Fundamentals of Fluid Dynamics: Elementary Viscous Flow

Introductory Course on Multiphysics Modelling

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Outline

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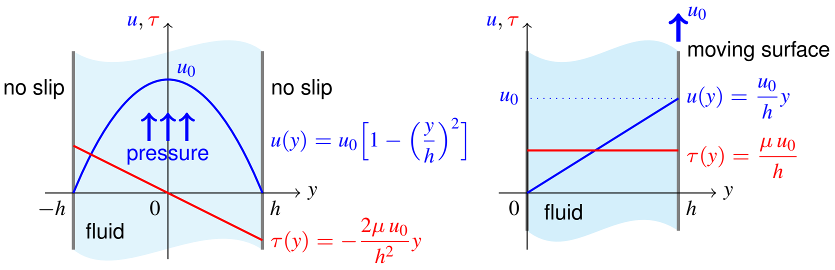
Newtonian fluids and viscosity

Definition (Newtonian fluid)

A **Newtonian fluid** is a viscous fluid for which the shear stress is proportional to the velocity gradient (i.e., to the rate of strain):

$$\tau = \mu \, \frac{\mathrm{d}u}{\mathrm{d}y} \; .$$

Here: τ [Pa] is the shear stress ("drag") exerted by the fluid,
μ $[\mathrm{Pa}\cdot\mathrm{s}]$ is the (**dynamic** or **absolute**) **viscosity**,
$\frac{\mathrm{d}u}{\mathrm{d}y}$ $\left[\frac{1}{\mathrm{s}}\right]$ is the velocity gradient perpendicular to the direction of shear.



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Definition (Kinematic viscosity)

The **kinematic viscosity** of a fluid is defined as the quotient of its absolute viscosity μ and density ϱ:

$$\nu = \frac{\mu}{\varrho} \qquad \left[\frac{\mathrm{m}^2}{\mathrm{s}}\right] \, .$$

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Definition (Kinematic viscosity)

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fluid	$\mu \ [10^{-5}\mathrm{Pa}\cdot\mathrm{s}]$	$\nu \ [10^{-5}\mathrm{m}^2/\mathrm{s}]$
air (at 20°C)	1.82	1.51
water (at 20°C)	100.2	0.1004

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Non-Newtonian fluids

For a non-Newtonian fluid the **viscosity changes with the applied strain rate** (velocity gradient). As a result, non-Newtonian fluids may not have a well-defined viscosity.

Constitutive relation for Newtonian fluids

The **stress tensor** can be decomposed into **spherical** and **deviatoric parts**:

$$\boldsymbol{\sigma} = \boldsymbol{\tau} - p\boldsymbol{I} \quad \text{or} \quad \sigma_{ij} = \tau_{ij} - p\,\delta_{ij}\,, \quad \text{where} \quad p = -\frac{1}{3}\,\mathrm{tr}\,\boldsymbol{\sigma} = -\frac{1}{3}\sigma_{ii}$$

is the (mechanical) **pressure** and $\boldsymbol{\tau}$ is the the **stress deviator** (**shear stress tensor**).

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is the (mechanical) **pressure** and $\boldsymbol{\tau}$ is the the **stress deviator** (**shear stress tensor**).

Using this decomposition Stokes (1845) deduced his constitutive relation for Newtonian fluids from three elementary hypotheses:

1. $\boldsymbol{\tau}$ should be **linear** function of the **velocity gradient**;
2. this relationship should be **isotropic**, as the physical properties of the fluid are assumed to show **no preferred direction**;
3. $\boldsymbol{\tau}$ should **vanish** if the flow involves **no deformation** of fluid elements.

Moreover, the **principle of conservation of moment of momentum** implies the **symmetry of stress tensor**: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}$, i.e., $\sigma_{ij} = \sigma_{ji}$. Therefore, the stress deviator $\boldsymbol{\tau}$ should also be symmetric: $\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathsf{T}}$, i.e., $\tau_{ij} = \tau_{ji}$ (since the spherical part is always symmetric).

Constitutive relation for Newtonian fluids

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3. $\boldsymbol{\tau}$ should **vanish** if the flow involves **no deformation** of fluid elements;
4. $\boldsymbol{\tau}$ is symmetric, i.e., $\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathsf{T}}$ or $\tau_{ij} = \tau_{ji}$.

Constitutive relation for Newtonian fluids

$$\boldsymbol{\sigma} = \underbrace{\mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{T}}\right)}_{\boldsymbol{\tau}\text{ for incompressible}} - p\,\boldsymbol{I} \quad \text{or} \quad \sigma_{ij} = \mu\left(u_{i|j} + u_{j|i}\right) - p\,\delta_{ij}\,.$$

This is a relation for incompressible fluid (i.e., when $\nabla\cdot\boldsymbol{u} = 0$).

