Vol. 49 (2002) REPORTS ON MATHEMATICAL PHYSICS No. 2/3

KLEIN-GORDON-DIRAC EQUATION: PHYSICAL JUSTIFICATION AND QUANTIZATION ATTEMPTS

Jan J. Sławianowski and Vasyl Kovalchuk

Institute of Fundamental Technological Research, Polish Academy of Sciences,
Świętokrzyska 21, 00-049 Warsaw, Poland
(e-mails: jslawian@ippt.gov.pl vkoval@ippt.gov.pl)

(*Received September 24, 2001 — Revised December 27, 2001*)

Discussed is the Klein-Gordon-Dirac equation, i.e. a linear differential equation with constant coefficients, obtained by superposing Dirac and d'Alembert operators. A general solution of KGD equation as a superposition of two Dirac plane harmonic waves with different masses has been obtained. The multiplication rules for Dirac bispinors with different masses have been found. Lagrange formalism has been applied to receive the energy-momentum tensor and 4-current. It appears, in particular, that the scalar product is a superposition of Klein-Gordon and Dirac scalar products. The primary approach to canonical formalism is suggested. The limit cases of equal masses and one zero mass have been calculated.

Keywords: Klein-Gordon-Dirac equation, plane harmonic waves with different masses, Dirac bispinors, Lagrange formalism, canonical formalism.

1. Introduction

The problem has originally arisen from the paper [1] and the more later one [2], where the Klein-Gordon-Dirac equation (KGD), i.e. a linear differential equation with constant coefficients, obtained by superposing Dirac and d'Alembert operators, appears from the U(2,2)-ruled gauge model of spinorial geometrodynamics in a natural and logical way. Another kind of motivation for this seemingly strange idea comes from the standard model of electroweak interactions with its mysterious pairing of fundamental fermions.

Let us consider the density of the Klein-Gordon-Dirac Lagrangian in the form

$$\pounds = ug^{\mu\nu}\partial_\mu\overline{\Psi}\partial_\nu\Psi + \frac{vi}{2}\left(\overline{\Psi}\gamma^\mu\partial_\mu\Psi - \partial_\mu\overline{\Psi}\gamma^\mu\Psi\right) - w\overline{\Psi}\Psi\,, \tag{1}$$

where $g^{\mu\nu}$ is the metric tensor, which in special-relativistic limit equals $\eta^{\mu\nu}$, i.e. the flat metric tensor on space-time manifold M with signature $(+,-,-,-)$ and constant coefficients, $\overline{\Psi} = \Psi^{+}\gamma^0$ introduces the rule of Dirac conjugation of bispinors and u, v, w are some real constants.

Since the density of common Dirac Lagrangian

$$\pounds = \frac{i}{2}\left(\overline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - \partial_{\mu}\overline{\Psi}\gamma^{\mu}\Psi\right) - m\overline{\Psi}\Psi$$

contains only the first order derivatives from $\overline{\Psi},\Psi$, the action $S = \int \pounds d^4x$ can have neither minimum nor maximum and the principle of the least action $\delta S = 0$ defines only a stationary point but not the extremum of the action integral [4]. In our opinion, adding d'Alembert operator into common Dirac theory is not only interesting by itself, but may also help to solve these difficulties, while deducing Dirac equation from the variation principle.

2. KGD equation of motion

Lagrange equations of motion for KGD Lagrangian take the following form:

$$vi\gamma^{\mu}\partial_{\mu}\Psi - w\Psi = ug^{\mu\nu}\partial_{\mu}\partial_{\nu}\Psi\,, \qquad vi\partial_{\mu}\overline{\Psi}\gamma^{\mu} + w\overline{\Psi} = -ug^{\mu\nu}\partial_{\mu}\partial_{\nu}\overline{\Psi}\,. \tag{2}$$

As it has already been said in [1], such an equation does not correspond to any irreducible representation of Poincaré group, and in this sense it is not admitted by Wigner-Bargmann classification as a relativistic wave equation for elementary particles. Nevertheless, there are no principle obstacles against considering a continuous dynamical system ruled by the KGD equation.

