

Mathematical Preliminaries

Introductory Course on Multiphysics Modelling

TOMASZ G. ZIELIŃSKI

`bluebox.ippt.pan.pl/~tzielins/`

**Institute of Fundamental Technological Research
of the Polish Academy of Sciences
Warsaw • Poland**



Outline

- 1 Vectors, tensors, and index notation**
 - Generalization of the concept of vector
 - Summation convention and index notation
 - Kronecker delta and permutation symbol
 - Tensors and their representations
 - Multiplication of vectors and tensors
 - Vertical-bar convention and Nabla operator

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- 2 Integral theorems**
 - General idea
 - Stokes' theorem
 - Gauss-Ostrogradsky theorem

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- 2 Integral theorems**
 - General idea
 - Stokes' theorem
 - Gauss-Ostrogradsky theorem
- 3 Time-harmonic approach**
 - Types of dynamic problems
 - Complex-valued notation
 - A practical example

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Vectors, tensors, and index notation

Generalization of the concept of vector

- A **vector** is a quantity that possesses both a **magnitude** and a **direction** and obeys certain laws (of **vector algebra**):
 - the vector addition and the commutative and associative laws,
 - the associative and distributive laws for the multiplication with scalars.
- The **vectors** are suited to **describe physical phenomena**, since they are **independent of any system of reference**.

The concept of a **vector** that is independent of any coordinate system **can be generalised** to higher-order quantities, which are called **tensors**. Consequently, vectors and scalars can be treated as lower-rank tensors.

Vectors, tensors, and index notation

Generalization of the concept of vector

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Scalars have a magnitude but no direction. They are tensors of order 0. *Example:* the mass density.

Vectors are characterised by their magnitude and direction. They are tensors of order 1. *Example:* the velocity vector.

Tensors of second order are quantities which multiplied by a vector give as the result another vector. *Example:* the stress tensor.

Higher-order tensors are often encountered in constitutive relations between second-order tensor quantities. *Example:* the fourth-order elasticity tensor.

Vectors, tensors, and index notation

Summation convention and index notation

Einstein's summation convention

A summation is carried out over repeated indices in an expression and the summation symbol is skipped.

Example

$$a_i b_i \equiv \sum_{i=1}^3 a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$A_{ii} \equiv \sum_{i=1}^3 A_{ii} = A_{11} + A_{22} + A_{33}$$

$$A_{ij} b_j \equiv \sum_{j=1}^3 A_{ij} b_j = A_{i1} b_1 + A_{i2} b_2 + A_{i3} b_3 \quad (i = 1, 2, 3) \quad [3 \text{ expressions}]$$

$$\begin{aligned} T_{ij} S_{ij} \equiv \sum_{i=1}^3 \sum_{j=1}^3 T_{ij} S_{ij} &= T_{11} S_{11} + T_{12} S_{12} + T_{13} S_{13} \\ &\quad + T_{21} S_{21} + T_{22} S_{22} + T_{23} S_{23} \\ &\quad + T_{31} S_{31} + T_{32} S_{32} + T_{33} S_{33} \end{aligned}$$

Vectors, tensors, and index notation

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The principles of index notation:

- **An index cannot appear more than twice in one term!**
If necessary, the standard summation symbol (\sum) must be used. A repeated index is called a **bound** or **dummy index**.

Example

$$A_{ii}, \quad C_{ijkl} S_{kl}, \quad A_{ij} b_i c_j \quad \leftarrow \text{Correct}$$

$$A_{ij} b_j c_j \quad \leftarrow \text{Wrong!}$$

$$\sum_j A_{ij} b_j c_j \quad \leftarrow \text{Correct}$$

A term with an index repeated more than two times is correct if:

- the summation sign is used: $\sum_i a_i b_i c_i = a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3$, or
- the dummy index is underlined: $a_{\underline{i}} b_{\underline{i}} c_{\underline{i}} = a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3$.