Constitutive relation for compressible viscous flow

Definition (Rate of strain)

$$\dot{\boldsymbol{\varepsilon}} = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathsf{T}}\right)$$

The deviatoric (shear) and volumetric strain rates are given as $\left(\dot{\boldsymbol{\varepsilon}} - \frac{1}{3}(\operatorname{tr}\dot{\boldsymbol{\varepsilon}})\,\boldsymbol{I}\right)$ and $\operatorname{tr}\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}} \cdot \boldsymbol{I} = \nabla \cdot \boldsymbol{u}$, respectively.

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Newtonian fluids are characterized by a linear, **isotropic relation** between stresses and strain rates. That requires **two constants**:

- the **viscosity** μ – to relate the deviatoric (shear) stresses to the deviatoric (shear) strain rates:

$$\boldsymbol{\tau} = 2\mu\Big(\dot{\boldsymbol{\varepsilon}} - \frac{1}{3}(\operatorname{tr}\dot{\boldsymbol{\varepsilon}})\,\boldsymbol{I}\Big)\,,$$

- the so-called **volumetric viscosity** κ – to relate the mechanical pressure (the mean stress) to the volumetric strain rate:

$$p \equiv -\frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} = -\kappa\,\operatorname{tr}\dot{\boldsymbol{\varepsilon}} + p_0\,.$$

  Here, p_0 is the **initial hydrostatic pressure** independent of the strain rate.

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Volumetric viscosity

There is little evidence about the existence of volumetric viscosity and Stokes made the hypothesis that $\boxed{\kappa = 0}$. This is frequently used though it has not been definitely confirmed.

Constitutive relation for compressible viscous flow

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Here, p_0 is the **initial hydrostatic pressure**.

Constitutive relation for compressible Newtonian fluids

$$\boldsymbol{\sigma} = 2\mu\Big(\dot{\boldsymbol{\varepsilon}} - \tfrac{1}{3}(\operatorname{tr}\dot{\boldsymbol{\varepsilon}})\,\boldsymbol{I}\Big) - p\,\boldsymbol{I} = 2\mu\,\dot{\boldsymbol{\varepsilon}} - \Big(p + \tfrac{2}{3}\mu\,\operatorname{tr}\dot{\boldsymbol{\varepsilon}}\Big)\boldsymbol{I}\,,$$

and after using the definition for strain rate:

$$\boldsymbol{\sigma} = \mu\,\Big(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{T}}\Big) - \Big(p + \tfrac{2}{3}\mu\,\nabla\cdot\boldsymbol{u}\Big)\boldsymbol{I} \quad \text{or} \quad \sigma_{ij} = \mu\,\big(u_{i|j} + u_{j|i}\big) - \Big(p + \tfrac{2}{3}\mu\,u_{k|k}\Big)\,\delta_{ij}\,.$$

Outline

Continuity equation

Continuity (or mass conservation) equation

The balance of mass flow entering and leaving an infinitesimal control volume is equal to the rate of change in density:

$$\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\,\nabla\cdot\boldsymbol{u} = \frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\,\boldsymbol{u}) = 0\,.$$

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For **incompressible flows** the density does not change ($\varrho = \varrho_0$ where ϱ_0 is the constant initial density) so

$$\frac{\mathrm{D}\varrho}{\mathrm{D}t} = 0 \qquad \rightarrow \qquad \nabla\cdot\boldsymbol{u} = 0\,.$$

This last **kinematic constraint** for the velocity field is called the **incompressibility condition**.

Cauchy's equation of motion

The general **equation of motion** valid for any continuous medium is obtained from the **principle of conservation of linear momentum**:

$$\frac{\mathrm{D}}{\mathrm{D}t}\int\limits_{\mathcal{V}} \varrho\,\boldsymbol{u}\;\mathrm{d}\mathcal{V} = \int\limits_{\mathcal{V}} \boldsymbol{b}\;\mathrm{d}\mathcal{V} + \int\limits_{\mathcal{S}} \boldsymbol{t}\;\mathrm{d}\mathcal{S}$$

where $\boldsymbol{b}$ is the body (or volume) force, and $\boldsymbol{t}$ is the surface traction.