In the momentum representation $\left(\Psi(x) = \int e^{-ip_{\mu}x^{\mu}}\varphi(p)\,dp\right)$ we obtain the equation of motion as follows,

$$v\gamma^{\mu}p_{\mu}\varphi(p) - w\varphi(p) = -ug^{\mu\nu}p_{\mu}p_{\nu}\varphi(p)\,.$$

Since $g^{\mu\nu}p_{\mu}p_{\nu} = p^2$, and in the case when $v \neq 0$, we may change the form of this equation into

$$\gamma^{\mu}p_{\mu}\varphi(p) = m\varphi(p)\,, \tag{3}$$

which formally looks like Dirac equation with mass $m^2 = p^2 = \left(\frac{w-up^2}{v}\right)^2$. Thus, we may write that the general solution of KGD equation is a superposition of two Dirac plane harmonic waves with masses

$$m_{\pm} = \frac{1}{\sqrt{2}\,|u|}\sqrt{v^2 + 2uw \pm |v|\sqrt{v^2+4uw}}\,. \tag{4}$$

For the existence of real non-negative (non-tachyonic situation) solutions for m^2 we should have $(v^2+4uw \geq 0) \wedge (v^2+2uw \geq 0)$, i.e. $(uw \geq 0,\ \forall v) \vee \left(uw < 0,\ v^2 \geq 4\,|uw|\right)$.

To complete our consideration we would like to add the analysis of such a situation given by J. J. Sławianowski in [1]. The appearance of two mass shells in a general solution of KGD equation does not have to be so embarrassing as it could seem, for the following reasons:

1) If the splitting of masses $m_+ - m_-$ is large, then, in normal conditions, it may be difficult to excite the m_+-states, because the frequency spectrum of external perturbations

will have to contain frequencies of the order $\frac{(m_+ - m_-)c^2}{h}$, e.g. if $u \to 0$, then $m_- \to \frac{|w|}{|v|}$, $m_+ \to \infty$ (compare this with the idea of Pauli-Villars-Rayski regularization [3]).

2) It is not excluded that superposition of states with two masses might be just desirable, e.g. one could try to explain in this way the mysterious kinship between heavy leptons and their neutrinos, or the corresponding pairing between quarks. If there is no algebraic term, $w = 0$, then $m_- = 0$, $m_+ = \frac{|v|}{|u|}$. Thus, in spite of a purely differential character of KGD equation, massive states appear and are paired with the massless ones.

3) For special values of u, v, w, i.e. when $v^2 + 4uw = 0$, the mass gap vanishes, $m_- = m_+ = \frac{|w|}{|u|}$, and KGD equation is exactly reduced to the common Dirac equation.

Thus, for the solution of Dirac equation we may write the expansion in eigenfunctions in accordance with the superposition principle in the following form:

$$\begin{aligned} \Psi(x) &= \sum_{s=1,2} \int d\mu(m,\boldsymbol{p}) \left(e^{-ipx} u_{\boldsymbol{p}}^{s,m} a_{\boldsymbol{p}}^{s,m} + e^{ipx} v_{\boldsymbol{p}}^{s,m} b_{\boldsymbol{p}}^{+s,m} \right) \\ &+ \sum_{s=1,2} \int d\mu(M,\boldsymbol{p}) \left(e^{-ipx} u_{\boldsymbol{p}}^{s,M} a_{\boldsymbol{p}}^{s,M} + e^{ipx} v_{\boldsymbol{p}}^{s,M} b_{\boldsymbol{p}}^{+s,M} \right), \end{aligned} \tag{5}$$

$$\begin{aligned} \overline{\Psi}(x) &= \sum_{r=1,2} \int d\mu(m,\boldsymbol{p}) \left(e^{ipx} \overline{u}_{\boldsymbol{p}}^{r,m} a_{\boldsymbol{p}}^{+r,m} + e^{-ipx} \overline{v}_{\boldsymbol{p}}^{r,m} b_{\boldsymbol{p}}^{r,m} \right) \\ &+ \sum_{r=1,2} \int d\mu(M,\boldsymbol{p}) \left(e^{ipx} \overline{u}_{\boldsymbol{p}}^{r,M} a_{\boldsymbol{p}}^{+r,M} + e^{-ipx} \overline{v}_{\boldsymbol{p}}^{r,M} b_{\boldsymbol{p}}^{r,M} \right), \end{aligned} \tag{6}$$

