Methods of IDT Charge Spatial Spectrum Evaluation

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Abstract. Interdigital transducers (IDTs) have found numerous applications in electronic devices for excitation and detection of surface acoustic waves (SAWs). This paper discusses the methods of solution of the electrostatic problem arising in modelling of the IDT most important characteristic — its frequency response. Numerical results are presented based on the so-called spectral approach. Computation difficulties are discussed and their solutions are proposed.

1. Introduction

The IDT frequency characteristics are mostly determined by the finger geometry. When the voltage is applied to fingers by means of the transducer bus-bars, a surface acoustic wave is generated in a piezoelectric substrate that supports IDT. It is known that the spatial spectrum of the induced electric charge distribution on the IDT approximates well the IDT frequency response. Thus the evaluation of the electric charge distribution on transducer’s fingers is important for modelling of SAW devices. For typical weak piezoelectric substrates, the so-called quasi-static approximation is used for evaluation of the charge. In this approximation the piezoelectric substrate is replaced by dielectric substrate, and electrodes have specified potentials or charges. The electrodes are assumed to be of infinitesimal thickness and infinitely long. Thus, the 2D problem is considered. The analytical form of the spatial spectrum of electric charge distribution on IDT fingers can be found only for the simplest topologies, for example consisting of a few strips (up to 3) [1], [2], or for an infinite periodic system of electrodes [2], [3]. But for practical cases it is important to analyze the longest possible systems of arbitrary electrodes. This problem can only be solved numerically.

Three known numerical methods may be mentioned to be most appropriate for the task. In the first method [4] [5], the topology of real IDT is approximated by the system of periodic narrow strips. Each transducer finger is modelled by a group of strips connected to each other, while spacings are represented by isolated strips. The analytical form of electric charge distribution evaluated in [3] for such a system of strips is used. The subsequent two methods use the analytical form of the solution of electrostatic problem for arbitrary systems of strips. Electrostatic problem is reduced to a mixed boundary value problem of the analytic functions theory [8], [9]. The second method [6] puts stress on evaluation of charge spatial distribution, then the spatial spectrum is evaluated by means of Fourier transformation. The third method [7] evaluates the spatial spectrum of electric charge distribution directly, providing powerful tool for modelling of IDTs, since there is no numerical evaluation of Fourier transformation (by means of FFT algorithm or similar) of the function that has square-root singularities at the electrode edges (this is a well known property of electric charge distribution on strips).

This paper shortly describes the method of solution of the electrostatic problem that arises in connection with IDTs modelling. Furthermore, numerical results are presented based on the spectral approach [7], and the fundamental difficulties of application of the algorithm in simulation of long IDTs are briefly discussed.

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2. Spatial Electrostatic Problem

A system of $N$ infinitely long strips is considered, located on the surface $y = 0$ of a homogeneous anisotropic dielectric substrate that is characterized by symmetric permittivity tensor $\epsilon$ (Fig. 1). Strips have infinitesimal thickness, their left and right edges are $a_n, b_n$ respectively, and have the specified potentials $\phi_k, k = 1..N$. The system of Maxwell’s equations in electrostatic approximation are

$$\nabla \times E = 0$$

$$\nabla \cdot D = 0$$

(1)

where $D$ and $E$ are the electric displacement and the electric field vectors, $\epsilon$ and $\epsilon_0$ are the dielectric permittivity of the substrate and vacuum, respectively.

The electrostatic potential $\phi$, satisfying equation

$$E = -\nabla \phi$$

must obey the system of partial elliptic differential equations resulting from (1)

$$\epsilon_{xx} \frac{\partial^2 \phi}{\partial x^2} + 2\epsilon_{xy} \frac{\partial^2 \phi}{\partial x \partial y} + \epsilon_{yy} \frac{\partial^2 \phi}{\partial y^2} = 0, \ y < 0,$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \ y > 0,$$

(3)

$$\epsilon_{xx} \epsilon_{yy} - \epsilon_{xy}^2 > 0, \ \epsilon_{xx} > 0.$$  

Using a coordinate transformation, the equation for $y < 0$ can be converted to the Laplace equation [2]

$$y' = \frac{\sqrt{\epsilon_{xx} \epsilon_{yy} - \epsilon_{xy}^2}}{\epsilon_{yy}} y, \ x' = x - \frac{\epsilon_{xy}}{\epsilon_{yy}} y \rightarrow \frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial y'^2} = 0, \ y < 0.$$ 

(4)