Vectors, tensors, and index notation

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If necessary, the standard summation symbol (\sum) must be used. A repeated index is called a **bound** or **dummy index**.
- If an index appears once, it is called a **free index**. The number of free indices determines the order of a tensor.

Example

$$\begin{array}{ll}
 A_{ii}, \quad a_i b_i, \quad T_{ij} S_{ij} & \leftarrow \text{scalars (no free indices)} \\
 A_{ij} b_j & \leftarrow \text{a vector (one free index: } i) \\
 C_{ijkl} S_{kl} & \leftarrow \text{a second-order tensor (two free indices: } i, j)
 \end{array}$$

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- The denomination of dummy index (in a term) is arbitrary, since it vanishes after summation, namely: $a_i b_i \equiv a_j b_j \equiv a_k b_k$, etc.

Example

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_j b_j$$

$$A_{ii} \equiv A_{jj}, \quad T_{ij} S_{ij} \equiv T_{kl} S_{kl}, \quad T_{ij} + C_{ijkl} S_{kl} \equiv T_{ij} + C_{ijmn} S_{mn}$$

Vectors, tensors, and index notation

Kronecker delta and permutation symbol

Definition (Kronecker delta)

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

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- The Kronecker delta can be used to substitute one index by another, for example: $a_i \delta_{ij} = a_1 \delta_{1j} + a_2 \delta_{2j} + a_3 \delta_{3j} = a_j$, i.e., here $i \rightarrow j$.
- When Cartesian coordinates are used (with orthonormal base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) the Kronecker delta δ_{ij} is the **(matrix) representation** of the **unity tensor** $\mathbf{I} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$.
- $\mathbf{A} \bullet \mathbf{I} = A_{ij} \delta_{ij} = A_{ii}$ which is the **trace** of the matrix (tensor) \mathbf{A} .

Vectors, tensors, and index notation

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Definition (Permutation symbol)

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for even permutations: } 123, 231, 312 \\ -1 & \text{for odd permutations: } 132, 321, 213 \\ 0 & \text{if an index is repeated} \end{cases}$$

Vectors, tensors, and index notation

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The permutation symbol (or tensor) is widely used in index notation to express the **vector** or **cross product** of two vectors:

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \rightarrow c_i = \epsilon_{ijk} a_j b_k \rightarrow \begin{cases} c_1 = a_2 b_3 - a_3 b_2 \\ c_2 = a_3 b_1 - a_1 b_3 \\ c_3 = a_1 b_2 - a_2 b_1 \end{cases}$$

Vectors, tensors, and index notation

Tensors and their representations

Informal definition of tensor

A **tensor** is a generalized **linear ‘quantity’** that can be expressed as a **multi-dimensional array** relative to a choice of basis of the particular space on which it is defined. Therefore:

- a tensor is independent of any chosen frame of reference,
- its representation behaves in a specific way under coordinate transformations.

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- its representation behaves in a specific way under coordinate transformations.

Cartesian system of reference

Let \mathcal{E}^3 be the three-dimensional **Euclidean space** with a **Cartesian coordinate system** with three **orthonormal base vectors** e_1, e_2, e_3 , so that

$$e_i \cdot e_j = \delta_{ij} \quad (i, j = 1, 2, 3).$$

Vectors, tensors, and index notation

Tensors and their representations

Cartesian system of reference

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$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad (i, j = 1, 2, 3).$$

- A **second-order tensor** $\mathbf{T} \in \mathcal{E}^3 \otimes \mathcal{E}^3$ is defined by

$$\begin{aligned} \mathbf{T} := T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j &= T_{11} \mathbf{e}_1 \otimes \mathbf{e}_1 + T_{12} \mathbf{e}_1 \otimes \mathbf{e}_2 + T_{13} \mathbf{e}_1 \otimes \mathbf{e}_3 \\ &+ T_{21} \mathbf{e}_2 \otimes \mathbf{e}_1 + T_{22} \mathbf{e}_2 \otimes \mathbf{e}_2 + T_{23} \mathbf{e}_2 \otimes \mathbf{e}_3 \\ &+ T_{31} \mathbf{e}_3 \otimes \mathbf{e}_1 + T_{32} \mathbf{e}_3 \otimes \mathbf{e}_2 + T_{33} \mathbf{e}_3 \otimes \mathbf{e}_3 \end{aligned}$$

where \otimes denotes the tensorial (or dyadic) product, and T_{ij} is the **(matrix) representation** of \mathbf{T} in the given frame of reference defined by the base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

