the proposed approach the amplitudes of the spatial harmonics in free space above and below the array are expanded in a Fourier series with coefficients being properly chosen Legendre polynomials. As a result the boundary and edge conditions are satisfied directly by field representation. The method results in a small system of linear equations for unknown expansion coefficients to be solved numerically. Some numerical examples are given, presenting a comparison to the mode matching technique.

1. INTRODUCTION

Radiation and scattering of the electromagnetic waves by periodic structures is a significant and important problem of the diffraction theory. Different systems and methods are described in the literature. In the practical aspect the perfectly electric conductor (PEC) periodic structures can simulate the phased arrays in micro and millimeter wave applications, such as filters [1], frequency selective structures [2], splitters and antennas [3], widely used in today’s communication and radar systems [4]. From the theoretical point of view it gives a deeper insight into the study of periodic structures. Recently, periodic systems with a particular arrangement of scatterers including conductors and dielectrics has received growing attention, because such the systems may behave like negative refractive index materials [5, 6] within a certain frequency range. Many approaches [7–9] have been proposed to analyze them, such as the mode-matching method, finite difference

Received 9 November 2010.
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In this paper, we present a method of scattering analysis in periodic structures, based on application of certain Fourier series expansion of spatial harmonics representing the scattered field [9, 10]. The corresponding expansion coefficients are properly chosen Legendre functions. This allows one to satisfy the boundary and edge conditions directly by field representation. An infinite periodic array of PEC bars is considered here, since such the system is of great importance in the modeling of multilayered structure being the fundamental part of metallic photonic crystals composed of arbitrary shapes or multilayered structures [11, 12]. The method is presented in details for the case of TM polarization of the incident wave. The TE incidence can be considered in a similar manner, therefore only the general remarks and final results are given.

2. FIELD SOLUTIONS FOR PERIODIC BARS

Let us consider an infinite system of perfectly conducting (PEC) bars shown in Fig. 1. The period of the structure is \( \Lambda \); \( d \) and \( h \) are the distance between bars and its height, respectively. We also introduce the following variables to shorten notation in the further analysis: \( \hat{\Lambda} = \Lambda / 2, \hat{d} = d / 2 \) and \( \hat{h} = h / 2 \). The system is homogeneous in the \( z \)-direction, and periodic in the \( x \)-direction. An incident plane harmonic wave of the angular frequency \( \omega \) impinges on the system at the angle \( \theta \) counted from the \( y \)-axis. In what follows, the term \( \exp(\jmath \omega t) \) will be omitted. The case of TM incidence will be considered in details. For TE incidence only the general remarks will be given to emphasize the main differences of both cases. In the case of TM incidence the only nonzero component of the magnetic field vector is \( H_z \), whereas \( E_z \) is the only nonzero component of the electric field in the case of TE polarization of the incident wave. The total field resulting from
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The Maxwell equations can be represented as follows

\[ \begin{align*}
\text{TM:} & & H_z, E_x = -j \nu H_{z,y}, & E_y = j \nu H_{z,x} \\
\text{TE:} & & E_z, H_x = j \nu E_{z,y}, & H_y = -j \nu E_{z,x},
\end{align*} \tag{1} \]

where \( \nu = (\omega \varepsilon_0)^{-1} \) for TM and \( \nu = (\omega \mu_0)^{-1} \) for TE case. The total field can be represented in the following form (TM)

\[ \begin{align*}
H_z &= H_z^+ + H_z^-, & E_i = E_i^+ + E_i^-, & y > h, \\
H_z &= H_z^-, & E_i = E_i^-, & y < h,
\end{align*} \tag{2} \]

where \( i = x, y \) and \( I \) denotes the field of given incident wave

\[ \begin{align*}
H_z^+ &= e^{-j(k x x - k y y)}, & k_x = k \sin \theta, & k_y = k \cos \theta, & k = 2\pi/\lambda, \\
E_x^+ &= \nu k_y H_z^+; & E_y^+ &= \nu k_x H_z^+,
\end{align*} \tag{3} \]