Hence, the solution of the problem for the case of electrodes placed in vacuum is fundamental. Having obtained the solution of the above problem, the original electric field components in anisotropic media are [1]

$$E_x(x, y) = E_x^{(0)}(x', y'),$$

$$E_y(x, y) = \frac{\epsilon}{\epsilon_{yy}} E_y^{(0)}(x', y') - \frac{\epsilon_{xy}}{\epsilon_{yy}} E_x^{(0)}(x', y'),$$

(5)

$$\epsilon = \sqrt{\epsilon_{xx} \epsilon_{yy} - (\epsilon_{xy})^2}.$$
Here, $E_x^{(0)}(x', y')$ and $E_y^{(0)}(x', y')$ denote the solutions in vacuum dependent on transformed coordinates. Some numerical examples showing the electric field in a system of metallic electrodes placed in vacuum and on the anisotropic substrate are presented in Appendix A; the anisotropy distorts the field making it asymmetric.

The electric charge distribution on strips is defined by the discontinuity of the normal component of the electric displacement vector $D_y$ at $y = 0$

$$\sigma(x) = D_y(x, +0) - D_y(x, -0); \quad (6)$$

after substitution of (5) into (1), this yields

$$\sigma(x) = \epsilon_0 E_y^{(0)}(x', +0) - \epsilon E_y^{(0)}(x', -0) = \epsilon_0 E_y^{(0)}(x, +0) - \epsilon E_y^{(0)}(x, -0). \quad (7)$$

As it was mentioned earlier, we need rather the charge spatial spectrum, that is the Fourier transform of charge

$$\sigma(r) = \int_{-\infty}^{\infty} \sigma(x)e^{-jrx} dx, \quad (8)$$

where $r$ is a spectral variable.

To this end, the solution of electrostatic problem in vacuum is sufficient, since it allows us to find the electric charge distribution and its spatial spectrum on the electrodes. Thus, further analysis concerns the electrodes placed in vacuum. For this case, the second equation in (1) may be written in terms of electric field vector in analogous form. As a consequence of this, the system of equations (1) can be interpreted as the Cauchy - Riemann conditions for analyticity of the function of complex variable defined as

$$E(z) = E_x(x, y) - iE_y(x, y), \quad z = x + iy. \quad (9)$$

And what is more, the symmetry of the problem implies

$$E_x(x, y) = E_x(x, -y), \quad E_y(x, y) = -E_y(x, -y). \quad (10)$$

On the $x$ axis the field components satisfy the following boundary conditions:

$$E_x(x, 0) \equiv Re(E) = 0, \quad for \ x \in \bigcup(a_k, b_k),$$

$$E_y(x, 0) \equiv Im(E) = 0, \quad for \ x \in R \setminus \bigcup(a_k, b_k). \quad (11)$$

The solution of the above electrostatic problem can thus be obtained by means of the theory of analytic functions. The solution vanishing at infinity of the so-called mixed boundary value problem for a half-plane is exploited, that is known as the Keldysh and Sedov formula [8], also presented in [9] and [10], and used in [1] for theoretical analysis of the simplest IDTs.

$$E(z) = \frac{P_{N-1}(z)}{\sqrt{R_N(z)}}, \quad (12)$$

where

$$P_{N-1}(z) = \sum_{n=0}^{N-1} \alpha_n z^n, \quad (13)$$

$$R_N(z) = \prod_{n=1}^{N}(z - a_n)(z - b_n),$$
\(a_n, b_n\) – coordinates of the left and right edges of the \(n\)th electrode, and \(N\) – number of electrodes. Coefficients \(\alpha_n\) of the polynomial \(P_N(z)\) are arbitrary real coefficients. In (13), \(\sqrt{R_N}\) denotes the branch that is positive on the real axis of the complex plane with Riemann’s cuts along \(a_n, b_n, n = 1 \ldots N\) for \(x > b_N\).

The electric field at \(y = 0\) is of our principal interest. Taking the limit \(y \to 0\) of (12) the surface field components are

\[
E_x(x, 0) = \begin{cases} 
0, & x \in (a_m, b_m) \quad m = 1..N, \\
(-1)^{N+m} \frac{P_{N-1}(x)}{H_N(x)}, & x \in (b_m, a_{m+1}) \quad m = 0..N,
\end{cases}
\]

\[
E_y(x, +0) = \begin{cases} 
(-1)^{N+m} \frac{P_{N-1}(x)}{H_N(x)}, & x \in (a_m, b_m) \quad m = 1..N, \\
0, & x \in (b_m, a_{m+1}) \quad m = 0..N,
\end{cases}
\]