where $M = m_+$, $m = m_-$, the normalized measure of these integrals is $d\mu(m,\boldsymbol{p}) = \frac{m d^3 p}{(2\pi)^3 E_{\boldsymbol{p}}^m}$, where the energy $E_{\boldsymbol{p}}^m = p_0 = \sqrt{m^2 + \boldsymbol{p}^2}$; $u_{\boldsymbol{p}}^{s,m}, v_{\boldsymbol{p}}^{r,m}$ are the amplitudes of plane harmonic waves with positive and negative frequencies (Dirac bispinors), which we may write in the common form:

$$\begin{aligned} u_{\boldsymbol{p}}^{s,m} &= \frac{1}{\sqrt{2m\left(m + E_{\boldsymbol{p}}^m\right)}} \begin{pmatrix} \left(m + E_{\boldsymbol{p}}^m\right)\omega^s \\ \boldsymbol{\sigma p}\,\omega^s \end{pmatrix}, \\ v_{\boldsymbol{p}}^{r,m} &= \frac{1}{\sqrt{2m\left(m + E_{\boldsymbol{p}}^m\right)}} \begin{pmatrix} \boldsymbol{\sigma p}\,\omega^r \\ \left(m + E_{\boldsymbol{p}}^m\right)\omega^r \end{pmatrix}, \end{aligned}$$

where ω^s is Dirac 3-spinor, which satisfies the normalization condition $\omega^{+s}\omega^r = \delta^{sr}$.

The multiplication rules for Dirac bispinors with the same mass (m or M) are as follows (the second table of multiplication rules can be received by substituting m with M):

$$\begin{array}{lllll} & u_{\boldsymbol{p}}^{s,m} & v_{\boldsymbol{p}}^{s,m} & u_{-\boldsymbol{p}}^{s,m} & v_{-\boldsymbol{p}}^{s,m} \\ (\overline{u})_{\boldsymbol{p}}^{r,m} & \delta^{rs} & 0 & \frac{1}{m} E_{\boldsymbol{p}}^m \delta^{rs} & -\frac{1}{m} p^A \sigma_A^{rs} \\ (\overline{v})_{\boldsymbol{p}}^{r,m} & 0 & -\delta^{rs} & \frac{1}{m} p^A \sigma_A^{rs} & -\frac{1}{m} E_{\boldsymbol{p}}^m \delta^{rs} \\ u_{\boldsymbol{p}}^{+r,m} & \frac{1}{m} E_{\boldsymbol{p}}^m \delta^{rs} & \frac{1}{m} p^A \sigma_A^{rs} & \delta^{rs} & 0 \\ v_{\boldsymbol{p}}^{+r,m} & \frac{1}{m} p^A \sigma_A^{rs} & \frac{1}{m} E_{\boldsymbol{p}}^m \delta^{rs} & 0 & \delta^{rs} \end{array}, \tag{7}$$

where $p^A\sigma_A^{rs}$ are elements of the matrix $\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$, which is given as a scalar product of Dirac vector-matrix $\boldsymbol{\sigma}$ and the 3-momentum $\boldsymbol{p}$.