and the superscripts +/- in (2) denote the scattered field above and below the bars, respectively. Due to the system periodicity the scattered field outside the bars can be represented by a series of spatial harmonics of the form according to the Floquet’s theorem:

\[ \begin{align*}
\psi_n^+ &= e^{-j(k x x + k y (y+\hat{h}+\hat{h}))}, & k^2 = k_{x n}^2 + k_{y n}^2, & k_{x n} = k_x + nK, \\
k_{y n} &= \begin{cases} 
(k^2 - k_{x n}^2)^{1/2}, & k \geq k_{x n}, \\
-j(k_{x n}^2 - k^2)^{1/2}, & k < k_{x n}, \end{cases} & K = 2\pi/\Lambda
\end{align*} \tag{4} \]

with amplitudes \( A_n^\pm \). Substituting (4) into (1) we obtain for the scattered field

\[ \begin{align*}
H_z^\pm &= \sum_n A_n^\pm \psi_n^\pm, & E_x^\pm &= -\nu \sum_n k_{y n} A_n^\pm \psi_n^\pm, & E_y^\pm &= \nu \sum_n k_{x n} A_n^\pm \psi_n^\pm, \tag{5} \end{align*} \]

where \( n \in \mathbb{Z} \) throughout the paper, unless otherwise stated. In the above Equations (2), (3) and (5) the \( H \) and \( E \) symbols and the +/- signs before \( \nu \) should be swapped over to address the TE incidence case. The scattered field between bars can be represented by a series of the parallel plate waveguide modes as follows (for \( -\hat{\Lambda} < x < \hat{\Lambda}, 0 < y < h \))

\[ \begin{align*}
H_z^b &= \sum_p (B_p^+ e^{j \eta_p y} + B_p^- e^{-j \eta_p y}) \cos(\xi_p(x+\hat{d})), \\
E_x^b &= \nu \sum_p \eta_p (B_p^+ e^{j \eta_p y} + B_p^- e^{-j \eta_p y}) \cos(\xi_p(x+\hat{d})), \\
E_y^b &= -j \nu \sum_p \xi_p (B_p^+ e^{j \eta_p y} + B_p^- e^{-j \eta_p y}) \cos(\xi_p(x+\hat{d}))
\end{align*} \tag{6} \]

where \( B_p^\pm \) are unknown mode amplitudes; \( p \in \mathbb{N} \cup 0 \) for TM incidence throughout the paper (for TE incidence \( p \in \mathbb{N} \)), unless otherwise stated. Swapping over \( H \) with \( E \) and the sine with cosine functions in (6) one obtains the expressions for the scattered field in the case
of TE polarization. The propagation constants $\xi_p$ and $\eta_p$ are defined below

$$\xi_p = p\pi/d, \eta_p = \begin{cases} \sqrt{(k^2 - (\xi_p)^2)} & \text{for real } \eta_p, \\ -j\sqrt{(\eta_p)^2 - k^2} & \text{for imaginary } \eta_p. \end{cases} \tag{7}$$

It is easy to verify that it is sufficient to consider the field distribution in one period only. The mode amplitudes in different periods are related to those in (6) by simple relation

$$B^\pm_{p} = B^\pm_{0} e^{-jm\Lambda}, B^\pm_{p} = B^\pm_{0}, m \in \mathbb{Z}. \tag{8}$$

3. BOUNDARY CONDITIONS

The tangential component of total electric field vector must vanish on the surface of the perfectly conducting bars. This yields (TM)

$$E_x = 0, \quad \hat{d} < |x| < \hat{\Lambda}, \quad y = +h, y = -0, \quad \text{outside bars},$$
$$E_y = 0, \quad x = \pm\hat{d}, \quad 0 < y < h, \quad \text{between bars}. \tag{9}$$

In the case of TE polarization the above conditions hold for the $E_z$ component. Besides, near the bars’ edges the tangential components of the electric (TM) and magnetic (TE) field vectors exhibit singular behavior

$$\text{TM: } E_i = O(q^{-\frac{1}{2}}), \quad \text{TE: } H_i = O(q^{-\frac{1}{2}}), \quad i = x, y,$$
$$q = \sqrt{(x \pm \hat{d})^2 + y^2}, \quad q \to 0. \tag{10}$$