(14)

where \(b_0, a_{N+1}\) stand for \(-\infty\) and \(\infty\) respectively and

\[
H_N(x) = \left| \prod_{n=1}^{N} (x - a_n)(x - b_n) \right|.
\]

(15)

The coefficients \(\alpha_n\) of the polynomial \(P_{N-1}(z)\) are to be determined from the Kirchhoff law yielding the conditions on voltages of the neighboring electrodes

\[
U_k = \phi_{k+1} - \phi_k, \quad k = 1..N - 1;
\]

(16)

here \(\phi_k, k = 1..N\) are the specified potentials on the strips

\[
\phi(x, 0) = \phi_k, \text{ for } x \in \bigcup (a_k, b_k).
\]

(17)

For real systems the total charge vanishes and the potential \(\phi(x) \to 0\) when \(x \to \pm\infty\). Therefore taking into account (2), the order of the polynomial \(P_{N-1}(z)\) in (12) must be reduced to \(N-2\) to obey the above condition. Concluding, there are only \(N-1\) unknown constants that should be found from \(N-1\) constraints (16).

3. Evaluation of Charge Spatial Spectrum

3.1 Fourier Transform of the Charge Distribution

This approach was used in [6] for modelling of non-periodic system of metal electrodes. Evaluation of the charge spatial spectrum is based on the above presented solution (12) of the electrostatic problem. Unknown coefficients \(\alpha_k, k = 1 \ldots N-1\) are determined as follows (see Eqs.(14), (16)):

\[
\phi_{k+1} - \phi_k = -\int_{b_k}^{a_{k+1}} E_x(x) \, dx = -\int_{b_k}^{a_{k+1}} (-1)^{N-k} \frac{P_{N-2}(x)}{\sqrt{H_N(x)}} \, dx.
\]

(18)

The following system of linear equations has to be solved:

\[
\sum_{m=0}^{N-2} A_{km} \alpha_m = \phi_{k+1} - \phi_k, \quad k = 1..N - 1,
\]

(19)
where

\[
A_{km} = (-1)^{N-k} \int_{b_k}^{a_k+1} x^m \, dx \sqrt{\prod_{n=1}^{N} |(x-a_n)(x-b_n)|}.
\]  

(20)

For numerical integration in (20) the Gauss formula is used

\[
\int_{a}^{b} \frac{f(x) \, dx}{\sqrt{(x-a)(b-x)}} \approx \frac{\pi M}{M} \sum_{k=1}^{M} \left[ \frac{a+b}{2} + \frac{b-a}{2} \cos \left( \frac{2k-1}{2M} \right) \right].
\]

(21)

where \( M \) is large enough to achieve the required accuracy. The charge spatial spectrum is evaluated by the Fourier transform of the charge spatial distribution (see Eq.(8)):

\[
\sigma(x) = 2\varepsilon_0 E_y(x, +0)
\]

(22)

(this results from Eq.(7)) and \( E_y(x, +0) \) is given by Eq.(14). Further technique is based on the expansion of the integrand function into a series of Tchebychev’s polynomials and exploiting the Gauss formula for numerical evaluation of integrals, yielding the following expression for charge spatial spectrum

\[
\sigma(r) = \sum_{n=1}^{N} e^{-j\xi_n r} \sum_{m=0}^{N-2} \alpha_m \sum_{k=0}^{M_1} (-1)^{j} j^{k} D_{nmk} J_{k}(dnr),
\]

(23)

where \( \xi_n, d_n \) are the \( n \)th electrode center coordinate and half-width respectively, \( M_1 \) is the number of terms in the numerical integration by means of the Gauss formula in (22) (see Eq.(21)), \( D_{nmk} \) - summation coefficients resulting from the Tchebychev expansion mentioned above.

### 3.2 The Electric Field in an Infinite System of Periodic Strips

In this approach used in [5] for floating-electrode unidirectional SAW transducers modelling, the IDTs topology is approximated by infinite system of relatively narrow periodic strips. The analytical form of the electric field evaluated in [3] is exploited in this approach [4], [5]. Electric field in a periodic structure can be written in the following form (referred to the plane \( y = 0 \)):

\[
E_x = \sum_{n=-\infty}^{\infty} E_n e^{-j(n+K)x}, \quad y = 0,
\]

\[
E_y = j \sum_{n=-\infty}^{\infty} S_{n+K} E_n e^{-j(n+K)x}, \quad y = +0,
\]