Cauchy's equation of motion

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{V}} \varrho \boldsymbol{u} \,\mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{b} \,\mathrm{d}\mathcal{V} + \int_{\mathcal{S}} \boldsymbol{t} \,\mathrm{d}\mathcal{S}$$

- Use the **Reynolds' transport theorem**

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{V}} f \,\mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \left(\frac{\mathrm{D}f}{\mathrm{D}t} + f\,\nabla\cdot\boldsymbol{u}\right) \mathrm{d}\mathcal{V}\,,$$

  and the **continuity equation**

$$\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\,\nabla\cdot\boldsymbol{u} = 0\,,$$

  for the inertial term:

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{V}} \varrho\boldsymbol{u} \,\mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \left[\frac{\mathrm{D}(\varrho\boldsymbol{u})}{\mathrm{D}t} + \varrho\boldsymbol{u}\,\nabla\cdot\boldsymbol{u}\right] \mathrm{d}\mathcal{V}$$

$$= \int_{\mathcal{V}} \left[\varrho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{u}\underbrace{\left(\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\,\nabla\cdot\boldsymbol{u}\right)}_{0}\right] \mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \varrho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\,\mathrm{d}\mathcal{V}\,.$$

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- Use the **Reynolds' transport theorem** and the **continuity equation** for the inertial term:

$$\frac{\mathrm{D}}{\mathrm{D}t}\int\limits_{\mathcal{V}} \varrho\,\boldsymbol{u}\;\mathrm{d}\mathcal{V} = \int\limits_{\mathcal{V}} \left[\frac{\mathrm{D}(\varrho\,\boldsymbol{u})}{\mathrm{D}t} + \varrho\,\boldsymbol{u}\,\nabla\cdot\boldsymbol{u}\right]\mathrm{d}\mathcal{V} = \int\limits_{\mathcal{V}} \varrho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\;\mathrm{d}\mathcal{V}\,.$$

- Apply the **Cauchy's formula**: $\boldsymbol{t} = \boldsymbol{\sigma}\cdot\boldsymbol{n}$, and the **divergence theorem** for the surface traction term:

$$\int\limits_{\mathcal{S}} \boldsymbol{t}\;\mathrm{d}\mathcal{S} = \int\limits_{\mathcal{S}} \boldsymbol{\sigma}\cdot\boldsymbol{n}\;\mathrm{d}\mathcal{S} = \int\limits_{\mathcal{V}} \nabla\cdot\boldsymbol{\sigma}\;\mathrm{d}\mathcal{V}\,.$$

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- Now, the **global (integral) form** of equation of motion is obtained:

$$\int\limits_{\mathcal{V}} \left(\varrho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} - \nabla \cdot \boldsymbol{\sigma} - \boldsymbol{b} \right) \mathrm{d}\mathcal{V} = 0 \, ,$$

which, being true for arbitrary $\mathcal{V}$ and provided that the integrand is continuous, yields the **local (differential) form**.

Cauchy's equation of motion

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{V}} \varrho \boldsymbol{u} \, \mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{b} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}} \boldsymbol{t} \, \mathrm{d}\mathcal{S}$$

- Use the **Reynolds' transport theorem** and the **continuity equation** for the inertial term:

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{V}} \varrho \boldsymbol{u} \, \mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \left[\frac{\mathrm{D}(\varrho \boldsymbol{u})}{\mathrm{D}t} + \varrho \boldsymbol{u} \, \nabla \cdot \boldsymbol{u}\right] \mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \varrho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \, \mathrm{d}\mathcal{V} \, .$$

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- Now, the global (integral) form of equation of motion is obtained which, being true for arbitrary $\mathcal{V}$ and provided that the integrand is continuous, yields the **local (differential) form** – the Cauchy's equation of motion.

Cauchy's equation of motion

$$\varrho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} \quad \text{or} \quad \varrho \frac{\mathrm{D}u_i}{\mathrm{D}t} = \sigma_{ij|j} + b_i$$

Navier–Stokes equations of motion

On applying the constitutive relations of Newtonian incompressible fluids to the Cauchy's equation of motion of continuous media, the so-called **incompressible Navier–Stokes equations** are obtained.

Incompressible Navier–Stokes equations

$$\varrho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \mu \, \triangle \boldsymbol{u} - \nabla p + \varrho_0 \, \boldsymbol{g} \quad \text{or} \quad \varrho_0 \frac{\mathrm{D}u_i}{\mathrm{D}t} = \mu \, u_{i|jj} - p_{|i} + \varrho_0 \, g_i \,,$$

$$\text{(+ the incompressibility constraint:)} \quad \nabla \cdot \boldsymbol{u} = 0 \quad \text{or} \quad u_{i|i} = 0 \,.$$

Here, the density is constant $\varrho = \varrho_0$, and the body force $\boldsymbol{b}$ has been substituted by the the gravitational force $\varrho_0 \, \boldsymbol{g}$, where $\boldsymbol{g}$ is the gravitaty acceleration. Now, on dividing by ϱ_0, using $\nu = \frac{\mu}{\varrho_0}$, and expanding the total-time derivative the main relations can be written as

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} = \nu \, \triangle \boldsymbol{u} - \frac{1}{\varrho_0} \nabla p + \boldsymbol{g} \quad \text{or} \quad \frac{\partial u_i}{\partial t} + u_j \, u_{i|j} = \nu \, u_{i|jj} - \frac{1}{\varrho_0} p_{|i} + g_i \,.$$

They differ from the Euler equations by virtue of the **viscous term**.