Conditions between the bars (9) are satisfied directly by (6). To obey the boundary conditions outside the bars, we follow a similar approach described in [9, 10] and try the solution for the $E_x$ in TM mode in the form

$$E_x(x, +h) = \nu \sum_{n,m} \alpha^+_m P^\mu_{n-m}(\cos \Delta)e^{-jkx_n x},$$
$$E_x(x, -0) = \nu \sum_{n,m} \alpha^-_m P^\mu_{n-m}(\cos \Delta)e^{-jkx_n x}, \tag{11}$$

where $\mu = -1/6$ hereinafter; $P$ is the Legendre polynomials; $\Delta = \pi d/\Lambda$. Comparing (11) with (2)–(4) we obtain the following simple relation between the corresponding amplitudes of spatial harmonics $A^\pm_n$ and the expansion coefficients $\alpha^\pm_m$

$$A^+_n = \delta_{n0} e^{jkyn} - k^{-1}_{yn} \sum_m \alpha^+_m P^\mu_{n-m},$$
$$A^-_n = k^{-1}_{yn} \sum_m \alpha^-_m P^\mu_{n-m}. \tag{12}$$
where $\delta_{nI}$ is Kronecker delta. In (12) the arguments of the Legendre polynomials were dropped to shorten notation. In analogous manner in the case of TE polarization we apply a similar expansion to the $x$-derivative of the $E_z$

$$
E_{z,x}(x, +h) = \sum_{n,m} \alpha^+_m P^\mu_{n-m} e^{-jk_{xn}x}, \ E_z(k\Lambda, +h) = 0,
$$

$$
E_{z,x}(x, -0) = \sum_{n,m} \alpha^-_m P^\mu_{n-m} e^{-jk_{xn}x}, \ E_z(k\Lambda, -0) = 0, \ k \in \mathbb{Z}.
$$

In (13) the additional condition is applied to obey the boundary conditions outside the bars (9) for $E_z$. Taking into account (2)–(4) the following relation between $A^\pm_n$ and $\alpha^\pm_m$ can be obtained for TM polarization

$$
A^+_n = -\delta_{n0}e^{jk_y h} + k^{-1}_{kn} \sum_m \alpha^+_m P^\mu_{n-m}, \ A^-_n = k^{-1}_{kn} \sum_m \alpha^-_m P^\mu_{n-m}. \ (14)
$$

4. EVALUATION OF EXPANSION COEFFICIENTS

In this section the detailed discussion concerning evaluation of the unknown coefficients $\alpha^\pm_m$ is given for the TM incidence case. First we use the continuity conditions of the tangential field components at the planes $y = 0, h$

$$
H_z = H^b_z, \ E_x = E^b_x, \ x \in (-\hat{d}, \hat{d}). \ (15)
$$

Substituting the corresponding expressions for the tangential components from (2)–(6) into (15) we next multiply the resulting equations by $\cos(\xi_p x + p\pi/2)$ and integrate with respect to $x$ from $-\hat{d}$ to $\hat{d}$ to obtain the following system

$$
(B^+_p + B^-_p)e^{j2\eta_p h} = L_p \sum_n k_{xn} F_{pn}(\delta_{n0}e^{jk_y h} + A^+_n), \ (16a)
$$

$$
(B^+_p - B^-_p)e^{j2\eta_p h} = L_p \eta^{-1}_p \sum_n k_{xn} F_{pn} k_{yn}(\delta_{n0}e^{jk_y h} - A^+_n) \ (16b)
$$

$$
B^+_p + B^-_p = L_p \sum_n k_{xn} F_{pn} A^-_n \ (16c)
$$

$$
B^+_p - B^-_p = L_p \eta^{-1}_p \sum_n k_{xn} F_{pn} k_{yn} A^-_n, \ (16d)
$$

where

$$
F_{pn} = \sin(\hat{d}(k_{xn} - \xi_p))/(k^2_{xn} - \xi^2_p) \ L_p = 2e^{jpr\pi/2}/(\hat{d}(1 + \delta_p 0)). \ (17)
$$

