Weighted Residual Methods

Introductory Course on Multiphysics Modelling

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Outline

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Boundary-Value Problem

Let $\mathcal{B}$ be a domain with the boundary $\partial\mathcal{B}$, and:

- $\mathcal{L}(.)$ be a (second order) differential operator,
- $f = f(\boldsymbol{x})$ be a known source term in $\mathcal{B}$,
- $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x})$ be the unit vector normal to the boundary $\partial\mathcal{B}$.

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Boundary-Value Problem

Find $u = u(\boldsymbol{x}) = ?$ satisfying **PDE**

$$\mathcal{L}(u) = f \quad \text{in } \mathcal{B}$$

and subject to (at least one of) the following **boundary conditions**

$$u = \hat{u} \text{ on } \partial\mathcal{B}_1\,, \qquad \frac{\partial u}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} = \hat{\gamma} \text{ on } \partial\mathcal{B}_2\,, \qquad \frac{\partial u}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} + \hat{\alpha}\, u = \hat{\beta} \text{ on } \partial\mathcal{B}_3\,,$$

where $\hat{u} = \hat{u}(\boldsymbol{x})$, $\hat{\gamma} = \hat{\gamma}(\boldsymbol{x})$, $\hat{\alpha} = \hat{\alpha}(\boldsymbol{x})$, and $\hat{\beta} = \hat{\beta}(\boldsymbol{x})$ are known fields prescribed on adequate parts of the boundary $\partial\mathcal{B} = \partial\mathcal{B}_1 \cup \partial\mathcal{B}_2 \cup \partial\mathcal{B}_3$

Remarks:

- the boundary parts are mutually disjoint,
- for $f \equiv 0$ the PDE is called *homogeneous*.

Types of boundary conditions

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3. the third kind or **Robin** b.c.:

$$\frac{\partial u}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} + \hat{\alpha}\, u = \hat{\beta} \quad \text{on } \partial\mathcal{B}_3 \,,$$

   also known as the **generalized Neumann** b.c., it can be presented as

$$\frac{\partial u}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} = \hat{\gamma} + \hat{\alpha}\big(\hat{u} - u\big) \quad \text{on } \partial\mathcal{B}_3 \,.$$

   Indeed, this form is obtained for $\hat{\beta} = \hat{\gamma} + \hat{\alpha}\,\hat{u}$.

Outline

General idea of the method

Weighted Residual Method (WRM) assumes that a solution can be approximated analytically or piecewise analytically. In general,

- a solution to a PDE can be expressed as **a linear combination of a base set of functions** where the coefficients are determined by a chosen method, and
- the method attempts to **minimize the approximation error**.

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- the method attempts to **minimize the approximation error**.

In fact, WRM represents a particular group of methods where **an integral error is minimized** in a certain way. Depending on this way the WRM can generate:

- the finite volume method,
- finite element methods,
- spectral methods,
- finite difference methods.

Approximation

Assumption: the exact solution, u, can be approximated by a linear combination of N (linearly-independent) analytical functions, that is,

$$u(\boldsymbol{x}) \approx \tilde{u}(\boldsymbol{x}) = \sum_{s=1}^{N} U_s \, \phi_s(\boldsymbol{x})$$

Here: $\tilde{u}$ is an approximated solution, and

- U_s are unknown coefficients, the so-called **degrees of freedom**,
- $\phi_s = \phi_s(\boldsymbol{x})$ form a base set of selected functions (often called as **trial functions** or **shape functions**). This set of functions generates the space of approximated solutions.
- $s = 1, \ldots N$ where N is the number of degrees of freedom.

Residua or error functions

In general, an approximated solution, $\tilde{u}$, does not satisfy exactly the PDE and/or some (or all) boundary conditions. The generated errors can be described by the following **error functions**:

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$$\mathcal{R}_0(\tilde{u}) = \mathcal{L}(\tilde{u}) - f\,,$$

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$$\mathcal{R}_2(\tilde{u}) = \frac{\partial \tilde{u}}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} - \hat{\gamma}\,,$$

3. the **Robin condition residuum**

$$\mathcal{R}_3(\tilde{u}) = \frac{\partial \tilde{u}}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} + \hat{\alpha}\tilde{u} - \hat{\beta}\,.$$

Minimization of errors

Requirement: Minimize the errors in a weighted integral sense

$$\int_{\mathcal{B}} \mathcal{R}_0(\tilde{u})\,\overset{0}{\psi}_r + \int_{\partial\mathcal{B}_1} \mathcal{R}_1(\tilde{u})\,\overset{1}{\psi}_r + \int_{\partial\mathcal{B}_2} \mathcal{R}_2(\tilde{u})\,\overset{2}{\psi}_r + \int_{\partial\mathcal{B}_3} \mathcal{R}_3(\tilde{u})\,\overset{3}{\psi}_r = 0\,.$$

Here, $\{\overset{0}{\psi}_r\}$, $\{\overset{1}{\psi}_r\}$, $\{\overset{2}{\psi}_r\}$, and $\{\overset{3}{\psi}_r\}$ ($r = 1, \ldots M$) are sets of **weight functions**.