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where \otimes denotes the tensorial (or dyadic) product, and T_{ij} is the **(matrix) representation** of \mathbf{T} in the given frame of reference defined by the base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

- The second-order tensor $\mathbf{T} \in \mathcal{E}^3 \otimes \mathcal{E}^3$ can be viewed as a **linear transformation** from \mathcal{E}^3 onto \mathcal{E}^3 , meaning that it transforms every vector $\mathbf{v} \in \mathcal{E}^3$ into another vector from \mathcal{E}^3 as follows

$$\begin{aligned} \mathbf{T} \cdot \mathbf{v} &= (T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j) \cdot (v_k \mathbf{e}_k) = T_{ij} v_k \overbrace{(\mathbf{e}_j \cdot \mathbf{e}_k)}^{\delta_{jk}} \mathbf{e}_i \\ &= T_{ij} v_k \delta_{jk} \mathbf{e}_i = T_{ij} v_j \mathbf{e}_i = w_i \mathbf{e}_i = \mathbf{w} \in \mathcal{E}^3 \quad \text{where} \quad w_i = T_{ij} v_j \end{aligned}$$

Vectors, tensors, and index notation

Tensors and their representations

- A tensor of order n is defined by

$$\mathbf{T}_n := \underbrace{T_{ijk\dots}}_{n \text{ indices}} \underbrace{\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \dots}_{n \text{ terms}},$$

where $T_{ijk\dots}$ is its (n -dimensional array) representation in the given frame of reference.

Example

Let $\mathbf{C} \in \mathcal{E}^3 \otimes \mathcal{E}^3 \otimes \mathcal{E}^3 \otimes \mathcal{E}^3$ and $\mathbf{S} \in \mathcal{E}^3 \otimes \mathcal{E}^3$. The fourth-order tensor \mathbf{C} describes a linear transformation in $\mathcal{E}^3 \otimes \mathcal{E}^3$:

$$\begin{aligned} \mathbf{C} \bullet \mathbf{S} &= \mathbf{C} : \mathbf{S} = (C_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) : (S_{mn} \mathbf{e}_m \otimes \mathbf{e}_n) \\ &= C_{ijkl} S_{mn} (\mathbf{e}_k \cdot \mathbf{e}_m) (\mathbf{e}_l \cdot \mathbf{e}_n) \mathbf{e}_i \otimes \mathbf{e}_j \\ &= C_{ijkl} S_{mn} \delta_{km} \delta_{ln} \mathbf{e}_i \otimes \mathbf{e}_j = C_{ijkl} S_{kl} \mathbf{e}_i \otimes \mathbf{e}_j \\ &= T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{T} \in \mathcal{E}^3 \otimes \mathcal{E}^3 \quad \text{where} \quad T_{ij} = C_{ijkl} S_{kl} \end{aligned}$$

Vectors, tensors, and index notation

Multiplication of vectors and tensors

Example

Let: s be a scalar (a zero-order tensor), \mathbf{v}, \mathbf{w} be vectors (first-order tensors), $\mathbf{R}, \mathbf{S}, \mathbf{T}$ be second-order tensors, \mathbf{D} be a third-order tensor, and \mathbf{C} be a fourth-order tensor. The order of tensors is shown explicitly in the expressions below.