To eliminate the coefficients $B^\pm_o$ we first subtract the sum of (16b) and (16d) from the sum of (16a) and (16c). Afterwards, we subtract
the sum of (16b) and (16c) from the sum of (16a) and (16d). This yields

\[ (\eta_p \tan \eta_p \hat{h}) \sum_n k_{xn} F_{pn}(\delta_{n0} e^{jk_y h} + a_n^+) \]

\[ = j \sum_n k_{xn} F_{pn} k_{yn}(\delta_{n0} e^{jk_y h} - a_n^+) \]

\[ (\eta_p \cot \eta_p \hat{h}) \sum_n k_{xn} F_{pn}(\delta_{n0} e^{jk_y h} + a_n^-) \]

\[ = \sum_n k_{xn} F_{pn} k_{yn}(\delta_{n0} e^{jk_y h} - a_n^-), \]

where the new field amplitudes have been introduced: \( a_n^\pm = A_n^+ \pm A_n^- \).

We note here, that the new coefficients \( a_n^\pm \) can be expanded in a similar way into the series like (12) and (14)

\[ a_n^\pm = \delta_{n0} e^{jk_y h} - k_{yn}^{-1} \sum_m \beta_m^\pm P_n^{m-m}, \]

\[ \beta_m^\pm = \frac{1}{2}(\alpha_m^+ \pm \alpha_m^-) \Rightarrow \alpha_m^\pm = \frac{1}{2}(\beta_m^+ \pm \beta_m^-). \]

Substituting the expansions (19) into (16), after straightforward algebraic manipulations, we obtain the following systems for unknown coefficients \( \beta_m^\pm \)

\[ \sum_m G_{pm}^\pm \beta_m^\pm = g_p^\pm, \]  

where the following notation is used

\[ G_p^+ = \sum_n P_n^{m=} F_{pn}(jk_{xn} \eta_p^{-1} \sin \eta_p \hat{h} - k_{xn} k_{yn}^{-1} \sin \eta_p \hat{h}), \]

\[ G_p^- = \sum_n P_n^{m=} F_{pn}(jk_{xn} \eta_p^{-1} \sin \eta_p \hat{h} - jk_{xn} k_{yn}^{-1} \cos \eta_p \hat{h}), \]

\[ g_p^+ = -2F_{p0} k_x \sin \eta_p \hat{h} e^{jsy h}, \quad g_p^- = -2F_{p0} jk_x \cos \eta_p \hat{h} e^{jsy h}. \]

Following the same steps a similar system of equations can be obtained for the case of TE incidence. The corresponding coefficients \( G_{pm}^\pm \) can be obtained from (21) by swapping over \( k_{xn} \eta_p^{-1} \) with \( k_{yn} k_{xn}^{-1} \) and \( g_p^\pm \) by swapping over \( k_x \) with \( k_y \) and sine with cosine functions, respectively. Besides, in the case of TE polarization the following equations should be added in order to obey the conditions in the center of the bar at the \( y = 0, h \) planes (13)

\[ \sum_m \beta_m^\pm \left( \sum_n (-1)^n P_n^{m-m}/k_{xn} \right). \]

The above doubly infinite systems of linear Equations (20)–(22) can be only solved numerically. If the coefficients \( \beta_m^\pm \) are known, the corresponding \( \alpha_m^\pm \) and \( A_n^\pm \) can be found from (19) and (12), (14), respectively. This yields the scattered field outside the bars from (5).
Between bars the scattered field can be found from (6), where $B_p^{\pm}$ can be readily obtained from (16). We give the final results without going into details for TM incidence

$$B_p^{\pm} = e^{\mp i \eta \hat{h}} (b_p^{+} \pm b_p^{-}) / 4,$$

$$b_p^{+} = \sec(\eta \hat{h}) L_p \sum_n k_{xn} F_{pn} (\delta_n \cos j k_y h + a_n^{+}),$$

$$b_p^{-} = j \csc(\eta \hat{h}) \eta^{-1} L_p \sum_n k_{yn} k_{xn} F_{pn} (a_n^{+} - \delta_n \cos j k_y h),$$

(23)

where $a_n^{+} = A_n^{+} + A_n^{-}$ as before. For TE polarization in the above expressions the $k_{xn}$ should be replaced with $-j \xi_p$.