Boundary conditions (for incompressible flow)

- Let $\boldsymbol{n}$ be the unit normal vector to the boundary, and $\boldsymbol{m}^{(1)}$, $\boldsymbol{m}^{(2)}$ be two (non-parallel) unit tangential vectors.
- Let $\hat{\boldsymbol{u}}$, $\hat{u}_n$, $\hat{p}$ be values prescribed on the boundary, namely, the prescribed velocity vector, normal velocity, and pressure, respectively.

Inflow/Outflow velocity or No-slip condition:

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \quad (\hat{\boldsymbol{u}} = \boldsymbol{0} \text{ for the no-slip condition})\,.$$

Slip or Symmetry condition:

$$\begin{cases} \boldsymbol{u} \cdot \boldsymbol{n} = \hat{u}_n \quad (\hat{u}_n = 0 \text{ for the symmetry condition})\,, \\ (\boldsymbol{\sigma}\,\boldsymbol{n}) \cdot \boldsymbol{m}^{(1)} = 0\,, \quad (\boldsymbol{\sigma}\,\boldsymbol{n}) \cdot \boldsymbol{m}^{(2)} = 0 \\ \qquad\qquad (\text{or:} \quad (\boldsymbol{\tau}\,\boldsymbol{n}) \cdot \boldsymbol{m}^{(1)} = 0\,, \quad (\boldsymbol{\tau}\,\boldsymbol{n}) \cdot \boldsymbol{m}^{(2)} = 0)\,. \end{cases}$$

Pressure condition:

$$\boldsymbol{\sigma}\,\boldsymbol{n} = -\hat{p}\,\boldsymbol{n} \quad (\text{or:} \quad p = \hat{p}\,, \quad \boldsymbol{\tau}\,\boldsymbol{n} = \boldsymbol{0})\,.$$

Normal flow:

$$\begin{cases} \boldsymbol{u} \cdot \boldsymbol{m}^{(1)} = 0\,, \quad \boldsymbol{u} \cdot \boldsymbol{m}^{(2)} = 0\,, \\ (\boldsymbol{\sigma}\,\boldsymbol{n}) \cdot \boldsymbol{n} = -\hat{p} \quad (\text{or:} \quad p = \hat{p}\,, \quad (\boldsymbol{\tau}\,\boldsymbol{n}) \cdot \boldsymbol{n} = 0)\,. \end{cases}$$

Compressible Navier–Stokes equations of motion

On applying the constitutive relations of Newtonian compressible flow to the Cauchy's equation of motion, the **compressible Navier–Stokes equations of motion** are obtained.

Compressible Navier–Stokes equations of motion

$$\varrho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \mu \,\triangle \boldsymbol{u} + \frac{\mu}{3} \nabla (\nabla \cdot \boldsymbol{u}) - \nabla p + \varrho \,\boldsymbol{g} \quad \text{or} \quad \varrho \frac{\mathrm{D}u_i}{\mathrm{D}t} = \mu \, u_{i|jj} + \frac{\mu}{3} u_{j|ji} - p_{|i} + \varrho \, g_i$$

(+ the continuity equation:) $\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho \nabla \cdot \boldsymbol{u} = 0$ or $\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho \, u_{i|i} = 0\,.$

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(+ the continuity equation:) $$\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho \nabla \cdot \boldsymbol{u} = 0 \quad \text{or} \quad \frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\, u_{i|i} = 0\,.$$

- These equations are **incomplete** – there are only 4 relations for 5 unknown fields: ϱ, $\boldsymbol{u}$, p.
- They can be completed by a **state relationship between ϱ and p**.
- However, this would normally introduce also another state variable: the temperature T, and that would involve the requirement for energy balance (yet another equations). Such approach is governed by the **complete Navier–Stokes equations for compressible flow**.
- More simplified yet complete set of equations can be used to describe an **isothermal flow with small compressibility**.

Small-compressibility Navier–Stokes equations

***Assumptions*:**

1. The problem is **isothermal**.
2. The **variation of ϱ with p is very small**, such that *in* product terms of $\boldsymbol{u}$ and ϱ the latter can be assumed constant: $\varrho = \varrho_0$.