Note that M weight functions yield M conditions (or equations) from which to determine the N coefficients U_s. To determine these N coefficients uniquely we need N independent conditions (equations).

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Note that M weight functions yield M conditions (or equations) from which to determine the N coefficients U_s. To determine these N coefficients uniquely we need N independent conditions (equations). Now, using the formulae for residua results in

$$\int_{\mathcal{B}} \mathcal{L}(\tilde{u}) \overset{0}{\psi}_r + \int_{\partial\mathcal{B}_1} \tilde{u} \overset{1}{\psi}_r + \int_{\partial\mathcal{B}_2} \frac{\partial \tilde{u}}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} \overset{2}{\psi}_r + \int_{\partial\mathcal{B}_3} \Big(\frac{\partial \tilde{u}}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} + \hat{\alpha}\, \tilde{u} \Big) \overset{3}{\psi}_r$$

$$= \int_{\mathcal{B}} f \overset{0}{\psi}_r + \int_{\partial\mathcal{B}_1} \hat{u} \overset{1}{\psi}_r + \int_{\partial\mathcal{B}_2} \hat{\gamma} \overset{2}{\psi}_r + \int_{\partial\mathcal{B}_3} \hat{\beta} \overset{3}{\psi}_r\,.$$

System of algebraic equations

By applying the approximation $\tilde{u} = \sum_{s=1}^{N} U_s \phi_s$, and using the properties of *linear* operators,

$$\mathcal{L}(\tilde{u}) = \sum_{s=1}^{N} U_s \, \mathcal{L}(\phi_s) \,, \qquad \frac{\partial \tilde{u}}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} = \sum_{s=1}^{N} U_s \, \frac{\partial \phi_s}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} \,,$$

the following **system of algebraic equations** is obtained:

$$\boxed{\sum_{s=1}^{N} A_{rs} \, U_s = B_r}$$

$$A_{rs} = \int\limits_{\mathcal{B}} \mathcal{L}(\phi_s) \, \overset{0}{\psi}_r + \int\limits_{\partial\mathcal{B}_1} \phi_s \, \overset{1}{\psi}_r + \int\limits_{\partial\mathcal{B}_2} \frac{\partial \phi_s}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} \, \overset{2}{\psi}_r + \int\limits_{\partial\mathcal{B}_3} \left(\frac{\partial \phi_s}{\partial \boldsymbol{x}} \cdot \boldsymbol{n} + \hat{\alpha} \, \phi_s \right) \overset{3}{\psi}_r \,,$$

$$B_r = \int\limits_{\mathcal{B}} f \, \overset{0}{\psi}_r + \int\limits_{\partial\mathcal{B}_1} \hat{u} \, \overset{1}{\psi}_r + \int\limits_{\partial\mathcal{B}_2} \hat{\gamma} \, \overset{2}{\psi}_r + \int\limits_{\partial\mathcal{B}_3} \hat{\beta} \, \overset{3}{\psi}_r \,.$$

Categories of WRM generated by weight functions

There are **four main categories** of weight functions which generate the following categories of WRM:

Subdomain method.

Collocation method.

Least squares method.

Galerkin method.

Categories of WRM generated by weight functions

***Main categories*:**

Subdomain method. Here the domain is divided in M subdomains $\Delta\mathcal{B}_r$ where

$$\overset{0}{\psi}_r(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in \Delta\mathcal{B}_r, \\ 0 & \text{outside}, \end{cases}$$

such that this method minimizes the residual error in each of the chosen subdomains. Note that the choice of the subdomains is free. In many cases an equal division of the total domain is likely the best choice. However, if higher resolution (and a corresponding smaller error) in a particular area is desired, a non-uniform choice may be more appropriate.

Collocation method.

Least squares method.

Galerkin method.

Categories of WRM generated by weight functions

***Main categories*:**

Subdomain method.