5. NUMERICAL RESULTS AND DISCUSSION

For numerical computations the system of Equations (20)–(22) should be truncated so as to take into account only the propagating modes and a finite number of lower order evanescent modes in the solution [9, 10]. The truncated system for unknown vectors $\beta_m^{\pm}, m = -M \ldots M, M$ is the truncation index, have the coefficients in the form of infinite series (summation over $n$). In order to calculate the matrix elements of the reduced system from (21), the corresponding series have also been truncated at some $N$ large enough to assure convergence. Taking into account that $P_{n}^\mu = O(n^{-2/3})$ for $\mu = -1/6$ (see e.g., [14] Sec. 3.9.1, Eq. (1) on page 162) and the term $F_{pn} = O(n^{-2})$ for large $n$ in (16) we readily obtain: the coefficients of the system (21) behave like $O(n^{-5/3})$ for TM and $O(n^{-8/3})$ for TE incidence, respectively, whereas that of (22) like $O(n^{-5/3})$. Generally, the series in (21) are fast converging [15] and even for $M = 15 \div 30$ the results assure acceptable accuracy, provided that $N$ is at least about one order of magnitude higher than $M$ [10].

In Fig. 2, a dependence of the transmission coefficient $A_0^{-}$ versus $\Lambda/\lambda$ is shown for different values of the incident angle. The above examples are analogous to those considered in [13] (for convenience they are referred in the embedded graphs) where a similar problem is solved by the mode matching method. The comparison reveals a good qualitative agreement of the results obtained by two different methods. Another example is given in Fig. 3 where a dependence of the transmission coefficient $A_0^{+}$ versus the $\Lambda/\lambda$ is shown for incidence close to normal: the angle of incidence $10^{-2}$ degrees is applied in the computations (compare with Figs. 86, 87 and Figs 70, 71 of [13]). It should be noted that for the case of $\theta = 0^\circ$ the corresponding systems of Equations (20)–(22) become near singular and numerical solution
Figure 2. Dependence of the transmission coefficient $A_0^-$ versus $\Lambda/\lambda$ for different values of the incident angle and $d = \Lambda/2$, (a) TM, and (b) TE polarization. The embedded graphs illustrate the results shown in Figs. 92, 82 of [13] for comparison.

Figure 3. Dependence of the transmission coefficient $A_0^-$ versus $\Lambda/\lambda$ for normal incidence and $d = \Lambda/2$, (a) TM, and (b) TE polarization; the values of $h$ are fixed: 1 — $h = 0$, 2 — $h = \Lambda/20$, 3 — $h = \Lambda/2$, colored 4 — $h = 2\Lambda$.

can not be obtained. But it works well for $\theta \rightarrow 0$ (10^{-6} degrees) provided that the double precision arithmetics is used for calculations.

6. CONCLUSION

In this paper the problem of plane EM wave scattering by a periodic system of PEC bars was solved by the properly modified method
presented in the previous work [9]. The field components outside the bars are represented by the series of spatial harmonics in accordance with the Floquet’s theorem. Between the bars the parallel-plate waveguide mode expansion is used. The key point of the presented method is the expansion of the amplitudes of spatial harmonics into the Fourier series with coefficients being properly chosen Legendre functions. This allows to satisfy the boundary conditions in both regions directly. Another important feature of the method is that the singular field behavior near the bars edges is taken into account explicitly by virtue of the form of expansion coefficients.

The presented method is quite straightforward due to the simple form of matrix elements (see (21)) being fast converging series, and can be easily implemented for an approximate evaluation of the scattered field. The numerical examples presented in the paper has revealed a good agreement with the results obtained in the literature.

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