(24)

where \( r \in (0, K), K = 2\pi/\Lambda, \Lambda \)–period of the structure. The solution of electrostatic problem that satisfies boundary conditions (11) may be written as follows [3]:

\[
E_x = \alpha(r) \sum_{n=-\infty}^{\infty} S_{n} P_n(\cos \Delta) e^{-j(n+K)x}, \quad y = 0,
\]

\[
E_y = -j\alpha(r) \sum_{n=-\infty}^{\infty} P_n(\cos \Delta) e^{-j(n+K)x}, \quad y = +0,
\]

(25)

where \( r \in (0, K), K = 2\pi/\Lambda, \Lambda \)–period of the structure, \( \Delta = \pi w/\Lambda, w \)–electrode’s width; \( K = 2\pi/\Lambda; w = \Lambda/2 \rightarrow \cos \Delta = 0. P_n \) denotes Legendre polynomials of the first kind. \( S_n \) – the sign function: \( S_n = -1 \) for \( n < 0 \) and \( S_n = 1 \) otherwise. Voltage between neighbor strips and strip’s charge are

\[
U_k(r) = \phi_{k+1} - \phi_k = \int_{k\Lambda}^{(k+1)\Lambda} E_x \, dx = \alpha(r) \Lambda P_{-\frac{1}{2}}(0) e^{-jr(k+\frac{1}{2})\Lambda},
\]

(26)
\( Q_k(r) = 2\varepsilon_0 \int_{(k-\frac{1}{2})A}^{(k+\frac{1}{2})A} E_y \, dx = 2\varepsilon_0 \alpha(r) \Lambda P_{-\frac{\pi}{\Lambda}}(0)e^{-jr\Lambda}; \) \hfill (27)

Here the function \( \alpha(r) \) is of our principal interest, because it describes the charge spatial spectrum behavior. The function \( \alpha(r) \) may be represented in the following form:

\[
\alpha(r) = -\frac{\sum Q_l e^{jrl\Lambda}}{2\varepsilon_0 \Lambda P_{-\frac{\pi}{\Lambda}}(0)},
\] \hfill (28)

\( Q_l \) denotes the charge residing at the \( l \)th narrow strip

\[
Q_l = \int_0^K Q_l(r) \, dr.
\] \hfill (29)

To evaluate the unknown \( Q_l \) the following system of linear equation may be deduced

\[
U_k = \frac{1}{K} \int_0^K U_k(r) \, dr = \frac{j}{2\pi\varepsilon_0} \sum_l Q_l \left( -\frac{k-1}{2} \right),
\] \hfill (30)

\( U_k \) - specified voltage between the \((k+1)\)th and the \( k \)th strips. Since for isolated electrodes \( Q_l = 0 \), the number of unknown charges \( Q_l \) is

\[
\sum_{k=1}^{N} N_k
\] \hfill (31)

where \( N_k \) is the number of narrow strips representing the \( k \)th IDT electrode. The number of voltages between the connected together strips \((U_k = 0)\) is

\[
\sum_{m=1}^{N} N_m - N.
\]

Besides there are \( N - 1 \) voltages between IDT electrodes (but not between narrow strips). To complete the system of linear equations (30), the condition on total system charge must be added

\[
\sum_l Q_l = 0
\] \hfill (32)

where \( l \) varies over the number of electrodes connected together (31).

### 3.3 Spectral Analysis

As mentioned earlier, the spatial spectrum of electric charge distribution on electrodes must be evaluated to approximate the frequency response of the transducer. It results from (7) and (14) that the normalized charge distribution for real finite system of electrodes can be expressed by

\[
\frac{\sigma(x)}{2\varepsilon_0} = E_y(x, +0) = \begin{cases} 
(-1)^{N+m} \frac{P_{N-2}(x)}{H_N(x)}, & x \in (a_m, b_m) \quad m = 1..N \\
0, & x \in (b_m, a_{m+1}) \quad m = 0..N
\end{cases}
\] \hfill (32)

while the \( E_x(x) \) is

\[
E_x(x, 0) = \begin{cases} 
0, & x \in (a_m, b_m) \quad m = 1..N \\
(-1)^{N+m} \frac{P_{N-2}(x)}{H_N(x)}, & x \in (b_m, a_{m+1}) \quad m = 0..N.
\end{cases}
\] \hfill (33)
In the approach presented in [7], the following function is introduced to enable direct evaluation of the charge spatial spectrum

\[ de(x) \equiv E_y(x, +0) + jE_x(x). \]  

(34)

The polynomial in Eqs.(32), (33) can be written in alternative form

\[ P_{N-2}(x) = \sum_{m=0}^{N-2} \alpha_m \prod_{i=1}^{m} (x - \xi_i) \]  

(35)

while the function \( H_N(x) \)

\[ H_N(x) = \sqrt{\prod_{n=1}^{N} |((x - \xi_n) - d_n)((x - \xi_n) + d_n)|} \]  

(36)

where \( \xi_n, d_n \) are the \( n \)th electrode center coordinate and half-width respectively.