$$\begin{aligned}
 s &= \underset{0}{\mathbf{v}} \bullet \underset{1}{\mathbf{w}} = \underset{1}{\mathbf{v}} \underset{1}{\mathbf{w}} = \underset{1}{\mathbf{v}} \cdot \underset{1}{\mathbf{w}} &\rightarrow & v_i w_i = s \\
 \mathbf{v} &= \underset{2}{\mathbf{T}} \underset{1}{\mathbf{w}} = \underset{2}{\mathbf{T}} \cdot \underset{1}{\mathbf{w}} &\rightarrow & T_{ij} w_j = v_i \\
 \mathbf{R} &= \underset{2}{\mathbf{T}} \underset{2}{\mathbf{S}} = \underset{2}{\mathbf{T}} \cdot \underset{2}{\mathbf{S}} &\rightarrow & T_{ij} S_{jk} = R_{ik} \\
 s &= \underset{2}{\mathbf{T}} \bullet \underset{2}{\mathbf{S}} = \underset{2}{\mathbf{T}} : \underset{2}{\mathbf{S}} &\rightarrow & T_{ij} S_{ij} = s \\
 \mathbf{T} &= \underset{4}{\mathbf{C}} \bullet \underset{2}{\mathbf{S}} = \underset{4}{\mathbf{C}} : \underset{2}{\mathbf{S}} &\rightarrow & C_{ijkl} S_{kl} = T_{ij} \\
 \mathbf{T} &= \underset{2}{\mathbf{v}} \underset{1}{\mathbf{D}} = \underset{1}{\mathbf{v}} \cdot \underset{3}{\mathbf{D}} &\rightarrow & v_k D_{kij} = T_{ij}
 \end{aligned}$$

Remark: Notice a vital difference between the two dot-operators ‘ \bullet ’ and ‘ \cdot ’. To avoid ambiguity, usually, the operators ‘ $:$ ’ and ‘ \cdot ’ are not used, and the dot-operator has the meaning of the (full) dot-product, so that:

$$C_{ijkl} S_{kl} \rightarrow \mathbf{C} \bullet \mathbf{S}, \quad T_{ij} S_{ij} \rightarrow \mathbf{T} \bullet \mathbf{S}, \quad \text{and} \quad T_{ij} S_{jk} \rightarrow \mathbf{T} \mathbf{S}.$$

Vectors, tensors, and index notation

Vertical-bar convention and Nabla operator

Vertical-bar convention

The **vertical-bar (or comma) convention** is used to facilitate the denotation of partial derivatives with respect to the Cartesian position vectors $\mathbf{x} \sim x_i$, for example,

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \rightarrow \frac{\partial u_i}{\partial x_j} =: u_{i|j}$$

Vectors, tensors, and index notation

Vertical-bar convention and Nabla operator

Vertical-bar convention

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \quad \rightarrow \quad \frac{\partial u_i}{\partial x_j} =: u_{i|j}$$

Definition (Nabla-operator)

$$\nabla \equiv (\cdot)_{|i} \mathbf{e}_i = (\cdot)_{|1} \mathbf{e}_1 + (\cdot)_{|2} \mathbf{e}_2 + (\cdot)_{|3} \mathbf{e}_3$$

The **gradient**, **divergence**, **curl (rotation)**, and **Laplacian** operations can be written using the **Nabla-operator**:

$$\mathbf{v} = \text{grad } s \equiv \nabla s \quad \rightarrow \quad v_i = s_{|i}$$

$$s = \text{div } \mathbf{v} \equiv \nabla \cdot \mathbf{v} \quad \rightarrow \quad s = v_{i|i}$$

$$\mathbf{w} = \text{curl } \mathbf{v} \equiv \nabla \times \mathbf{v} \quad \rightarrow \quad w_i = \epsilon_{ijk} v_{k|j}$$

$$\text{lapl}(\cdot) \equiv \Delta(\cdot) \equiv \nabla^2(\cdot) \quad \rightarrow \quad (\cdot)_{|ii}$$

