Exercise 1.1

One dimensional wave equation (transverse vibrations of an **infinite** string)

Initial-value problem (IVP)

Find $u = u(x,t) = ?$ satisfying for $x \in (-\infty, +\infty)$ and $t \in [0, \infty)$:

$$\textbf{PDE:}\ u_{tt} = c^2\, u_{xx}\,, \qquad \textbf{ICs:}\ \begin{cases} u(x,0) = f(x)\,, \\ u_t(x,0) = g(x)\,. \end{cases}$$

(Here, c is the speed of wave propagation.)

Find the so-called *d'Alembert solution*. To this end:

- Replace x and t by new canonical coordinates: $\xi(x,t) = x + ct$ and $\eta(x,t) = x - ct$, to obtain a canonical form $u_{\xi\eta} = 0$.
- Solve the transformed equation (by two straightforward integrations).
- Transform back to the original coordinates to get a general solution.
- Substitute the general solution into the ICs.

Consider two cases of ICs:

1. $f(x) = 1$ for $x \in (-1, 1)$ and $f(x) = 0$ otherwise, and $g(x) \equiv 0$;
2. $f(x) \equiv 0$ (initial position at equilibrium) and $g(x) = \sin(x)$ (initial velocity as in a piano string).

Exercise 1.2

One dimensional wave equation (transverse vibrations of a **finite** string)

Initial-boundary-value problem (IBVP)

Find $u = u(x,t) =?$ satisfying for $x \in [0, L]$ and $t \in [0, \infty)$:

$$\textbf{PDE:}\ u_{tt} = c^2\, u_{xx}\,, \quad \textbf{BCs:}\ \begin{cases} u(0,t) = 0\,, \\ u(L,t) = 0\,, \end{cases} \quad \textbf{ICs:}\ \begin{cases} u(x,0) = f(x)\,, \\ u_t(x,0) = g(x)\,. \end{cases}$$

Hints and comments:

- The PDE as well as BCs are linear and homogeneous.
- The solution can be found by the standard technique of *separation of variables* as the infinite sum: $u(x,t) = \sum_{i=1}^{\infty} a_i\, X_i(x)\, T_i(t)$.
- $X_i(x)$ will be fundamental shapes of the so-called *standing waves*.

Consider two cases of ICs:

1. $f(x) = \sin(\pi\, x/L) + \frac{1}{2}\sin(3\pi\, x/L)$ and $g(x) \equiv 0$;
2. $f(x) \equiv 0$ and $g(x) = \sin(3\pi\, x/L)$.