Introduction to Partial Differential Equations

Introductory Course on Multiphysics Modelling

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(after: S.J. FARLOW's "*Partial Differential Equations for Scientists and Engineers*")

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Outline

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Basic notions and notations

Motivation: most physical phenomena, whether in the domain of fluid dynamics or solid mechanics, electricity, magnetism, optics or heat flow, can be in general (and actually are) described by *partial differential equations*.

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1. has an *unknown function* depending on *at least two variables*,
2. contains some *partial derivatives* of the unknown function.

- A **solution to PDE** is, generally speaking, any function (in the independent variables) that satisfies the PDE.
- From this family of functions one may be uniquely selected by imposing adequate **initial** and/or **boundary conditions**.
- A PDE with initial and boundary conditions constitutes the so-called **initial-boundary-value problem (IBVP)**. Such problems are mathematical models of most physical phenomena.

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2. contains some *partial derivatives* of the unknown function.

The following notation will be used throughout this lecture:

- t, x, y, z (or, e.g.: r, θ, ϕ) – the **independent variables** (here, t represents time while the other variables are space coordinates),
- $u = u(t, x, \ldots)$ – the **dependent variable** (the unknown function),
- the *partial derivatives* will be denoted as follows

$$u_t = \frac{\partial u}{\partial t}\,, \qquad u_{tt} = \frac{\partial^2 u}{\partial t^2}\,, \qquad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}\,, \qquad \text{etc.}$$

Methods and techniques for solving PDEs

Separation of variables. A PDE in n independent variables is reduced to n ODEs.

Methods and techniques for solving PDEs

Separation of variables.

Integral transforms. A PDE in n independent variables is reduced to one in $(n-1)$ independent variables. Hence, a PDE in two variables can be changed to an ODE.

Methods and techniques for solving PDEs

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Change of coordinates. A PDE can be changed to an ODE or to an easier PDE by changing the coordinates of the problem (rotating the axes, etc.).

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Transformation of the dependent variable. The unknown of a PDE is transformed into a new unknown that is easier to find.

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Numerical methods. A PDE is changed to a system of *difference equations* that can be solved by means of iterative techniques (*Finite Difference Methods*). These methods can be divided into two main groups, namely: **explicit** and **implicit** methods. There are also other methods that attempt to approximate solutions by polynomial functions (eg., *Finite Element Method*).

Methods and techniques for solving PDEs

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Numerical methods.

Perturbation methods. A nonlinear problem (a nonlinear PDE) is changed into a sequence of linear problems that approximates the nonlinear one.

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Perturbation methods.

Impulse-response technique. Initial and boundary conditions of a problem are decomposed into simple impulses and the response is found for each impulse. The overall response is then obtained by adding these simple responses.

Methods and techniques for solving PDEs

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Integral equations. A PDE is changed to an integral equation (that is, an equation where the unknown is inside the integral). The integral equations is then solved by various techniques.

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Variational methods. The solution to a PDE is found by reformulating the equation as a minimization problem. It turns out that the minimum of a certain expression (very likely the expression will stand for total energy) is also the solution to the PDE.

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Variational methods.

Eigenfunction expansion. The solution of a PDE is as an infinite sum of eigenfunctions. These eigenfunctions are found by solving the so-called eigenvalue problem corresponding to the original problem.

Well-posed and ill-posed problems

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Importance of well-posedness:

- In practice, the initial and boundary data are measured and so small errors occur.
- Very often the problem must be solved numerically which involves truncation and round-off errors.
- If the problem is well-posed then these unavoidable small errors produce only slight errors in the computed solution, and, hence, useful results are obtained.

Outline

Basic classifications of PDEs

Order of the PDE. The order of a PDE is the order of the highest partial derivative in the equation.

Example

first order: $u_t = u_x \, ,$

second order: $u_t = u_{xx} \, , \quad u_{xy} = 0 \, ,$

third order: $u_t + u \, u_{xxx} = \sin(x)$

fourth order: $u_{xxxx} = u_{tt} \, .$

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Number of variables. PDEs may be classified by the number of their independent variables, that is, the number of variables the unknown function depends on.

Example

PDE in two variables: $u_t = u_{xx}\,, \qquad \big(u = u(t,x)\big),$

PDE in three variables: $u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\,, \qquad \big(u = u(t,r,\theta)\big),$

PDE in four variables: $u_t = u_{xx} + u_{yy} + u_{zz}\,, \qquad \big(u = u(t,x,y,z)\big).$

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Linearity. PDE is linear if the dependent variable and all its derivatives appear in a linear fashion.