Vectors, tensors, and index notation

Nabla-operator and vector calculus identities

$$\nabla \equiv (\cdot)_{|i} \mathbf{e}_i = (\cdot)_{|1} \mathbf{e}_1 + (\cdot)_{|2} \mathbf{e}_2 + (\cdot)_{|3} \mathbf{e}_3$$

$$\mathbf{v} = \text{grad } s \equiv \nabla s \quad \rightarrow \quad v_i = s_{|i}$$

$$\mathbf{T} = \text{grad } \mathbf{v} \equiv \nabla \otimes \mathbf{v} \quad \rightarrow \quad T_{ij} = v_{i|j}$$

$$s = \text{div } \mathbf{v} \equiv \nabla \cdot \mathbf{v} \quad \rightarrow \quad s = v_{i|i}$$

$$\mathbf{v} = \text{div } \mathbf{T} \equiv \nabla \cdot \mathbf{T} \quad \rightarrow \quad v_i = T_{ji|j}$$

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$$\text{lapl}(\cdot) \equiv \Delta(\cdot) \equiv \nabla^2(\cdot) \quad \rightarrow \quad (\cdot)_{|ii}$$

Some vector calculus identities:

- $\nabla \times (\nabla s) = \mathbf{0}$ (curl grad = 0)
- $\nabla \cdot (\nabla \times \mathbf{v}) = 0$ (div curl = 0)
- $\nabla \cdot (\nabla s) = \nabla^2 s$ (div grad = lapl)
- $\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$ (curl curl = grad div - lapl)

Vectors, tensors, and index notation

Vector calculus identities

- $\nabla \times (\nabla s) = \mathbf{0}$ (curl grad = $\mathbf{0}$)

Proof.

$$\nabla \times (\nabla s) = \epsilon_{ijk} (s_{|k})_{|j} = \epsilon_{ijk} s_{|kj} = \begin{cases} \text{for } i = 1: s_{|23} - s_{|32} = 0 \\ \text{for } i = 2: s_{|31} - s_{|13} = 0 \\ \text{for } i = 3: s_{|12} - s_{|21} = 0 \end{cases}$$

QED

- $\nabla \cdot (\nabla \times \mathbf{v}) = 0$ (div curl = 0)
- $\nabla \cdot (\nabla s) = \nabla^2 s$ (div grad = lapl)
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Vectors, tensors, and index notation

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- $\nabla \times (\nabla s) = \mathbf{0}$ (curl grad = $\mathbf{0}$)
- $\nabla \cdot (\nabla \times \mathbf{v}) = 0$ (div curl = 0)

Proof.

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{v}) &= (\epsilon_{ijk} v_{k|j})_{|i} = \epsilon_{ijk} v_{k|ji} \\ &= (v_{3|21} - v_{3|12}) + (v_{1|32} - v_{1|23}) + (v_{2|13} - v_{2|31}) = 0 \end{aligned}$$

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- $\nabla \cdot (\nabla s) = \nabla^2 s$ (div grad = lapl)
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Proof.

$$\nabla \cdot (\nabla s) = (s_{|i})_{|i} = s_{|ii} = s_{11} + s_{22} + s_{33} \equiv \nabla^2 s$$

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Proof.

$$\nabla \times (\nabla \times \mathbf{v}) \rightarrow \epsilon_{mni} (\epsilon_{ijk} v_{k|j})_{|n} = \epsilon_{mni} \epsilon_{ijk} v_{k|jn}$$

$$\begin{aligned} \text{for } m = 1: \quad \epsilon_{1ni} \epsilon_{ijk} v_{k|jn} &= \epsilon_{123} (\epsilon_{312} v_{2|12} + \epsilon_{321} v_{1|22}) + \epsilon_{132} (\epsilon_{213} v_{3|13} + \epsilon_{231} v_{1|33}) \\ &= (v_{2|2} + v_{3|3})_{|1} - (v_{1|22} + v_{1|33}) \\ &= (v_{1|1} + v_{2|2} + v_{3|3})_{|1} - (v_{1|11} + v_{1|22} + v_{1|33}) \\ &= (v_{i|i})_{|1} - v_{1|i\ddot{i}} = (\nabla \cdot \mathbf{v})_{|1} - \nabla^2 v_1 \end{aligned}$$