For bispinors with different masses m and M we may write the multiplication rule tables in the following way:

$$\begin{array}{lllll} & u_{\boldsymbol{p}}^{s,M} & v_{\boldsymbol{p}}^{s,M} & u_{-\boldsymbol{p}}^{s,M} & v_{-\boldsymbol{p}}^{s,M} \\ (\overline{u})_{\boldsymbol{p}}^{r,m} & A\delta^{rs} & -Bp^A\sigma_A^{rs} & C\delta^{rs} & -Dp^A\sigma_A^{rs} \\ (\overline{v})_{\boldsymbol{p}}^{r,m} & Bp^A\sigma_A^{rs} & -A\delta^{rs} & Dp^A\sigma_A^{rs} & -C\delta^{rs} \\ u_{\boldsymbol{p}}^{+r,m} & C\delta^{rs} & Dp^A\sigma_A^{rs} & A\delta^{rs} & Bp^A\sigma_A^{rs} \\ v_{\boldsymbol{p}}^{+r,m} & Dp^A\sigma_A^{rs} & C\delta^{rs} & Bp^A\sigma_A^{rs} & A\delta^{rs} \end{array} \tag{8}$$

and

$$\begin{array}{lllll} & u_{\boldsymbol{p}}^{s,m} & v_{\boldsymbol{p}}^{s,m} & u_{-\boldsymbol{p}}^{s,m} & v_{-\boldsymbol{p}}^{s,m} \\ (\overline{u})_{\boldsymbol{p}}^{r,M} & A\delta^{rs} & Bp^A\sigma_A^{rs} & C\delta^{rs} & -Dp^A\sigma_A^{rs} \\ (\overline{v})_{\boldsymbol{p}}^{r,M} & -Bp^A\sigma_A^{rs} & -A\delta^{rs} & Dp^A\sigma_A^{rs} & -C\delta^{rs} \\ u_{\boldsymbol{p}}^{+r,M} & C\delta^{rs} & Dp^A\sigma_A^{rs} & A\delta^{rs} & -Bp^A\sigma_A^{rs} \\ v_{\boldsymbol{p}}^{+r,M} & Dp^A\sigma_A^{rs} & C\delta^{rs} & -Bp^A\sigma_A^{rs} & A\delta^{rs} \end{array} , \tag{9}$$

where the coefficients are as follows:

$$A = \frac{(m+p_0)(M+P_0) - p^2}{2\sqrt{mM}\sqrt{(m+p_0)(M+P_0)}} > 0, \quad B = \frac{M + P_0 - m - p_0}{2\sqrt{mM}\sqrt{(m+p_0)(M+P_0)}} \geq 0,$$

$$C = \frac{(m+p_0)(M+P_0) + p^2}{2\sqrt{mM}\sqrt{(m+p_0)(M+P_0)}} > 0, \quad D = \frac{M + P_0 + m + p_0}{2\sqrt{mM}\sqrt{(m+p_0)(M+P_0)}} > 0,$$

and $p_0 = E_{\boldsymbol{p}}^m$, $P_0 = E_{\boldsymbol{p}}^M$. In the case of equal masses $m = M$ we obtain limit values:

$$A = 1\,, \qquad B = 0\,, \quad C = \frac{1}{m}E_{\boldsymbol{p}}^m\,, \quad D = \frac{1}{m}\,.$$

3. Lagrange formalism

To apply Lagrange formalism we calculate the following derivatives:

$$\frac{\partial \pounds}{\partial \Psi_{,\mu}} = ug^{\mu\lambda}\overline{\Psi}_{,\lambda} + \frac{vi}{2}\overline{\Psi}\gamma^\mu, \qquad \frac{\partial \pounds}{\partial \overline{\Psi}_{,\mu}} = ug^{\mu\lambda}\Psi_{,\lambda} - \frac{vi}{2}\gamma^\mu\Psi\,.$$

Then the energy-momentum tensor and 4-current have the following forms:

$$t^\mu_\nu = \overline{\Psi}_{,\nu}\frac{\partial \pounds}{\partial \overline{\Psi}_{,\mu}} + \frac{\partial \pounds}{\partial \Psi_{,\mu}}\Psi_{,\nu} - \pounds\delta^\mu_\nu = ug^{\mu\lambda}\left(\overline{\Psi}_{,\lambda}\Psi_{,\nu} + \overline{\Psi}_{,\nu}\Psi_{,\lambda}\right) + \frac{vi}{2}\left(\overline{\Psi}\gamma^\mu\Psi_{,\nu} - \overline{\Psi}_{,\nu}\gamma^\mu\Psi\right), \tag{10}$$

$$j^\mu = i\left(\overline{\Psi}\frac{\partial \pounds}{\partial \overline{\Psi}_{,\mu}} + \frac{\partial \pounds}{\partial \Psi_{,\mu}}\Psi\right) = uig^{\mu\lambda}\left(\overline{\Psi}\Psi_{,\lambda} - \overline{\Psi}_{,\lambda}\Psi\right) + v\overline{\Psi}\gamma^\mu\Psi_{,\nu}\,. \tag{11}$$