Example

$$\begin{aligned}\text{linear:}\quad & u_{tt} + \exp(-t)\, u_{xx} = \sin(t)\,,\\ \text{nonlinear:}\quad & u\, u_{xx} + u_t = 0\,,\\ \text{linear:}\quad & x\, u_{xx} + y\, u_{yy} = 0\,,\\ \text{nonlinear:}\quad & u_x + u_y + u^2 = 0\,.\end{aligned}$$

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Kinds of coefficients. PDE can be with constant or variable coefficients (if at least one of the coefficients is a function of (some of) independent variables).

Example

constant coefficients: $u_{tt} + 5u_{xx} - 3u_{xy} = \cos(x)\,,$

variable coefficients: $u_t + \exp(-t)\, u_{xx} = 0\,.$

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Homogeneity. PDE is homogeneous if the free term (the right-hand side term) is zero.

Example

$$\text{homogeneous:} \quad u_{tt} - u_{xx} = 0\,,$$

$$\text{nonhomogeneous:} \quad u_{tt} - u_{xx} = x^2 \sin(t)\,.$$

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Kind of PDE. All linear second-order PDEs are either:

- **hyperbolic** (e.g., $u_{tt} - u_{xx} = f(t, x, u, u_t, u_x)$),
- **parabolic** (e.g., $u_{xx} = f(t, x, u, u_t, u_x)$),
- **elliptic** (e.g., $u_{xx} + u_{yy} = f(x, y, u, u_x, u_y)$).

Kinds of nonlinearity

Definition (Semi-linearity, quasi-linearity, and full nonlinearity)

A partial differential equation is:

semi-linear – if the highest derivatives appear in a linear fashion and their coefficients do not depend on the unknown function or its derivatives;

quasi-linear – if the highest derivatives appear in a linear fashion;

fully nonlinear – if the highest derivatives appear in a nonlinear fashion.

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Let: $u = u(\boldsymbol{x})$ and $\boldsymbol{x} = (x, y)$.

Example (semi-linear PDE)

$$C_1(\boldsymbol{x})\, u_{xx} + C_2(\boldsymbol{x})\, u_{xy} + C_3(\boldsymbol{x})\, u_{yy} + C_0(\boldsymbol{x}, u, u_x, u_y) = 0$$

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Let: $u = u(\boldsymbol{x})$ and $\boldsymbol{x} = (x, y)$.

Example (quasi-linear PDE)

$$C_1(\boldsymbol{x}, u, u_x, u_y)\, u_{xx} + C_2(\boldsymbol{x}, u, u_x, u_y)\, u_{xy} + C_0(\boldsymbol{x}, u, u_x, u_y) = 0$$

Kinds of nonlinearity

Definition (Semi-linearity, quasi-linearity, and full nonlinearity)

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fully nonlinear – if the highest derivatives appear in a nonlinear fashion.

Let: $u = u(\boldsymbol{x})$ and $\boldsymbol{x} = (x, y)$.

Example (fully non-linear PDE)

$$u_{xx}\, u_{xy} = 0$$

Types of second-order linear PDEs

A **second-order linear PDE in two variables** can be in general written in the following form

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D\,u_x + E\,u_y + F\,u = G$$

where A, B, C, D, E, and F are coefficients, and G is a right-hand side (i.e., non-homogeneous) term. All these quantities are constants, or at most, functions of (x, y).

Types of second-order linear PDEs

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D\,u_x + E\,u_y + F\,u = G$$

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hyperbolic: if $B^2 - 4AC > 0$

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Example

$$u_{tt} - u_{xx} = 0 \quad \rightarrow \quad B^2 - 4AC = 0^2 - 4 \cdot (-1) \cdot 1 = 4 > 0\,,$$

$$u_{tx} = 0 \quad \rightarrow \quad B^2 - 4AC = 1^2 - 4 \cdot 0 \cdot 0 = 1 > 0\,.$$

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Example

$$u_{xx} + u_{yy} = 0 \quad \rightarrow \quad B^2 - 4AC = 0^2 - 4 \cdot 1 \cdot 1 = -4 < 0\,.$$

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elliptic: if $B^2 - 4AC < 0$ (eg., $u_{xx} + u_{yy} = 0$).