Substituting (35) and (36) into Eqs.(32), (33) and taking into account (34), the function \( de(x) \) can be written

\[ de(x) = \sum_{m=0}^{N-2} (-j)^{N-1} \alpha_m \prod_{k=1}^{m} \frac{(x - \xi_k)}{\sqrt{d_k^2 - (x - \xi_k)^2}} \prod_{k=m+1}^{N} \frac{1}{\sqrt{d_k^2 - (x - \xi_k)^2}} \]  

(37)

or in a more compact form

\[ de(x) = \sum_{k=0}^{N-2} (-j)^{N-1} \alpha_k de^{(N,k)}(x) \]  

(38)

where the so-called 'generating' functions \( de^{(N,k)}(x) \) are introduced representing the expressions in brackets in Eq.(37). The Fourier transform of the \( de(x) \) thus can be defined as

\[ De(r) = \mathcal{F}\left\{ de(x) = \sum_{k=0}^{N-2} (-j)^{N-1} \alpha_k de^{(N,k)}(x) \right\} = \sum_{k=0}^{N-2} (-j)^{N-1} \alpha_k De^{(N,k)}(r) \]  

(39)

where \( r \) is a spectral variable, and the Fourier transforms \( De^{(N,k)} \) of the 'generating functions' \( de^{(N,k)} \) are in the form of multiple convolutions

\[ De^{(N,k)}(r) = De'_1(r) \ast \ldots \ast De'_k(r) \ast De_{k+1} \ast \ldots \ast De_N(r) \]  

(40)

of terms

\[ De_i(r) = \mathcal{F}\left\{ \frac{1}{\sqrt{(d_i^2 - (x - \xi_i)^2)}} \right\} = \begin{cases} e^{-jr\xi_i} J_0(rd_i), & r \geq 0 \\ 0, & r < 0 \end{cases} \]  

(41)

\[ De'_i(r) = \mathcal{F}\left\{ \frac{(x - \xi_i)}{\sqrt{(d_i^2 - (x - \xi_i)^2)}} \right\} = \begin{cases} je^{-jr\xi_i}[\delta(r) - d_i J_1(rd_i)], & r \geq 0 \\ 0, & r < 0 \end{cases} \]  

(42)

Coefficient \( \alpha_k \) can be evaluated from the constraints analogous to those of Eq.(18) which are modified due to Eq.(34)

\[ \dot{\phi}_{k+1} - \phi_k = -\int_{b_k}^{a_k+1} E_x(x) \, dx = -\int_{b_k}^{a_k+1} \text{Im} \{ de(x) \} \, dx. \]  

(43)

This yields the system of linear equations

\[ A\alpha = U \]  

(44)
where $\mathbf{U}$ – a vector of voltages between neighboring electrodes

$$\phi_{k+1} - \phi_k = U_k, \quad k = 1 \ldots N - 1$$

(45)

and the elements of matrix $\mathbf{A}$ are the integrals

$$A_{i k} = -\int_{\xi_i+a_i}^{\xi_{i+1} - a_{i+1}} \text{Im} \left\{ \frac{d e^{(N,k-1)}}{N,k-1} \right\} dx, \quad i, k = 1 \ldots N - 1$$

(46)

or in a more detailed form

$$A_{i k} = -\int_{\xi_i+a_i}^{\xi_{i+1} - a_{i+1}} \left\{ (-j)^{N-1} \frac{1}{\prod_{m=1}^{k-1} (x - \xi_m)} \prod_{n=1}^{N} (d_n^2 - (x - \xi_n)^2) \right\} dx, \quad i, k = 1 \ldots N - 1.$$  

(47)

Some numerical examples are presented below. In Figs. (2), (3) the electric charge spatial spectrum and the spatial distribution of electric potential referred to the plane $y = 0$ in the system of 5 periodic electrodes are shown. The electrode’s width equals the half of the system’s period.

**Figure 2: Spatial spectrum of electric charge in the system of 5 strips.**

It should be remarked here that the form of spatial spectrum of electric charge on IDTs electrodes described by (39) is too difficult for direct numerical calculations for many reasons thoroughly discussed below. Generally, evaluation of the function $D e(r)$ like it stands in (39) – (42) gives reasonable results for the number of electrodes not exceeding 20 (periodic system of strips). For longer systems of strips the numerical inaccuracies inevitably lead to severe distortions of the charge spatial spectrum.