$$\text{for } m = 2: \quad \epsilon_{2ni} \epsilon_{ijk} v_{k|jn} = (v_{i|i})_{|2} - v_{2|i\ddot{i}} = (\nabla \cdot \mathbf{v})_{|2} - \nabla^2 v_2$$

$$\text{for } m = 3: \quad \epsilon_{3ni} \epsilon_{ijk} v_{k|jn} = (v_{i|i})_{|3} - v_{3|i\ddot{i}} = (\nabla \cdot \mathbf{v})_{|3} - \nabla^2 v_3$$

QED

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Integral theorems

General idea

Integral theorems of vector calculus, namely:

- the **classical (Kelvin-)Stokes' theorem** (the curl theorem),
- **Green's theorem**,
- **Gauss theorem** (the Gauss-Ostrogradsky divergence theorem),

are special cases of the **general Stokes' theorem**, which generalizes the **fundamental theorem of calculus**.

Integral theorems

General idea

Integral theorems of vector calculus, namely:

- the **classical (Kelvin-)Stokes' theorem** (the curl theorem),
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- **Gauss theorem** (the Gauss-Ostrogradsky divergence theorem),

are special cases of the **general Stokes' theorem**, which generalizes the **fundamental theorem of calculus**.

Fundamental theorem of calculus relates scalar integral to boundary points:

$$\int_a^b f'(x) \, dx = f(b) - f(a)$$

Stokes's (curl) theorem relates surface integrals to line integrals.

Applications: for example, conservative forces.

Green's theorem is a two-dimensional special case of the Stokes' theorem.

Gauss (divergence) theorem relates volume integrals to surface integrals.

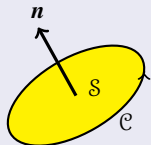
Applications: analysis of flux, pressure.

Integral theorems

Stokes' theorem

Theorem (Stokes' curl theorem)

Let \mathcal{C} be a simple closed curve spanned by a surface \mathcal{S} with unit normal \mathbf{n} . Then, for a continuously differentiable vector field \mathbf{f} :



$$\int_{\mathcal{S}} (\nabla \times \mathbf{f}) \cdot \underbrace{\mathbf{n}}_{d\mathcal{S}} d\mathcal{S} = \int_{\mathcal{C}} \mathbf{f} \cdot d\mathbf{r}$$

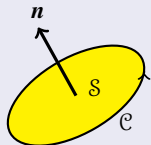
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- Formal requirements:** the surface \mathcal{S} must be open, orientable and piecewise smooth with a correspondingly orientated, simple, piecewise and smooth boundary curve \mathcal{C} .
- Green's theorem in the plane** may be viewed as a special case of Stokes' theorem (with $\mathbf{f} = [u(x, y), v(x, y), 0]$):

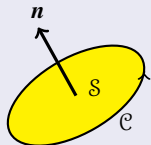
$$\int_{\mathcal{S}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy = \int_{\mathcal{C}} u dx + v dy$$

Integral theorems

Stokes' theorem

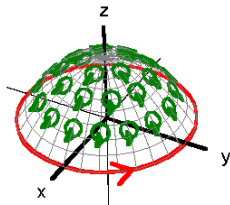
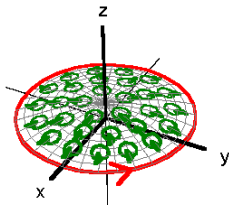
Theorem (Stokes' curl theorem)

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- Stokes' theorem implies that **the flux** of $\nabla \times \mathbf{f}$ **through a surface \mathcal{S}** depends only on the boundary \mathcal{C} of \mathcal{S} and is therefore **independent of the surface's shape**.