- The mathematical solutions to these three types of equations are quite different.
- The three major classifications of linear PDEs essentially classify physical problems into three basic types:
  1. vibrating systems and **wave** propagation (**hyperbolic** case),
  2. heat flow and **diffusion** processes (**parabolic** case),
  3. **steady-state** phenomena (**elliptic** case).

Types of second-order linear PDEs

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elliptic: if $B^2 - 4AC < 0$ (eg., $u_{xx} + u_{yy} = 0$).

In general, $(B^2 - 4A\,C)$ is a function of the independent variables (x, y). Hence, an equation can change from one basic type to another.

Example

$$y\,u_{xx} + u_{yy} = 0 \quad \to \quad B^2 - 4AC = -4y \begin{cases} > 0 & \text{for } y < 0 \text{ (hyperbolic)}, \\ = 0 & \text{for } y = 0 \text{ (parabolic)}, \\ < 0 & \text{for } y > 0 \text{ (elliptic)}. \end{cases}$$

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Second-order linear equations in *three or more variables* can also be classified except that matrix analysis must be used.

Example

$u_t = u_{xx} + u_{yy}$ $\leftarrow$ parabolic equation,

$u_{tt} = u_{xx} + u_{yy} + u_{zz}$ $\leftarrow$ hyperbolic equation.

Classic linear PDEs

Hyperbolic PDEs:

- Vibrating string (1D wave equation): $u_{tt} - c^2\, u_{xx} = 0$
- Wave equation with damping (if $h \neq 0$): $u_{tt} - c^2\, \nabla^2 u + h\, u_t = 0$
- Transmission line equation: $u_{tt} - c^2\, \nabla^2 u + h\, u_t + k\, u = 0$

Parabolic PDEs:

- Diffusion-convection equation: $u_t - \alpha^2\, u_{xx} + h\, u_x = 0$
- Diffusion with lateral heat-concentration loss:
  $u_t - \alpha^2\, u_{xx} + k\, u = 0$

Elliptic PDEs:

- Laplace's equation: $\nabla^2 u = 0$
- Poisson's equation: $\nabla^2 u = k$
- Helmholtz's equation: $\nabla^2 u + \lambda^2 u = 0$
- Shrödinger's equation: $\nabla^2 u + k\,(E - V)\, u = 0$

Higher-order PDEs:

- Airy's equation (third order): $u_t + u_{xxx} = 0$
- Bernouli's beam equation (fourth order): $\alpha^2\, u_{tt} + u_{xxxx} = 0$
- Kirchhoff's plate equation (fourth order): $\alpha^2\, u_{tt} + \nabla^4 u = 0$

(Here: ∇^2 is the Laplace operator, $\nabla^4 = \nabla^2\nabla^2$ is the biharmonic operator.)

Outline

Canonical forms of second order PDEs

Any second-order linear PDE (in two variables)

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D\,u_x + E\,u_y + F\,u = G$$

(where A, B, C, D, E, F, and G are constants or functions of (x, y)) **can be transformed into** the so-called **canonical form**.

This can be achieved **by introducing new coordinates**:

$$\xi = \xi(x, y) \quad \text{and} \quad \eta = \eta(x, y)$$

(in place of x, y) which simplify the equation to its canonical form.

Canonical forms of second order PDEs

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The type of PDE determines the canonical form:

- **for hyperbolic PDE** (that is, when $B^2 - 4A\,C > 0$) there are, in fact, two possibilities:

$$u_{\xi\xi} - u_{\eta\eta} = f(\xi,\eta,u,u_\xi,u_\eta) \quad \big(\tilde{B}^2 - 4\tilde{A}\,\tilde{C} = 0^2 - 4\cdot 1\cdot(-1) = 4 > 0\big),$$

$$\text{or} \quad u_{\xi\eta} = f(\xi,\eta,u,u_\xi,u_\eta) \quad \big(\tilde{B}^2 - 4\tilde{A}\,\tilde{C} = 1^2 - 4\cdot 0\cdot 0 = 1 > 0\big);$$

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- **for elliptic PDE** (that is, when $B^2 - 4A\,C < 0$):

$$u_{\xi\xi} + u_{\eta\eta} = f(\xi,\eta,u,u_\xi,u_\eta) \quad \left(\tilde{B}^2 - 4\tilde{A}\,\tilde{C} = 0^2 - 4\cdot 1\cdot 1 = -4 < 0\right).$$

Reduction to a canonical form

Step 1. Introduce new coordinates $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$.