Numerical integration in Eq.(46) is required to evaluate all the elements $A_{i k}$. This may be performed using the method [12] by applying the iterative numerical integration scheme based on the so-called ’extended midpoint rule’

$$\int_{x_1}^{x_2} f(x) dx = h \left[ f_{3/2} + f_{5/2} + \ldots + f_{M-3/2} + f_{M-1/2} \right] + O(1/M^2)$$

(48)
with different steps $h$ and combined as

$$S = (9S_M - S_M)/8$$

what results in cancellation of error terms of the second and third orders. The functions in Eq.(46) are square-root singular at both limits of integration. The appropriate technique needs to be implemented into numerical integration algorithm based on splitting the integral at the interior breakpoint $\gamma_m \in (\xi_m + a_m; \xi_{m+1} - a_{m+1})$

$$A_{ij} = -\text{Im} \int_{\xi_i + a_i}^{\gamma_i} d^{(N,j-1)}(x)dx - \text{Im} \int_{\gamma_i}^{\xi_{i+1} - a_{i+1}} d^{(N,j-1)}(x)dx$$

(49)

and introducing the variable transform, that allows to remove the above mentioned singularities [11], [12]

$$\int_a^b f(x)dx = \int_0^{\sqrt{b-a}} 2tf(a + t^2)dt, \quad (b > a)$$

for singularity at $a$, and

$$\int_a^b f(x)dx = \int_0^{\sqrt{b-a}} 2tf(b - t^2)dt, \quad (b > a)$$

for singularity at $b$. The system of linear equations is usually solved by means of the LU (lower-upper) matrix decomposition algorithm. The above approach to coefficients $\alpha_k$, $k = 1 \ldots N - 1$ evaluation encounter a number of difficulties. Namely, the integrals, as they are in Eq.(47) are poorly converging ones. The accuracy of iterative integration scheme is determined by the so-called 'converging factor' that is the fractional accuracy. Poor convergence means that the required accuracy of integration can not be achieved with the numerical routine because of the lost of convergence due to the numerical errors accumulation while the number of iterations increases. Secondly, the system of linear equations (43) becomes ill-conditioned for large number of electrodes $N$. In other words, the matrix of the system of equations, the elements of which are the mentioned above poorly converging integrals (47), becomes numerically close to singular, so the LU decomposition algorithm or a similar one can not be applied to get reasonable results. Both these factors limit the applicability of the above approach for analyzing the long IDTs.
Once all the coefficients \( \alpha_k \), \( k = 1 \ldots N - 1 \) are determined, the spectral function \( D e(r) \) can be evaluated from Eqs. (39) – (42). Evaluation of convolutions in (40) was performed using the ”convolution theorem” (property of the Fourier transformation)

\[
\mathcal{F}^{-1} \left\{ \int_{-\infty}^{\infty} G(r - r') F(r')dr' \right\} = g(x)f(x)
\]

(50)

if

\[
G(r) = \mathcal{F}\{g(x)\} \quad F(r) = \mathcal{F}\{f(x)\};
\]

here \( \mathcal{F} \) denotes the Fourier transformation and \( \mathcal{F}^{-1} \) its inverse. For numerical evaluation of convolutions the FFT algorithm was exploited for Fourier transforms evaluation. The algorithm operates over the discrete representations of the functions that are to be convoluted. The sampling step \( \Delta r \) and the interval over which the function should be sampled are critical. The problems that arise here are the following. First of all, since functions (41) and (42) are determined over the semi-infinite interval \((0, \infty)\), one should truncate the interval to \((0, r_{up})\), assuming that the functions take zero values outside the one. Such neglecting introduce initial inaccuracy into the convolution evaluation scheme. The so-called ‘aliasing phenomenon’ takes place here. Namely, the part of the function being truncated is spuriously reflected into the interval \((0, r_{up})\). To avoid this, each data set should be padded with zeros, doubled at least \([11]\). Secondly, the functions (41) and (42) for larger values of \( i \) become faster oscillating due to the presence of exponential term \( e^{-ir\xi_i} \), since \( \xi_i \) (ith electrode displacement) increases. Besides, the above functions are slowly fading ones due to the presence of Bessel functions \( J_0(rd_i) \) and \( J_1(rd_i) \). The former factor requires that the sampling step \( \Delta r \) be diminished when \( i \) increases. The latter one, on the other hand, requires the value of \( r_{up} \) to increase, in other words, the interval over which the functions should be sampled have to be extended. Thus, the data sets become very large, and this leads to instability of the FFT algorithm.