The term $\pounds\delta^\mu_\nu$ may be reduced with the help of KGD equation to the 4-divergence, but the density of Lagrangian is determined only to the accuracy of the 4-divergence of space coordinates and time function, so we may neglect this term. This is in accordance with the fact, that the density of Dirac Lagrangian on the solutions of Dirac equation equals 0 [4, 5]. Then we may receive the forms of Hamiltonian, the 3-momentum and total charge from (10–11):

$$H = \int t_{00} d^3x = \int \left\{ 2u\overline{\Psi}_{,0}\Psi_{,0} + \frac{vi}{2}\left(\Psi^+\Psi_{,0} - \Psi^+_{,0}\Psi\right) \right\} d^3x\,, \tag{12}$$

$$P_i = \int t_{0i} d^3x = \int \left\{ u\left(\overline{\Psi}_{,0}\Psi_{,i} + \overline{\Psi}_{,i}\Psi_{,0}\right) + \frac{vi}{2}\left(\Psi^+\Psi_{,i} - \Psi^+_{,i}\Psi\right) \right\} d^3x\,, \quad i = \overline{1,3}\,, \tag{13}$$

$$Q = \int j^0 d^3x = \int \left\{ ui\left(\overline{\Psi}\Psi_{,0} - \overline{\Psi}_{,0}\Psi\right) + v\Psi^+\Psi \right\} d^3x\,. \tag{14}$$

From the form of total charge $Q(\psi) = \langle \psi \mid \psi \rangle$ we may write the rule of the scalar product of two different wave functions $\psi(x)$ and $\varphi(x)$,

$$\begin{aligned} \langle \psi \mid \varphi \rangle &= \frac{1}{4}\left[Q(\psi+\varphi) - Q(\psi-\varphi) - iQ(\psi+i\varphi) + iQ(\psi-i\varphi)\right] \\ &= ui\int \left(\overline{\psi}\varphi_{,0} - \overline{\psi}_{,0}\varphi\right) + v\int \psi^+\varphi d^3x = u\langle \psi \mid \varphi \rangle_{KG} + v\langle \psi \mid \varphi \rangle_D\,, \end{aligned} \tag{15}$$

which is the superposition of Klein-Gordon and Dirac scalar products.

4. Canonical formalism

Now if we define the field momenta π_Ψ and $\pi_{\overline{\Psi}}$ in accordance with Hamilton formalism as:

$$\pi_\Psi = \frac{\partial \pounds}{\partial \Psi_{,0}} = u\overline{\Psi}_{,0} + \frac{vi}{2}\Psi^+, \qquad \pi_{\overline{\Psi}} = \frac{\partial \pounds}{\partial \overline{\Psi}_{,0}} = u\Psi_{,0} - \frac{vi}{2}\gamma^0\Psi\,,$$

we may find the same form of Hamiltonian as in (12) with the help of the formula

$$H = \int \left\{ \pi_\Psi \Psi_{,0} + \overline{\Psi}_{,0}\pi_{\overline{\Psi}} - \pounds \right\} d^3x\,.$$