Reduction to a canonical form

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- Compute the partial derivatives:

$$u_x = u_\xi\, \xi_x + u_\eta\, \eta_x\,, \qquad u_y = u_\xi\, \xi_y + u_\eta\, \eta_y\,,$$

$$u_{xx} = u_{\xi\xi}\, \xi_x^2 + 2u_{\xi\eta}\, \xi_x\, \eta_x + u_{\eta\eta}\, \eta_x^2 + u_\xi\, \xi_{xx} + u_\eta\, \eta_{xx}\,,$$

$$u_{yy} = u_{\xi\xi}\, \xi_y^2 + 2u_{\xi\eta}\, \xi_y\, \eta_y + u_{\eta\eta}\, \eta_y^2 + u_\xi\, \xi_{yy} + u_\eta\, \eta_{yy}\,,$$

$$u_{xy} = u_{\xi\xi}\, \xi_x\, \xi_y + u_{\xi\eta}\, \left(\xi_x\, \eta_y + \xi_y\, \eta_x\right) + u_{\eta\eta}\, \eta_x\, \eta_y + u_\xi\, \xi_{xy} + u_\eta\, \eta_{xy}\,.$$

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$$u_{xx} = u_{\xi\xi} \, \xi_x^2 + 2u_{\xi\eta} \, \xi_x \, \eta_x + u_{\eta\eta} \, \eta_x^2 + u_\xi \, \xi_{xx} + u_\eta \, \eta_{xx} \,,$$

$$u_{yy} = u_{\xi\xi} \, \xi_y^2 + 2u_{\xi\eta} \, \xi_y \, \eta_y + u_{\eta\eta} \, \eta_y^2 + u_\xi \, \xi_{yy} + u_\eta \, \eta_{yy} \,,$$

$$u_{xy} = u_{\xi\xi} \, \xi_x \, \xi_y + u_{\xi\eta} \left(\xi_x \, \eta_y + \xi_y \, \eta_x \right) + u_{\eta\eta} \, \eta_x \, \eta_y + u_\xi \, \xi_{xy} + u_\eta \, \eta_{xy} \,.$$

- Substitute these values into the original equation to obtain a new form:

$$\widetilde{A} \, u_{\xi\xi} + \widetilde{B} \, u_{\xi\eta} + \widetilde{C} \, u_{\eta\eta} + \widetilde{D} \, u_\xi + \widetilde{E} \, u_\eta + F \, u = G$$

  where the new coefficients are as follows

$$\widetilde{A} = A \, \xi_x^2 + B \, \xi_x \, \xi_y + C \, \xi_y^2 \,, \quad \widetilde{B} = 2A \, \xi_x \, \eta_x + B \left(\xi_x \, \eta_y + \xi_y \, \eta_x \right) + 2C \, \xi_y \, \eta_y \,,$$

$$\widetilde{C} = A \, \eta_x^2 + B \, \eta_x \, \eta_y + C \, \eta_y^2 \,, \quad \widetilde{D} = A \, \xi_{xx} + B \, \xi_{xy} + C \, \xi_{yy} + D \, \xi_x + E \, \xi_y \,,$$

$$\widetilde{E} = A \, \eta_{xx} + B \, \eta_{xy} + C \, \eta_{yy} + D \, \eta_x + E \, \eta_y \,.$$

Reduction to a canonical form

Step 1. **Introduce new coordinates** $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$.

$$\widetilde{A}\, u_{\xi\xi} + \widetilde{B}\, u_{\xi\eta} + \widetilde{C}\, u_{\eta\eta} + \widetilde{D}\, u_{\xi} + \widetilde{E}\, u_{\eta} + F\, u = G$$

Step 2. **Impose the requirements onto coefficients** $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, and solve for ξ and η.