Small-compressibility Navier–Stokes equations

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1. The problem is **isothermal**.
2. The **variation of ϱ with p is very small**, such that *in* product terms of $\boldsymbol{u}$ and ϱ the latter can be assumed constant: $\varrho = \varrho_0$.

Small compressibility is allowed: density changes are, as a consequence of elastic deformability, related to pressure changes:

$$\mathrm{d}\varrho = \frac{\varrho_0}{K}\,\mathrm{d}p \quad \rightarrow \quad \frac{\partial \varrho}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} \quad \text{where} \;\; c = \sqrt{\frac{K}{\varrho_0}}$$

is the acoustic wave velocity, and K is the elastic bulk modulus. This relation can be used for the continuity equation yielding the following small-compressibility equation:

$$\frac{\partial p}{\partial t} = -\underbrace{c^2\varrho_0}_{K}\,\nabla\cdot\boldsymbol{u}\,,$$

where the density term standing by $\boldsymbol{u}$ has been assumed constant: $\varrho = \varrho_0$. This also applies now to the Navier-Stokes momentum equations of compressible flow ($\nu = \frac{\mu}{\varrho_0}$).

Small-compressibility Navier–Stokes equations

***Assumptions*:**

1. The problem is **isothermal**.
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Small compressibility is allowed: density changes are, as a consequence of elastic deformability, related to pressure changes.

Navier–Stokes equations for nearly incompressible flow

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu \triangle \boldsymbol{u} + \frac{\nu}{3}\nabla(\nabla \cdot \boldsymbol{u}) - \frac{1}{\varrho_0}\nabla p + \boldsymbol{g}$$

or $$\frac{\partial u_i}{\partial t} + u_j\, u_{i|j} = \nu\, u_{i|jj} + \frac{\nu}{3} u_{j|ji} - \frac{1}{\varrho_0} p_{|i} + g_i \,,$$

(+ small-compressibility equation:) $\dfrac{\partial p}{\partial t} = -K\, \nabla \cdot \boldsymbol{u}$ or $\dfrac{\partial p}{\partial t} = -K\, u_{i|i}$.

- These are 4 equations for 4 unknown fields: $\boldsymbol{u}$, p.
- After solution the density can be computed as $\varrho = \varrho_0\left(1 + \frac{p - p_0}{K}\right)$.

Complete Navier–Stokes equations

Mass conservation: $\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\,\nabla\cdot\boldsymbol{u} = \frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\,\boldsymbol{u}) = 0.$

This is also called the *continuity equation*.

Complete Navier–Stokes equations

Mass conservation: $\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\,\nabla\cdot\boldsymbol{u} = \frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\,\boldsymbol{u}) = 0.$

Momentum conservation: $\varrho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla\cdot\boldsymbol{\sigma} + \varrho\,\boldsymbol{g}\,,$ (here: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}$).

These are 3 equations of motion (a.k.a. balance or equilibrium equations). The symmetry of stress tensor (additional 3 equations) results from the conservation of angular momentum.

Complete Navier–Stokes equations

Mass conservation: $\frac{D\varrho}{Dt} + \varrho \nabla \cdot \boldsymbol{u} = \frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \boldsymbol{u}) = 0.$

Momentum conservation: $\varrho \frac{D\boldsymbol{u}}{Dt} = \nabla \cdot \boldsymbol{\sigma} + \varrho \boldsymbol{g}\,,$ (here: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}$).

Energy conservation: $\frac{D}{Dt}\left(\varrho e + \frac{1}{2}\varrho \boldsymbol{u} \cdot \boldsymbol{u}\right) = -\nabla \cdot \boldsymbol{q} + \nabla \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{u}) + \varrho \boldsymbol{g} \cdot \boldsymbol{u} + h.$

Here: e is the *intrinsic energy* per unit mass, $\boldsymbol{q}$ is the *heat flux vector*, and h is the *power of heat source* per unit volume. Moreover, notice that the term $\frac{1}{2}\varrho \boldsymbol{u} \cdot \boldsymbol{u}$ is the *kinetic energy*, $\nabla \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{u})$ is the *energy change due to internal stresses*, and $\varrho \boldsymbol{g} \cdot \boldsymbol{u}$ is the change of *potential energy* of gravity forces.

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Equations of state and constitutive relations:

- **Thermal equation of state**: $\varrho = \varrho(p, T)$.

  For a perfect gas: $\varrho = \frac{p}{R\,T}$, where R is the *universal gas constant*.

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  For *Newtonian fluids*: $\boldsymbol{\tau} = \mu\big(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{T}}\big) - \frac{2}{3}\mu\,(\nabla\cdot\boldsymbol{u})\,\boldsymbol{I}\,.$

  Other relations may be used, for example: $\boldsymbol{\tau} = \boldsymbol{0}$ for an inviscid fluid, or some nonlinear relationships for non-Newtonian fluids.