Collocation method. In this method the weight functions are chosen to be Dirac delta functions

$$\overset{0}{\psi}_r(\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{x}_r).$$

such that the error is zero at the chosen nodes $\boldsymbol{x}_r$.

Least squares method.

Galerkin method.

Categories of WRM generated by weight functions

***Main categories*:**

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Least squares method. This method uses derivatives of the residual itself as weight functions in the form

$$\overset{0}{\psi}_r(\boldsymbol{x}) = \frac{\partial \mathcal{R}_0(\tilde{u}(\boldsymbol{x}))}{\partial U_r} .$$

The motivation for this choice is to minimize $\int_{\mathcal{B}} \mathcal{R}_0^2$ of the computational domain. Note that (if the boundary conditions are satisfied) this choice of the weight function implies

$$\frac{\partial}{\partial U_r}\left(\int\limits_{\mathcal{B}} \mathcal{R}_0^2\right) = 0$$

for all values of U_r.

Galerkin method.

Categories of WRM generated by weight functions

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Subdomain method.

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Galerkin method. In this method the weight functions are chosen to be identical to the base functions.

$$\overset{0}{\psi}_r(\boldsymbol{x}) = \phi_r(\boldsymbol{x})\,.$$

In particular, if the base function set is orthogonal (i.e., $\int_{\mathcal{B}} \phi_r\,\phi_s = 0$ if $r \neq s$), this choice of weight functions implies that the residual $\mathcal{R}_0$ is rendered orthogonal with the minimization condition

$$\int\limits_{\mathcal{B}} \mathcal{R}_0\,\overset{0}{\psi}_r = 0$$

for all base functions.

Outline

A simple BVP approached by WRM

Boundary Value Problem (for an ODE)

Find $u = u(x) = ?$ satisfying

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \frac{\mathrm{d}u}{\mathrm{d}x} = 0 \quad \text{in } \mathcal{B} = [a, b]\,,$$

subject to boundary conditions on $\partial\mathcal{B} = \partial\mathcal{B}_1 \cup \partial\mathcal{B}_2 = \{a\} \cup \{b\}$:

$$u\Big|_{x=a} = \hat{u} \quad \text{(Dirichlet)}\,, \qquad \frac{\mathrm{d}u}{\mathrm{d}x}\Bigg|_{x=b} = \hat{\gamma} \quad \text{(Neumann)}\,.$$

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WRM approach:

- Residua for an approximated solution $\tilde{u}$

$$\mathcal{R}_0(\tilde{u}) = \frac{\mathrm{d}^2 \tilde{u}}{\mathrm{d}x^2} - \frac{\mathrm{d}\tilde{u}}{\mathrm{d}x}\,, \qquad \mathcal{R}_1(\tilde{u}) = \tilde{u}\big|_{x=a} - \hat{u}\,, \qquad \mathcal{R}_2(\tilde{u}) = \frac{\mathrm{d}\tilde{u}}{\mathrm{d}x}\bigg|_{x=b} - \hat{\gamma}\,.$$

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- Minimization of weighted residual error

$$\int_a^b \left(\frac{\mathrm{d}^2 \tilde{u}}{\mathrm{d}x^2} - \frac{\mathrm{d}\tilde{u}}{\mathrm{d}x} \right) \overset{0}{\psi}_r + \Big[(\tilde{u} - \hat{u}) \overset{1}{\psi}_r \Big]_{x=a} + \left[\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}x} - \hat{\gamma} \right) \overset{2}{\psi}_r \right]_{x=b} = 0 \,.$$

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WRM approach:

- System of algebraic equations: $\displaystyle\sum_{s=1}^{N} A_{rs}\, U_s = B_r,$

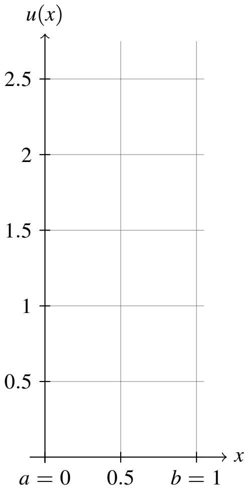
$$A_{rs} = \int_a^b \left(\frac{\mathrm{d}^2\phi_s}{\mathrm{d}x^2} - \frac{\mathrm{d}\phi_s}{\mathrm{d}x} \right) \overset{0}{\psi}_r + \left[\phi_s \overset{1}{\psi}_r \right]_{x=a} + \left[\frac{\mathrm{d}\phi_s}{\mathrm{d}x} \overset{2}{\psi}_r \right]_{x=b}\,,$$

$$B_r = \left[\hat{u} \overset{1}{\psi}_r \right]_{x=a} + \left[\hat{\gamma} \overset{2}{\psi}_r \right]_{x=b}\,.$$