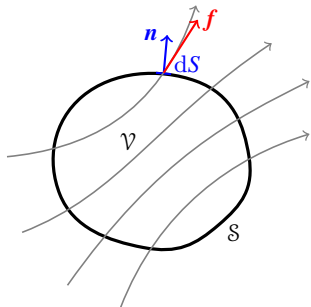
Integral theorems

Gauss-Ostrogradsky theorem

Theorem (Gauss divergence theorem)

Let the region \mathcal{V} be bounded by a simple surface \mathcal{S} with unit outward normal \mathbf{n} . Then, for a continuously differentiable vector field \mathbf{f} :

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{f} \, dV = \int_{\mathcal{S}} \mathbf{f} \cdot \underbrace{\mathbf{n}}_{d\mathbf{S}}; \quad \text{in particular} \quad \int_{\mathcal{V}} \nabla f \, dV = \int_{\mathcal{S}} f \mathbf{n} \, dS.$$



- The divergence theorem is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the vector field inside the surface.
- Intuitively, it states that the sum of all sources minus the sum of all sinks gives the net flow out of a region.

Outline

- 1 Vectors, tensors, and index notation**
 - Generalization of the concept of vector
 - Summation convention and index notation
 - Kronecker delta and permutation symbol
 - Tensors and their representations
 - Multiplication of vectors and tensors
 - Vertical-bar convention and Nabla operator
- 2 Integral theorems**
 - General idea
 - Stokes' theorem
 - Gauss-Ostrogradsky theorem
- 3 Time-harmonic approach**
 - Types of dynamic problems
 - Complex-valued notation
 - A practical example

Time-harmonic approach

Types of dynamic problems

Dynamic problems. In dynamic problems, the field variables depend upon position \mathbf{x} and time t , for example, $u = u(\mathbf{x}, t)$.

Time-harmonic approach

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Separation of variables. In many cases, the governing PDEs can be solved by expressing u as a product of functions that each depend only on one of the independent variables: $u(\mathbf{x}, t) = \hat{u}(\mathbf{x}) \check{u}(t)$.

Time-harmonic approach

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- Dynamic problems.** In dynamic problems, the field variables depend upon position x and time t , for example, $u = u(x, t)$.
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Time-harmonic approach

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Time-harmonic solution. If the time-dependent function $\check{u}(t)$ is a time-harmonic function (with the frequency f), the solution can be written as

$$u(x, t) = \hat{u}(x) \cos(\omega t + \alpha(x))$$

where: $\omega = 2\pi f$ is called the **angular** (or **circular**) **frequency**, $\alpha(x)$ is the **phase-angle shift**, and $\hat{u}(x)$ can be interpreted as a **spatial amplitude**.

Time-harmonic approach

Complex-valued notation

Time-harmonic solution:

$$u(\mathbf{x}, t) = \hat{u}(\mathbf{x}) \cos(\omega t + \alpha(\mathbf{x}))$$

Here: ω – the angular frequency, $\alpha(\mathbf{x})$ – the phase-angle shift,
 $\hat{u}(\mathbf{x})$ – the spatial amplitude.

A complex-valued notation for time-harmonic problems

A convenient way to handle time-harmonic problems is in the **complex notation** with the real part as a physically meaningful solution:

$$\begin{aligned} u(\mathbf{x}, t) &= \hat{u}(\mathbf{x}) \cos(\omega t + \alpha(\mathbf{x})) = \hat{u} \operatorname{Re} \left\{ \overbrace{\cos(\omega t + \alpha) + i \sin(\omega t + \alpha)}^{\exp[i(\omega t + \alpha)]} \right\} \\ &= \hat{u} \operatorname{Re} \left\{ \exp[i(\omega t + \alpha)] \right\} = \operatorname{Re} \left\{ \hat{u} \underbrace{\exp(i\alpha)}_{\tilde{u}} \exp(i\omega t) \right\} \\ &= \operatorname{Re} \left\{ \tilde{u} \exp(i\omega t) \right\} \end{aligned}$$

where the so-called **complex amplitude** (or **phasor**) is introduced:

$$\tilde{u} = \tilde{u}(\mathbf{x}) = \hat{u}(\mathbf{x}) \exp(i\alpha(\mathbf{x})) = \hat{u}(\mathbf{x}) (\cos \alpha(\mathbf{x}) + i \sin \alpha(\mathbf{x}))$$