It should be added, that the ‘generating functions’ described by Eqs. (40)–(42) spans large range of values for large number of electrodes. For example, for IDT having 25 fingers that range reaches the 12-th order of magnitude. Thus, numerical evaluation of \( D e(r) \) by means of Eq. (39) may become severely inaccurate.

All this restricts correct evaluation of the electric charge spatial spectrum to the number of electrodes not exceeding 20. For larger number of electrodes, numerical inaccuracies increase and distort the charge spatial spectrum. Generally, there are three main sources of numerical errors, that should be overcome to improve the numerical evaluation of the spatial spectrum of electric charge distribution (39) for longer IDTs (number of electrodes greater than 20).

- The first and perhaps the most substantial source of numerical inaccuracies may be associated with evaluation of the ‘generating functions’, that is evaluation of convolutions in (40) of the functions given by Eqs. (41) and (42). To overcome the difficulty, a higher order interpolation scheme for approximation of the function given by its samples was implemented into the convolution evaluation algorithm based on the ”convolution theorem” (50). The Fourier integral in this case is evaluated by means of FFT algorithm \([11]\)

\[
I = \int_a^b e^{jrx} f(r)dr \rightarrow I(x_n) \approx \Delta e^{jx_n a} [\text{FFT}(f_0 \ldots f_{M-1})]_n
\]

(51)

Interpolation implemented can be viewed as an approximation of the function by a sum of kernel functions (depending on interpolation scheme) times sample values (depending on the function)

\[
f(r) \approx \sum_{i=0}^{M} f_i \psi \left( \frac{r - r_i}{\Delta r} \right) + \sum_{i=\text{endpoints}} f_i \varphi \left( \frac{r - r_i}{\Delta r} \right)
\]

(52)
where $\psi(s)$ is the kernel function of an interior point, and $\varphi(s)$ is the kernel function for the subintervals closest to end points. The set $f_i$ forms the discrete representation of the function $f(r)$ sampled over the interval $(a, b)$ with the step $\Delta r$. Substituting (52) into (51) one obtain

$$I \approx \Delta e^{jxa} \left[ W(\Theta) \sum_{i=0}^{M} f_i e^{j i \Theta} + \sum_{i=\text{endpoints}} f_i \alpha_i(\Theta) \right], \Theta = x \Delta$$  \hspace{1cm} (53)

where

$$W(\Theta) \equiv \int_{-\infty}^{\infty} ds e^{j \Theta s} \psi(s), s = \left( \frac{r - r_i}{\Delta} \right), \Theta = x \Delta$$

and

$$\alpha_i(\Theta) \equiv \int_{-\infty}^{\infty} ds e^{j \Theta s} \varphi_i(s - i), s = \left( \frac{r - a}{\Delta} \right), \Theta = x \Delta$$

Interpolation is considered to be left-right symmetric

$$\varphi_{M-i}(s) = \varphi_i(-s) \quad \alpha_{M-i}(\Theta) = e^{jM \Theta} \alpha_i^*(\Theta)$$

Finally, the algorithm of evaluation of the integral like (53) may be written as

$$I(x_n) = \Delta e^{jxa} \left\{ W(\Theta) \left[ FFT(f_0 \ldots f_{M-1}) \right]_{n} + \sum_{i=0}^{M_1} \left( \alpha_i(\Theta)f_i + \alpha_i^*(\Theta)f_{M-i} \right) \right\}$$  \hspace{1cm} (54)

For the case of cubic interpolation (that was implemented) $M_1 = 3$.

- The second source of numerical inaccuracies is connected with the coefficients $\alpha_k, \ k = 1 \ldots N - 1$ evaluation, described above in this subsection. The solution was proposed, based on the following property of the Fourier transform:

$$\mathcal{F} \left\{ \int_{-\infty}^{x} f(t) dt \right\} = \frac{-j}{r} F(r) \quad \text{if} \quad \mathcal{F} \{ f(x) \} = F(r)$$  \hspace{1cm} (55)

and

$$d e^{(N,k)}(x) = \mathcal{F}^{-1}(D e^{(N,k)}(r)).$$  \hspace{1cm} (56)

Thus, since all the functions $D e^{(N,k)}$ are evaluated, the elements $A_{i,k}$ in (46) can be found by means of Eqs.(55) and (56) without application of iterative integration algorithm.