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The requirements depend on the type of the PDE, namely:

- set $\widetilde{A} = \widetilde{C} = 0$ for the **hyperbolic** PDE (when $B^2 - 4A\,C > 0$);

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- set either $\widetilde{A} = 0$ or $\widetilde{C} = 0$ for the **parabolic** PDE; in this case another necessary requirement $\widetilde{B} = 0$ will follow automatically (since $B^2 - 4A\,C = 0$);

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- set $\widetilde{A} = \widetilde{C} = 0$ for the **hyperbolic** PDE (when $B^2 - 4A\,C > 0$);
- set either $\widetilde{A} = 0$ or $\widetilde{C} = 0$ for the **parabolic** PDE; in this case another necessary requirement $\widetilde{B} = 0$ will follow automatically (since $B^2 - 4A\,C = 0$);
- for the **elliptic** PDE (when $B^2 - 4A\,C < 0$), firstly, proceed as in the hyperbolic case: set $\widetilde{A} = \widetilde{C} = 0$ to find the *complex conjugate coordinates* ξ, η (which would lead to a form of *complex hyperbolic equation* $u_{\xi\eta} = f(\xi,\eta,u,u_{\xi},u_{\eta})$); then, transform ξ and η as follows:

$$\alpha \leftarrow \frac{\xi+\eta}{2}, \qquad \beta \leftarrow \frac{\xi-\eta}{2\mathrm{i}}.$$

(Here, α is the real part of ξ and η, while β is the imaginary part.)

The new real coordinates, α and β, allow to write the final *canonical* elliptic form: $u_{\alpha\alpha} + u_{\beta\beta} = f(\alpha,\beta,u,u_{\alpha},u_{\beta})$.

Reduction to a canonical form

Step 1. **Introduce new coordinates** $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$.

$$\widetilde{A}\, u_{\xi\xi} + \widetilde{B}\, u_{\xi\eta} + \widetilde{C}\, u_{\eta\eta} + \widetilde{D}\, u_{\xi} + \widetilde{E}\, u_{\eta} + F\, u = G$$

Step 2. **Impose the requirements onto coefficients** $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, and solve for ξ and η.

Step 3. **Use the new coordinates** for the coefficients and homogeneous term of the new canonical form (i.e., replace $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$).

Transforming the hyperbolic equation

For **hyperbolic equation** the canonical form

$$u_{\xi\eta} = f(\xi, \eta, u, u_\xi, u_\eta)$$

is achieved by setting $\boxed{\widetilde{A} = \widetilde{C} = 0}$,

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$$\widetilde{A} = A\,\xi_x^2 + B\,\xi_x\,\xi_y + C\,\xi_y^2 = 0\,, \qquad \widetilde{C} = A\,\eta_x^2 + B\,\eta_x\,\eta_y + C\,\eta_y^2 = 0\,,$$

which can be rewritten as

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\,\frac{\xi_x}{\xi_y} + C = 0\,, \qquad A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\,\frac{\eta_x}{\eta_y} + C = 0\,.$$

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Solving these equations for $\frac{\xi_x}{\xi_y}$ and $\frac{\eta_x}{\eta_y}$ one finds the so-called **characteristic equations**:

$$\frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4A\,C}}{2A}\,, \qquad \frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{B^2 - 4A\,C}}{2A}\,.$$

Transforming the hyperbolic equation

The new coordinates equated to constant values define the parametric lines of the new system of coordinates. That means that the total derivatives are zero, i.e.,

$$\xi(x,y) = \text{const.} \quad \rightarrow \quad \mathrm{d}\xi = \xi_x \,\mathrm{d}x + \xi_y \,\mathrm{d}y = 0 \quad \rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\xi_x}{\xi_y}\,,$$

$$\eta(x,y) = \text{const.} \quad \rightarrow \quad \mathrm{d}\eta = \eta_x \,\mathrm{d}x + \eta_y \,\mathrm{d}y = 0 \quad \rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\eta_x}{\eta_y}\,,$$

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Therefore, the **characteristic equations** are

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\xi_x}{\xi_y} = \frac{B - \sqrt{B^2 - 4A\,C}}{2A} \,, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\eta_x}{\eta_y} = \frac{B + \sqrt{B^2 - 4A\,C}}{2A} \,,$$

and can be easily integrated to find the implicit solutions, $\xi(x,y) = \text{const.}$ and $\eta(x,y) = \text{const.}$, that is, the new coordinates ensuring the simple canonical form of the PDE.

Example

Rewriting a hyperbolic equation in canonical form

$$y^2\, u_{xx} - x^2\, u_{yy} = 0 \qquad x \in (0, +\infty)\,, \quad y \in (0, +\infty)\,.$$

(In the first quadrant this is a hyperbolic equation, since $B^2 - 4A\,C = 4y^2\,x^2 > 0$ for $x \neq 0$ and $y \neq 0$.)