After the substitution of Ψ and $\overline{\Psi}$ in (12), the 3-momentum (13) and total charge (14) by their expansion in plane harmonic waves, we obtain them as expressed by the terms of the amplitudes $a^{s,m}_{\boldsymbol{p}}$, $a^{+s,m}_{\boldsymbol{p}}$ and $b^{s,m}_{\boldsymbol{p}}$, $b^{+s,m}_{\boldsymbol{p}}$:

$$\begin{aligned} H = &\int \frac{d^3p}{(2\pi)^3} \sum_s \left\{ F\left(a^{+s,m}_{\boldsymbol{p}} a^{s,m}_{\boldsymbol{p}} - b^{s,m}_{\boldsymbol{p}} b^{+s,m}_{\boldsymbol{p}}\right) + G(M,M) + I\left[(m,M) + (M,m)\right] \right\} \\ + &\int \frac{d^3p}{(2\pi)^3} \sum_{r,s} p^A \sigma^{rs}_A \left\{ J\left(a^{+r,m}_{\boldsymbol{p}} b^{+s,m}_{-\boldsymbol{p}} - b^{r,m}_{\boldsymbol{p}} a^{s,m}_{-\boldsymbol{p}}\right) + K(M,M) + L\left[(m,M) + (M,m)\right] \right\}, \end{aligned} \tag{16}$$

$$\begin{aligned}\boldsymbol{P} = \int \frac{d^3p}{(2\pi)^3}\boldsymbol{p}\sum_s\{N\left(a_{\boldsymbol{p}}^{+s,m}a_{\boldsymbol{p}}^{s,m} - b_{\boldsymbol{p}}^{s,m}b_{\boldsymbol{p}}^{+s,m}\right) + O(M,M) + R\left[(m,M)+(M,m)\right]\}\\ + \int \frac{d^3p}{(2\pi)^3}\sum_{r,s} S(a_{\boldsymbol{p}}^{+r,m}b_{-\boldsymbol{p}}^{+s,M} - b_{\boldsymbol{p}}^{r,m}a_{-\boldsymbol{p}}^{s,M} - a_{\boldsymbol{p}}^{+r,M}b_{-\boldsymbol{p}}^{+s,m} + b_{\boldsymbol{p}}^{r,M}a_{-\boldsymbol{p}}^{s,m}),\end{aligned} \tag{17}$$

$$\begin{aligned}Q = \int \frac{d^3p}{(2\pi)^3}\sum_s\{T\left(a_{\boldsymbol{p}}^{+s,m}a_{\boldsymbol{p}}^{s,m} + b_{\boldsymbol{p}}^{s,m}b_{\boldsymbol{p}}^{+s,m}\right) + V(M,M) + U\left[(m,M)+(M,m)\right]\}\\ + \int \frac{d^3p}{(2\pi)^3}\sum_{r,s} W(a_{\boldsymbol{p}}^{+r,m}b_{-\boldsymbol{p}}^{+s,M} + b_{\boldsymbol{p}}^{r,m}a_{-\boldsymbol{p}}^{s,M} - a_{\boldsymbol{p}}^{+r,M}b_{-\boldsymbol{p}}^{+s,m} - b_{\boldsymbol{p}}^{r,M}a_{-\boldsymbol{p}}^{s,m}),\end{aligned} \tag{18}$$

where $(\bullet,\bullet)$ means the same term as the previous one but with different masses and the coefficients are as follows:

$$F = m(2mu+v)\,, \qquad G = M(2Mu+v)\,, \qquad I = mM\left(2Au + \frac{Cv}{2}\frac{E_{\boldsymbol{p}}^m + E_{\boldsymbol{p}}^M}{E_{\boldsymbol{p}}^m E_{\boldsymbol{p}}^M}\right),$$

$$J = 2mu\,, \qquad K = 2Mu\,, \qquad L = mM\left(2Du + \frac{Bv}{2}\frac{E_{\boldsymbol{p}}^m - E_{\boldsymbol{p}}^M}{E_{\boldsymbol{p}}^m E_{\boldsymbol{p}}^M}\right),$$

$$N = \frac{m\,(2mu-v)}{E_{\boldsymbol{p}}^m}\,, \qquad O = \frac{M\,(2Mu-v)}{E_{\boldsymbol{p}}^M}\,,$$