Numerical (and exact) solution

- Boundary limits and values:

$$a = 0\,, \quad \hat{u} = 1\,, \qquad b = 1\,, \quad \hat{\gamma} = 2\,.$$



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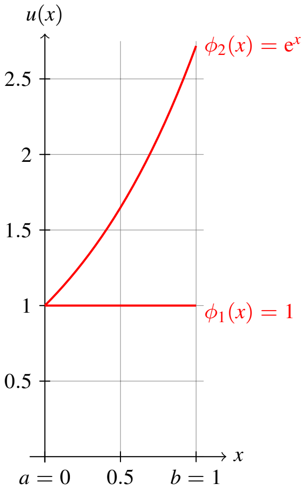

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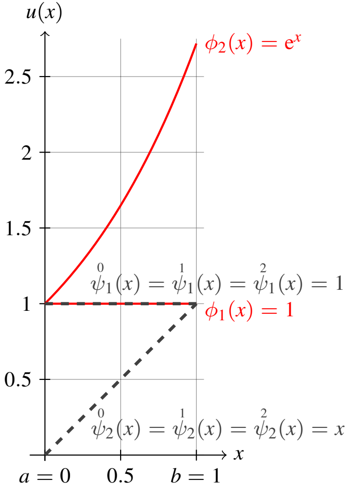

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- System of equations:

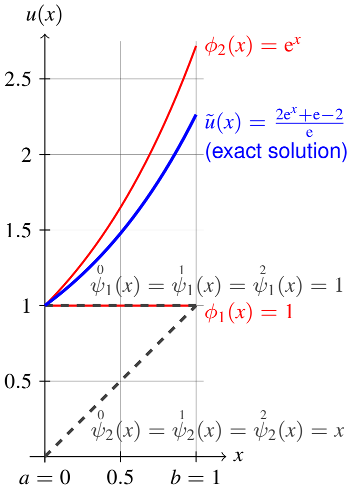
$$\begin{bmatrix} 1 & (1+\mathrm{e}) \\ 0 & \mathrm{e} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}\,.$$

- Coefficients:

$$U_1 = 1 - \frac{2}{\mathrm{e}}\,, \quad U_2 = \frac{2}{\mathrm{e}}\,.$$

- Approximated solution:

$$\tilde{u} = U_1 + U_2\,\mathrm{e}^x = \frac{2\mathrm{e}^x + \mathrm{e} - 2}{\mathrm{e}}\,.$$



Another numerical solution

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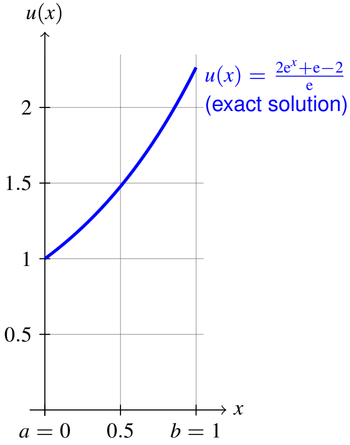

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$$a = 0\,, \quad \hat{u} = 1\,, \qquad b = 1\,, \quad \hat{\gamma} = 2\,.$$

- Shape and weight functions ($s = 1, 2, 3$):

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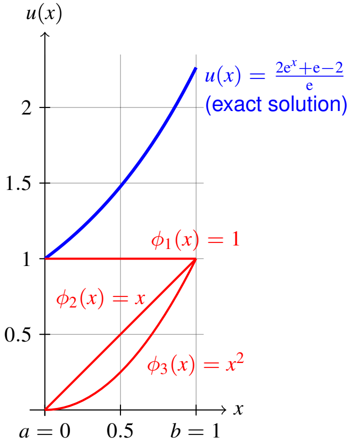

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- System of equations:

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & \frac{1}{2} & \frac{7}{3} \\ 0 & \frac{2}{3} & \frac{13}{6} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}.$$

- Coefficients:

$$U_1 = \frac{15}{17}\,, \quad U_2 = \frac{12}{17}\,, \quad U_3 = \frac{12}{17}\,.$$

- Approximated solution:

$$\tilde{u} = U_1 + U_2\,x + U_3\,x^2 = \frac{3}{17}\left(5 + 4x + 4x^2\right).$$