- The third source of numerical inaccuracies is associated with the large range of values spanned by ‘generating functions’ (40). To overcome this a proper modification of ‘generating functions’ Eq.(40) was implemented. Namely, Eq.(37) was modified so that in terms

$$\prod_{k=1}^{m} \frac{x - \xi_k}{\sqrt{(d_k^2 - (x - \xi_k)^2)}}$$

the value of $k$ is no longer chosen in consecutive order, but is selected ‘quasi-equidistantly’ from the set of values $< 1 \ldots N >$.

Introduction of all these improvements into the above scheme of charge spatial spectrum evaluation allowed us to improve the numerical analysis significantly. Numerical example in Fig. (4) shows the electric charge spatial spectrum for the case of 25 periodic electrodes. Here, the dashed curve represents the case when none of the above advanced techniques was implemented (distorted spectrum), while the solid one corresponds to the charge spatial spectrum evaluated by means of the improved algorithm.
Three different approaches to the electric charge spatial spectrum evaluation were discussed. In Fig. 5 the charge spatial spectrum on the system of 15 periodic electrodes calculated by means of these methods described in the previous section is shown. The black curve corresponds to the approach [7] of the Section 3.3, the red one - to the approach [4], [5] of the Section 3.2, and the green one - to the approach [6] of the Section 3.1. For the case of 15 periodic electrodes, the black and the green curves coincide giving the results that are in good agreement, while the red one differs slightly. The method, described in Section 3.1 evaluates the electric charge spatial distribution in the form (6) and (14). The spatial spectrum (8) is evaluated then by means of numerical integration (the Gauss formula), where the function is expanded into a series of Tchebychev’s polynomials (usually 4 terms of the series are used). For the number of elec-
trodes larger than 25, the computation time becomes too large and the accuracy of spectrum evaluation is insufficient. The advantage of the method is that it can be applied for modelling non-periodic systems of electrodes. The main disadvantage here is connected with numerical evaluation of the Fourier transform of the electric charge spatial distribution. The second approach shortly presented in Section 3.2 deals with the known solution of the electrostatic problem for an infinite periodic system of narrow strips. Once their connections are properly arranged, the real IDT topology can be approximated, as discussed above. The main disadvantage of the method is that the accuracy of such the approximation is dependent on the number of narrow strips per IDT electrode/spacing. Thus for longer IDT one has to deal with huge system of linear equations. Yet another disadvantage is connected with suitability of the non-periodic IDT’s topology approximation by the system of periodic narrow electrodes. But the main advantage of this approach is that for a periodic system of electrodes it gives reasonable results at minimal costs such as computation time and algorithm complexity. The last of the above methods evaluates the electric charge spatial spectrum directly. It seems to be the most promising for non-periodic IDTs analysis. There is no numerical evaluation of the Fourier transform of the electric charge distribution in contrast to the first method. The main disadvantage of this approach is its algorithm complexity, that in turn is mainly connected with multiple convolutions evaluation, as discussed in Section 3.3. To make it possible to apply this algorithm in practice for the analysis of the IDTs with the number of electrodes larger than 20 (for the case of periodic IDTs), the advanced numerical methods and techniques should be implemented. Such techniques were proposed and verified, and this allowed us to analyze the electric charge spatial spectrum for longer periodic and non-periodic IDTs. The numerical example in Fig. 6 illustrates the charge spatial spectrum, evaluated by three above methods for the system of 25 periodic electrodes. The one evaluated by the first method (green curve) is distorted by numerical inaccuracies. Both the second and third methods for comparable values of input parameters (sampling step, sampling domain, dimensions of the data sets and so on) give sufficiently accurate results close to each other.

Figure 6: Spatial spectrum of electric charge in the system of 15 strips.

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Appendix A

In Figs. (7) and (8) the electric field is shown for the system of two electrodes in vacuum. The arrows in Fig. (7) correspond to the direction of the electric field vector. In Fig. (8) its amplitude is presented in logarithmic scale.

Figure 7: Electric field for the case of two electrodes in vacuum (directions).

Figure 8: Electric field for the case of two electrodes in vacuum (amplitude, logarithmic scale).
In Figs. (9) and (10) the electric field is shown for the system of two electrodes on an anisotropic substrate (for numerical calculations the Rochelle Salt with relative permittivity constants $\varepsilon_{xx}/\varepsilon_0 = 205$, $\varepsilon_{yy}/\varepsilon_0 = 9.6$, $\varepsilon_{zz}/\varepsilon_0 = 9.5$ was used as the substrate material).

Figure 9: Electric field for the case of two electrodes on the anisotropic substrate (directions).

Figure 10: Electric field for the case of two electrodes on the anisotropic substrate (amplitude, logarithmic scale).
References