$$R = \frac{mM\left(Au\left[E_{\boldsymbol{p}}^m + E_{\boldsymbol{p}}^M\right] - Cv\right)}{E_{\boldsymbol{p}}^m E_{\boldsymbol{p}}^M}\,, \qquad S = \frac{mM\left(Du\left[E_{\boldsymbol{p}}^m - E_{\boldsymbol{p}}^M\right] - Bv\right)}{E_{\boldsymbol{p}}^m E_{\boldsymbol{p}}^M}\,,$$

$$T = \frac{m\,(2mu+v)}{E_{\boldsymbol{p}}^m}\,, \qquad V = \frac{M\,(2Mu+v)}{E_{\boldsymbol{p}}^M}\,,$$

$$U = \frac{mM\left(Au\left[E_{\boldsymbol{p}}^m + E_{\boldsymbol{p}}^M\right] + Cv\right)}{E_{\boldsymbol{p}}^m E_{\boldsymbol{p}}^M}\,, \qquad W = \frac{mM\left(Du\left[E_{\boldsymbol{p}}^m - E_{\boldsymbol{p}}^M\right] + Bv\right)}{E_{\boldsymbol{p}}^m E_{\boldsymbol{p}}^M}\,.$$

In the case of equal masses $m = M = \frac{|w|}{|u|}$ when u, v, w have special values, i.e. $v^2 + 4uw = 0$, we may obtain the limit values for Hamiltonian, 3-momentum and total charge:

$$\begin{aligned}H &= 4m\,(2mu+v)\int \frac{d^3p}{(2\pi)^3}\sum_s\left(a_{\boldsymbol{p}}^{+s,m}a_{\boldsymbol{p}}^{s,m} - b_{\boldsymbol{p}}^{s,m}b_{\boldsymbol{p}}^{+s,m}\right)\\ &\quad + 8mu\int \frac{d^3p}{(2\pi)^3}\sum_{r,s} p^A\sigma_A^{rs}\left(a_{\boldsymbol{p}}^{+r,m}b_{-\boldsymbol{p}}^{+s,m} - b_{\boldsymbol{p}}^{r,m}a_{-\boldsymbol{p}}^{s,m}\right),\end{aligned} \tag{19}$$

$$\boldsymbol{P} = 4m\,(2mu-v)\int \frac{d^3p}{(2\pi)^3}\frac{\boldsymbol{p}}{E_{\boldsymbol{p}}^m}\sum_s\left(a_{\boldsymbol{p}}^{+s,m}a_{\boldsymbol{p}}^{s,m} + b_{\boldsymbol{p}}^{s,m}b_{\boldsymbol{p}}^{+s,m}\right), \tag{20}$$

$$Q = 4m\,(2mu+v)\int \frac{d^3p}{(2\pi)^3\,E_{\boldsymbol{p}}^m}\sum_s\left(a_{\boldsymbol{p}}^{+s,m}a_{\boldsymbol{p}}^{s,m} + b_{\boldsymbol{p}}^{s,m}b_{\boldsymbol{p}}^{+s,m}\right). \tag{21}$$

We may consider another limit case when $w = 0$, and therefore, $m = 0$, $M = \frac{|v|}{|u|}$. In this case all coefficients in (16–18), except of G, K, O, V, are equal to 0 and then the Hamiltonian, the 3-momentum and the total charge contain only terms which describe the states with mass M, and have the same form as (20–21), but with the following substitutions of coefficients before the integrals:

$$4m(2mu+v) \to 4M(2Mu+v)\,, \qquad 8mu \to 8Mu\,, \qquad 4m(2mu-v) \to 4M(2Mu-v)\,.$$