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- **Thermodynamic relation** for state variables: $e = e(p, T)$.

  For a calorically perfect fluid: $e = c_V\, T$, where c_V is the *specific heat at constant volume*. This equation is sometimes called the *caloric equation of state*.

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  Fourier's law of thermal conduction with convection: $\boldsymbol{q} = -k\,\nabla T + \varrho\,c\,\boldsymbol{u}\,T$, where k is the *thermal conductivity* and c is the *thermal capacity* (the *specific heat*).

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- Using the constitutive and state relations for the conservation equations leaves only **5** equations in **5** unknowns: ϱ (or p), $\boldsymbol{u}$, T.

Complete Navier–Stokes equations

Boundary conditions for compressible flow

Density condition:

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Temperature or heat flux condition:

$$T = \hat{T} \quad \text{on } \mathcal{S}_T \,, \qquad \text{or} \qquad \boldsymbol{q} \cdot \boldsymbol{n} = \hat{q} \quad \text{on } \mathcal{S}_q \,, \qquad \text{(or mixed)},$$

where $\hat{T}$ is the temperature and $\hat{q}$ is the inward heat flux prescribed on the boundary.

Outline

Reynolds number

Definition (Reynolds number)

The **Reynolds number** is a dimensionless parameter defined as

$$Re = \frac{U\,L}{\nu}$$

where: U denotes a typical **flow speed**,
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The Reynolds number gives a rough indication of the **relative amplitudes of two key terms** in the equations of motion, namely,

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- Derivatives of the velocity components, such as $\frac{\partial u}{\partial x}$, will typically be of order U/L, that is, the components of $\boldsymbol{u}$ change by amounts of order U over distances of order L.

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- Typically, these derivatives of velocity will themselves change by amounts of order U/L over distances of order L so the second derivatives, such as $\frac{\partial^2 u}{\partial x^2}$, will be of order U/L^2.

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$$\frac{|\text{inertial term}|}{|\text{viscous term}|} = O\left(\frac{U^2/L}{\nu\,U/L^2}\right) = O(Re)\,.$$

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There are two extreme cases of viscous flow:

1. **High Reynolds number flow** – for $Re \gg 1$: a flow of a fluid of **small viscosity**, where **viscous effects** can be on the whole **negligible**.
2. **Low Reynolds number flow** – for $Re \ll 1$: a **very viscous flow**.

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1. **High Reynolds number flow** – for $Re \gg 1$: a flow of a fluid of **small viscosity**, where **viscous effects** can be on the whole **negligible**.
   - Even then, however, viscous effects become important in **thin boundary layers**, where the unusually large velocity gradients make the viscous term much larger than the estimate $\nu\,U/L^2$. The larger the Reynolds number, the thinner the boundary layer: $\delta/L = O(1/\sqrt{Re})$ (δ – typical thickness of boundary layer).
   - A large Reynolds number is *necessary* for **inviscid theory** to apply over most of the flow field, but it is not *sufficient*.
   - At high Reynolds number ($Re \sim 2000$) steady flows are often **unstable** to small disturbances, and may, as a result become **turbulent** (in fact, Re was first employed in this context).
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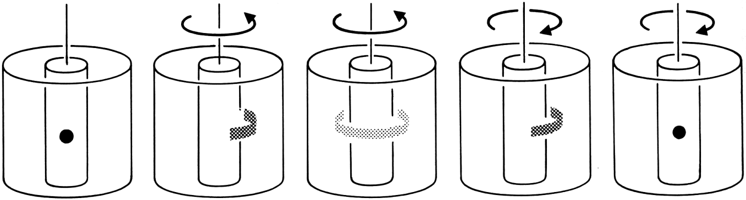
2. **Low Reynolds number flow** – for $Re \ll 1$: a **very viscous flow**.
   - There is **no turbulence** and the flow is extremely **ordered** and nearly **reversible** ($Re \sim 10^{-2}$).



Outline

Plane parallel shear flow

Plane parallel shear flow

$$\boldsymbol{u} = \boldsymbol{u}(y,t) = \big[u(y,t), 0, 0\big]$$

Such flow automatically satisfies the incompressibility condition: $\nabla \cdot \boldsymbol{u} = 0$, and in the absence of gravity the incompressibile Navier–Stokes equations of motion reduce to:

$$\frac{\partial u}{\partial t} = -\frac{1}{\varrho_0}\,\frac{\partial p}{\partial x} + \nu\,\frac{\partial^2 u}{\partial y^2}\,, \qquad \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0\,.$$

(The gravity can be ignored if it simply modifies the pressure distribution in the fluid and does nothing to change the velocity.)