Time-harmonic approach

A practical example

Consider a **linear dynamic system** characterized by the matrices of stiffness K , damping C , and mass M :

$$K q(t) + C \dot{q}(t) + M \ddot{q}(t) = Q(t)$$

where $Q(t)$ is the dynamic excitation (a time-varying force) and $q(t)$ is the system's response (displacement).

Time-harmonic approach

A practical example

$$K q(t) + C \dot{q}(t) + M \ddot{q}(t) = Q(t)$$

- Let the driving force $Q(t)$ be harmonic with the angular frequency ω and the (real-valued) amplitude \hat{Q} :

$$Q(t) = \hat{Q} \cos(\omega t) = \hat{Q} \operatorname{Re} \{ \cos(\omega t) + i \sin(\omega t) \} = \operatorname{Re} \{ \hat{Q} \exp(i \omega t) \}$$

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- Since the system is linear the response $q(t)$ will be also harmonic and with the same angular frequency but shifted by the phase angle α :

$$\begin{aligned} q(t) &= \hat{q} \cos(\omega t + \alpha) = \hat{q} \operatorname{Re} \{ \cos(\omega t + \alpha) + i \sin(\omega t + \alpha) \} \\ &= \hat{q} \operatorname{Re} \{ \exp[i(\omega t + \alpha)] \} = \operatorname{Re} \{ \underbrace{\hat{q} \exp(i \alpha)}_{\tilde{q}} \exp(i \omega t) \} \\ &= \operatorname{Re} \{ \tilde{q} \exp(i \omega t) \} \end{aligned}$$

Here, \hat{q} and \tilde{q} are the real and complex amplitudes, respectively. The real amplitude \hat{q} and the phase angle α are unknowns; thus, unknown is the complex amplitude $\tilde{q} = \hat{q}(\cos \alpha + i \sin \alpha)$.

Time-harmonic approach

A practical example

$$K q(t) + C \dot{q}(t) + M \ddot{q}(t) = Q(t)$$

- Now, one can substitute into the system's equation

$$Q(t) \leftarrow \hat{Q} \exp(i \omega t),$$

$$q(t) \leftarrow \tilde{q} \exp(i \omega t), \quad \dot{q}(t) = \tilde{q} i \omega \exp(i \omega t), \quad \ddot{q}(t) = -\tilde{q} \omega^2 \exp(i \omega t)$$

to obtain the following algebraic equation for the unknown complex amplitude \tilde{q} :

$$[K + i \omega C - \omega^2 M] \tilde{q} = \hat{Q}$$

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- For the Rayleigh damping model, where $C = \beta_K K + \beta_M M$ (β_K and β_M are real-valued constants), this equation can be presented as follows:

$$[\tilde{K} - \omega^2 \tilde{M}] \tilde{q} = \hat{Q}, \quad \text{where} \quad \tilde{K} = K(1 + i\omega \beta_K), \quad \tilde{M} = M \left(1 + \frac{\beta_M}{i\omega} \right)$$

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- Having computed the complex amplitude \tilde{q} for the given frequency ω , one can finally find the time-harmonic response as the real part of the complex solution:

$$q(t) = \operatorname{Re} \{ \tilde{q} \exp(i\omega t) \} = \hat{q} \cos(\omega t + \alpha), \quad \text{where} \quad \begin{cases} \hat{q} = |\tilde{q}| \\ \alpha = \arg(\tilde{q}) \end{cases}$$