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Rewriting a hyperbolic equation in canonical form

$$y^2\, u_{xx} - x^2\, u_{yy} = 0 \qquad x \in (0, +\infty)\,, \quad y \in (0, +\infty)\,.$$

- Writing the two **characteristic equations**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{B - \sqrt{B^2 - 4A\,C}}{2A} = -\frac{x}{y}\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{B + \sqrt{B^2 - 4A\,C}}{2A} = \frac{x}{y}\,.$$

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- **Solving** these equations – by separating the variables

$$y\ \mathrm{d}y = -x\ \mathrm{d}x\,, \qquad y\ \mathrm{d}y = x\ \mathrm{d}x\,,$$

and integrating

$$\xi(x,y) = y^2 + x^2 = \text{const.}\,, \qquad \eta(x,y) = y^2 - x^2 = \text{const.}$$

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Rewriting a hyperbolic equation in canonical form

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- **Solving** these equations – by separating the variables and integrating

$$y\ \mathrm{d}y = -x\ \mathrm{d}x\,, \qquad y\ \mathrm{d}y = x\ \mathrm{d}x\,,$$

$$\xi(x,y) = y^2 + x^2 = \text{const.}\,, \qquad \eta(x,y) = y^2 - x^2 = \text{const.}$$

- **Using the new coordinates** for the (non-zero) coefficients

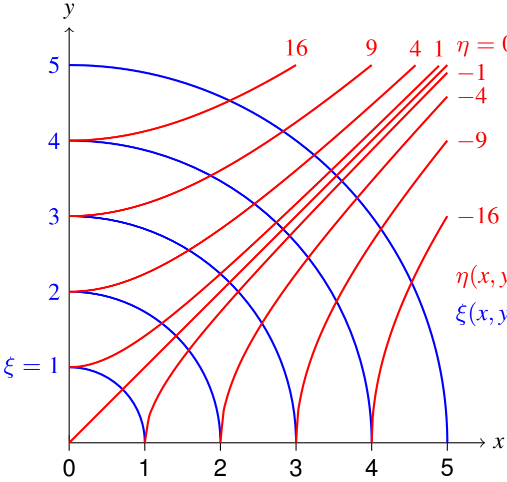
$$\widetilde{B} = -16x^2\, y^2 = 4(\eta^2 - \xi^2)\,, \quad \widetilde{D} = -2(y^2 + x^2) = -2\xi\,, \quad \widetilde{E} = 2(y^2 - x^2) = 2\eta\,,$$

to **present the PDE in the canonical form**:

$$u_{\xi\eta} = \frac{\widetilde{D}\, u_\xi + \widetilde{E}\, u_\eta}{\widetilde{B}} = \frac{\xi\, u_\xi - \eta\, u_\eta}{2(\xi^2 - \eta^2)}\,.$$

Example

New coordinates for the canonical form of the hyperbolic PDE



PDE in (x, y): $y^2\, u_{xx} - x^2\, u_{yy} = 0$

PDE in (ξ, η): $u_{\xi\eta} = \dfrac{\xi\, u_\xi - \eta\, u_\eta}{2(\xi^2 - \eta^2)}$

$\eta(x, y) =$ const. $\leftarrow$ **hyperbolas**

$\xi(x, y) =$ const. $\leftarrow$ **circles**

$\xi(x, y) = y^2 + x^2 =$ const. $\in (0, +\infty)$, $\eta(x, y) = y^2 - x^2 =$ const. $\in (-\infty, +\infty)$.

Outline

Separation of variables

Necessary assumptions

This technique applies to problems which satisfy two requirements.

1. The PDE is *linear* and *homogeneous* (not necessary constant coefficients).
2. The boundary conditions are *linear* and *homogeneous*.

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1. **The PDE is *linear* and *homogeneous*.**
   A second-order PDE in two variables (x and t) is *linear* and *homogeneous*, if it can be written in the following form

$$A\, u_{xx} + B\, u_{xt} + C\, u_{tt} + D\, u_x + E\, u_t + F\, u = 0$$

   where the coefficients A, B, C, D, E, and F do not depend on the dependent variable $u = u(x,t)$ or any of its derivatives though can be functions of independent variables (x,t).
2. The boundary conditions are *linear* and *homogeneous*.