5. Conclusions

After the quantization we may consider $a_{\boldsymbol{p}}^{s,\bullet}$, $a_{\boldsymbol{p}}^{+s,\bullet}$ and $b_{\boldsymbol{p}}^{s,\bullet}$, $b_{\boldsymbol{p}}^{+s,\bullet}$ as operators of creation and annihilation of a particle with the m or M, 3-momentum $\boldsymbol{p}$ and spin s: $\widehat{a}_{\boldsymbol{p}}^{s,\bullet}$, $\widehat{a}_{\boldsymbol{p}}^{+s,\bullet}$ and $\widehat{b}_{\boldsymbol{p}}^{s,\bullet}$, $\widehat{b}_{\boldsymbol{p}}^{+s,\bullet}$. We may find the commutation laws for these operators from the form of Hamilton operator, the 3-momentum operator and the total charge operator (as in [4]), which are obtained from (16–18) by substitution $a_{\boldsymbol{p}}^{s,\bullet}$, $a_{\boldsymbol{p}}^{+s,\bullet}$ ($b_{\boldsymbol{p}}^{s,\bullet}$, $b_{\boldsymbol{p}}^{+s,\bullet}$) with operators $\widehat{a}_{\boldsymbol{p}}^{s,\bullet}$, $\widehat{a}_{\boldsymbol{p}}^{+s,\bullet}$ ($\widehat{b}_{\boldsymbol{p}}^{s,\bullet}$, $\widehat{b}_{\boldsymbol{p}}^{+s,\bullet}$). The eigenvalues of operators $\widehat{a}_{\boldsymbol{p}}^{+s,\bullet}\widehat{a}_{\boldsymbol{p}}^{s,\bullet}$ and $\widehat{b}_{\boldsymbol{p}}^{+s,\bullet}\widehat{b}_{\boldsymbol{p}}^{s,\bullet}$ are equal to the positive numbers $N_{\boldsymbol{p}}^{s,\bullet}$ and $\overline{N}_{\boldsymbol{p}}^{s,\bullet}$, which are the numbers of particles and anti-particles with mass m or M, 3-momentum $\boldsymbol{p}$ and spin s. From the condition of positivity of the energy (the eigenvalue of Hamilton operator) and the conservation law of total charge (18) we may receive anti-commutation laws for the following operators (the second set with M instead of m):

$$\{\widehat{a}_{\boldsymbol{p}}^{r,m},\ \widehat{a}_{\boldsymbol{p}}^{+s,m}\} = \delta^{rs}, \qquad \{\widehat{b}_{\boldsymbol{p}}^{s,m},\ \widehat{b}_{\boldsymbol{p}}^{+s,m}\} = \delta^{rs}. \tag{22}$$

This means that we may consider the particles described by KGD wave function (6) as fermions. There arises the question: "Why our wave function does not describe any bosons in spite of the fact that our KGD equation contains Klein-Gordon term?" One of possible answers may be that the wave function (6) is not complete because we have obtained our equation (3) with the essential restriction $v \neq 0$, which means that any proper passage from (1) to Klein-Gordon Lagrangian is impossible. We hope to complete this theory in future papers.

In spite of dealing with the superposition of d'Alembert and Dirac operators, our model has nothing to do with the supersymmetric mixing of spinors and bosons. It is based on the $U(2,2)$-gauge formulation of gravitation. The last programme is a modification of the Poincaré-gauge theory of gravitation, simply the Poincaré group (or rather its $SL(2,\mathbb{C}) \times \mathbb{R}^4$-covering) is replaced by the $SU(2,2)$-covering of the conformal group. Similarity to the Seiberg-Witten model is superficial.

Acknowledgements

The second author is very grateful to Professor Jan J. Sławianowski for inspiration and encouragement in the work and is also thankful to the Institute of Physics, Nicholas Copernicus University, Kronenberg Foundation and the Institute of Fundamental Technological Research for the financial support of his participation in the 33-rd Symposium on Mathematical Physics.

REFERENCES

[1] J. J. Sławianowski: *Rep. Math. Phys.* **38**, 375–397 (1996).

[2] J. J. Sławianowski: *Rep. Math. Phys.* **46**, 93–104 (2000).

[3] J. Rayski: *Bulletin de L'Académie Polonaise des Sciences, Série des sciences physiques et astron.* **27**, 27–32 (1979).

[4] V. B. Berestetskii, E. M. Lifshytz and L. P. Pitaevskii: *Quantum Electrodynamics*, in Course of Theoretical Physics, vol. IV, Moscow, Nauka 1989.

[5] N. N. Bogolubov and D. V. Shyrkov: *Introduction to Theory of Quantizated Fields*, Moscow, Nauka 1984.