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- The first equation implies that $\frac{\partial p}{\partial x}$ cannot depend on x, while the remaining two equations imply that $p = p(x,t)$; therefore, $\frac{\partial p}{\partial x}$ may only depend on t.
- There are important circumstances when the flow is *not* being driven by any externally applied pressure gradient, which permits to assert that the pressures at $x = \pm\infty$ are equal. All this means that $\frac{\partial p}{\partial x} = 0$.

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Diffusion equation for viscous incompressible flow

For a gravity-independent plane parallel shear flow, not driven by any externally applied pressure gradient, the velocity $u(y,t)$ must satisfy the **one-dimensional diffusion equation**:

$$\frac{\partial u}{\partial t} = \nu\,\frac{\partial^2 u}{\partial y^2}\;.$$

Viscous diffusion of vorticity

Example (The flow due to impulsively moved plane boundary)

- Viscous fluid lies at rest in the region:

  (Problem A) $0 < y < \infty,$ **(Problem B)** $0 < y < h$.

- At $t = 0$ the rigid boundary at $y = 0$ is suddenly jerked into motion in the x-direction with constant speed U.
- By virtue of the no-slip condition the fluid elements in contact with the boundary will immediately move with velocity U.

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▶ *Mathematical statement of the problem*

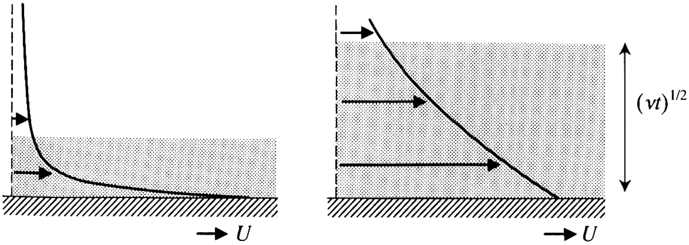
The flow velocity $u(y,t)$ must satisfy the one-dimensional **diffusion equation** $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$, together with the following conditions:

1. **initial condition:**
   - $u(y,0) = 0$ (for $y \geq 0$),
2. **boundary conditions:**
   - **(Problem A)** $u(0,t) = U$ and $u(\infty,t) = 0$ (for $t \geq 0$),
   - **(Problem B)** $u(0,t) = U$ and $u(h,t) = 0$ (for $t \geq 0$).

Viscous diffusion of vorticity

- *Solution to Problem A*:

$$u = U\left[1 - \frac{1}{\sqrt{\pi}} \int_0^{\eta} \exp\left(\frac{-s^2}{4}\right) \mathrm{d}s\right] \text{ with } \eta = \frac{y}{\sqrt{\nu\, t}}, \qquad \omega = -\frac{\partial u}{\partial y} = \frac{U}{\sqrt{\pi \nu\, t}} \exp\left(\frac{-y^2}{4\nu\, t}\right).$$

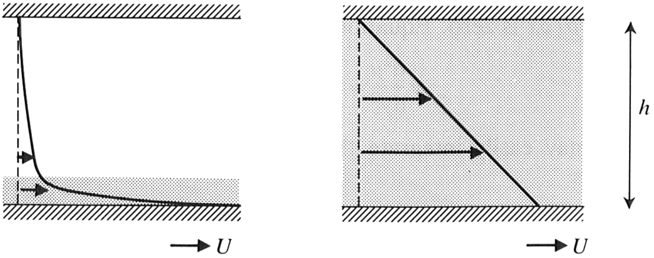


- The flow is largely confined to a distance of order $\sqrt{\nu\, t}$ from the moving boundary: the velocity and vorticity are very small beyond that region.
- **Vorticity diffuses** a distance of order $\sqrt{\nu\, t}$ in time t. Equivalently, the time taken for vosticity to diffuse a distance h is of the order $\frac{h^2}{\nu}$.

Viscous diffusion of vorticity

- *Solution to Problem B*:

$$u = \underbrace{U\left(1 - \frac{y}{h}\right)}_{\text{steady state}} - \frac{2U}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \exp\left(-n^2\pi^2 \frac{\nu t}{h^2}\right).$$



$t \ll \frac{h^2}{\nu}$ $\qquad$ $t \gtrsim \frac{h^2}{\nu}$

- For times greater than $\frac{h^2}{\nu}$ the flow has almost reached its steady state and the vorticity is almost distributed uniformly throughout the fluid.