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   where the coefficients A, B, C, D, E, and F do not depend on the dependent variable $u = u(x,t)$ or any of its derivatives though can be functions of independent variables (x,t).
2. **The boundary conditions are *linear* and *homogeneous*.**
   In the case of the second-order PDE, a general form of such boundary conditions is

   $$G_1\,u_x(x_1,t) + H_1\,u(x_1,t) = 0\,,$$
   $$G_2\,u_x(x_2,t) + H_2\,u(x_2,t) = 0\,,$$

   where G_1, G_2, H_1, H_2 are constants.

Separation of variables

Scheme of the method

Main procedure:

1. break down the initial conditions into simple components,

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The separation of variables technique looks first for the so-called **fundamental solutions**. They are simple-type solutions of the form

$$u_i(x,t) = X_i(x)\, T_i(t)\,,$$

where $X_i(x)$ is a sort of "shape" of the solution i whereas $T_i(t)$ scales this "shape" for different values of time t.

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The **fundamental solution will**:

- always **retain its basic "shape"**,
- at the same time, **satisfy the BCs** which puts a requirement only on the "shape" function $X_i(x)$ since the BCs are linear and homogeneous.

The general idea is that it is possible to find an infinite number of these fundamental solutions (everyone corresponding to an adequate simple component of initial conditions).

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The **solution of the problem** is found by adding the simple fundamental solutions in such a way that the resulting sum

$$u(x,t) = \sum_{i=1}^{n} a_i\, u_i(x,t) = \sum_{i=1}^{n} a_i\, X_i(x)\, T_i(t)$$

satisfies the initial conditions which is attained by a proper selection of the coefficients a_i.

Example

Solving a parabolic IBVP by the separation of variables method

IBVP for heat flow (or diffusion process)

Find $u = u(x,t) = ?$ satisfying for $x \in [0,1]$ and $t \in [0,\infty)$:

PDE: $u_t = \alpha^2 u_{xx}$, **BCs:** $\begin{cases} u(0,t) = 0\,, \\ u_x(1,t) + h\,u(1,t) = 0\,, \end{cases}$ **IC:** $u(x,0) = f(x)$,

where α, h, and $f(x)$ are some known constants or functions.

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where α, h, and $f(x)$ are some known constants or functions.

Step 1. Separating the PDE into two ODEs.

- Substituting the separated form (of the fundamental solution),

$$u(x,t) = u_i(x,t) = X_i(x)\,T_i(t)\,,$$

into the PDE gives (after division by $\alpha^2 X_i(x)\,T_i(t)$)

$$\frac{T_i'(t)}{\alpha^2\,T_i(t)} = \frac{X_i''(x)}{X_i(x)}\,.$$

- Both sides of this equation must be constant (since they depend only on x or t which are *independent*). Setting them both equal to μ_i results in two ODEs:

$$T_i'(t) - \mu_i\,\alpha^2\,T_i(t) = 0\,, \qquad X_i''(x) - \mu_i\,X_i(x) = 0\,.$$

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where α, h, and $f(x)$ are some known constants or functions.

Step 1. **Separating the PDE into two ODEs.**

Step 2. **Finding the separation constant and fundamental solutions.**

- If $\mu_i = 0$ then: (after using the BCs) a trivial solution $u(x,t) \equiv 0$ is obtained.
- For $\mu_i > 0$: $T(t)$ (and so $u(x,t) = X(x)\,T(t)$) will grow exponentially to infinity which can be rejected on physical grounds.
- Therefore: $\mu_i = -\lambda_i^2 < 0$.

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where α, h, and $f(x)$ are some known constants or functions.

Step 1. Separating the PDE into two ODEs.

Step 2. Finding the separation constant and fundamental solutions.

- Now, the two ODEs can be written as

$$T_i'(t) + \lambda_i^2\,\alpha^2\,T_i(t) = 0\,, \qquad X_i''(x) + \lambda_i^2\,X_i(x) = 0\,,$$

and solutions to them are

$$T_i(t) = \tilde{C}_0\,\exp\big(-\lambda_i^2\,\alpha^2\,t\big)\,, \qquad X_i(x) = \tilde{C}_1\,\sin(\lambda_i\,x) + \tilde{C}_2\,\cos(\lambda_i\,x)\,,$$

where $\tilde{C}_0$, $\tilde{C}_1$, and $\tilde{C}_2$ are constants.