Convection and diffusion of vorticity

Vorticity equation for viscous flows

In general:

$$\text{Incompress. Navier–Stokes} \quad \xrightarrow{\nabla\times} \quad \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} + \nu\triangle\boldsymbol{\omega}.$$

For a two-dimensional flow ($\boldsymbol{\omega} \perp \boldsymbol{u}$):

$$\frac{\partial \omega}{\partial t} + \underbrace{(\boldsymbol{u}\cdot\nabla)\omega}_{\text{convection}} = \underbrace{\nu\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right)}_{\text{diffusion}}.$$

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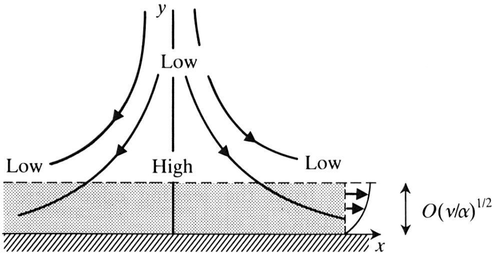
Observation: In general, there is both **convection and diffusion of vorticity** in a viscous flow.

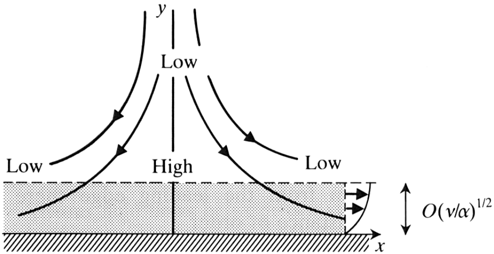
Convection and diffusion of vorticity



Example (Plane flow towards a stagnation point)

- There is an inviscid 'mainstream' flow: $u = \alpha x$, $v = -\alpha y$ (here, $\alpha > 0$ is a constant), towards a stagnation boundary at $y = 0$.
- This fails to satisfy the no-slip condition at the boundary, but the mainstream flow speed $\alpha|x|$ increases with distance $|x|$ along the boundary. By the Bernoulli's theorem, the mainstream pressure decreases with distance along the boundary in the flow direction.
- Thus, one may hope for a thin, unseparated boundary layer which adjusts the velocity to satisfy the no-slip condition.

Convection and diffusion of vorticity



Example (Plane flow towards a stagnation point)

- The boundary layer, in which all the vorticity in concentrated, has thickness of order $\sqrt{\frac{\nu}{\alpha}}$.
- In this boundary layer there is a steady state balance between the viscous diffusion of vorticity from the wall and the convection of vorticity towards the wall by the flow.
- If ν decreases the diffusive effect is weakened, while if α increases the convective effect is enhanced (in either case the boundary layer becomes thinner).

Boundary layers

- Steady flow past a fixed wing may seem to be wholly accounted for by inviscid theory. In particular, the **fluid in contact with the wing appears to slip** along the boundary.
- In fact, **there is no such slip**. Instead there is a very thin **boundary layer** where the inviscid theory fails and **viscous effects are very important**.

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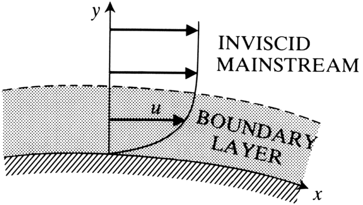
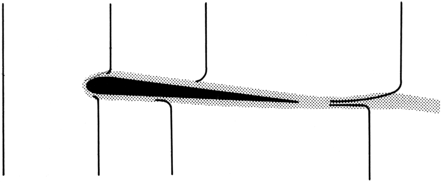
Boundary layer

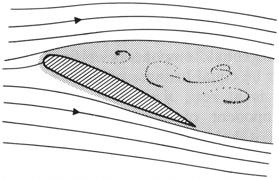
A **boundary layer** is a very thin layer along the boundary across which the flow velocity undergoes a smooth but rapid adjustment to precisely zero (i.e. *no-slip*) on the boundary itself.



Boundary layers

Layer separation

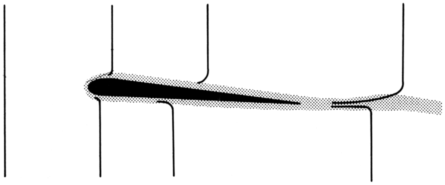


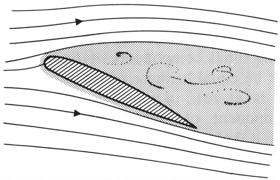


- In certain circumstances **boundary layers may separate** from the boundary, thus causing the whole flow of low-viscosity fluid to be quite different to that predicted by inviscid theory.

Boundary layers

Layer separation





- In certain circumstances **boundary layers may separate** from the boundary, thus causing the whole flow of low-viscosity fluid to be quite different to that predicted by inviscid theory.
- The behaviour of a **fluid of even very small viscosity** may, on account of boundary layer separation, be **completely different** to that of a (hypothetical) **fluid of no viscosity** at all.