- That leads to the following fundamental solution (with constants C_1, C_2)

$$u_i(x,t) = X_i(x)\,T_i(t) = \big[C_1\,\sin(\lambda_i\,x) + C_2\,\cos(\lambda_i\,x)\big]\,\exp(-\lambda_i^2\,\alpha^2\,t)\,.$$

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Solving a parabolic IBVP by the separation of variables method

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Find $u = u(x,t) = ?$ satisfying for $x \in [0,1]$ and $t \in [0,\infty)$:

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where α, h, and $f(x)$ are some known constants or functions.

Step 1. **Separating the PDE into two ODEs.**

Step 2. **Finding the separation constant and fundamental solutions.**

- Applying the boundary conditions

at $x=0$: $C_2\, \exp(-\lambda_i^2\, \alpha^2\, t) = 0 \quad \to \quad C_2 = 0\,,$

at $x=1$: $C_1\, \exp(-\lambda_i^2\, \alpha^2\, t)\, \big[\lambda_i\, \cos(\lambda_i) + h\, \sin(\lambda_i)\big] = 0 \quad \to \quad \tan\lambda_i = -\dfrac{\lambda_i}{h}\,.$

That gives a desired condition on λ_i SOLVE (they are **eigenvalues** for which there exists a nonzero solution).

- The **fundamental solutions** are as follows PLOT

$$u_i(x,t) = \sin(\lambda_i\, x)\, \exp(-\lambda_i^2\, \alpha^2\, t)\,.$$

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Solving a parabolic IBVP by the separation of variables method

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Step 1. **Separating the PDE into two ODEs.**

Step 2. **Finding the separation constant and fundamental solutions.**

Step 3. **Expansion of the IC as a sum of eigenfunctions.**

▶ The final solution is such linear combination (with coefficients a_i) of infinite number of fundamental solutions,

$$u(x,t) = \sum_{i=1}^{\infty} a_i\, u_i(x,t) = \sum_{i=1}^{\infty} a_i\, \sin(\lambda_i\, x)\, \exp(-\lambda_i^2\, \alpha^2\, t)\,,$$

that satisfies the initial condition:

$$f(x) \equiv u(x,0) = \sum_{i=1}^{\infty} a_i\, \sin(\lambda_i\, x)\,.$$

Example

Solving a parabolic IBVP by the separation of variables method

Step 1. **Separating the PDE into two ODEs.**

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- The final solution is such linear combination (with coefficients a_i) of infinite number of fundamental solutions,

$$u(x,t) = \sum_{i=1}^{\infty} a_i \, u_i(x,t) = \sum_{i=1}^{\infty} a_i \, \sin(\lambda_i \, x) \, \exp(-\lambda_i^2 \, \alpha^2 \, t) \, ,$$

that satisfies the initial condition:

$$f(x) \equiv u(x,0) = \sum_{i=1}^{\infty} a_i \, \sin(\lambda_i \, x) \, .$$

- The coefficients a_i in the eigenfunction expansion are found by multiplying both sides of the IC equation by $\sin(\lambda_j \, x)$ and integrating using the orthogonality property, i.e.,

$$\int_0^1 f(x) \, \sin(\lambda_j \, x) \, \mathrm{d}x = \sum_{i=1}^{\infty} a_i \int_0^1 \sin(\lambda_i \, x) \, \sin(\lambda_j \, x) \, \mathrm{d}x$$

Example

Solving a parabolic IBVP by the separation of variables method

Step 1. Separating the PDE into two ODEs.

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Step 3. Expansion of the IC as a sum of eigenfunctions.

- The final solution is such linear combination (with coefficients a_i) of infinite number of fundamental solutions,

$$u(x,t) = \sum_{i=1}^{\infty} a_i\, u_i(x,t) = \sum_{i=1}^{\infty} a_i\, \sin(\lambda_i\, x)\, \exp(-\lambda_i^2\, \alpha^2\, t)\,,$$

that satisfies the initial condition:

$$f(x) \equiv u(x,0) = \sum_{i=1}^{\infty} a_i\, \sin(\lambda_i\, x)\,.$$

- The coefficients a_i in the eigenfunction expansion are found by multiplying both sides of the IC equation by $\sin(\lambda_j\, x)$ and integrating using the orthogonality property, i.e.,

$$\int_0^1 f(x)\, \sin(\lambda_j\, x)\, \mathrm{d}x = a_j \int_0^1 \sin^2(\lambda_j\, x)\, \mathrm{d}x$$

Example

Solving a parabolic IBVP by the separation of variables method

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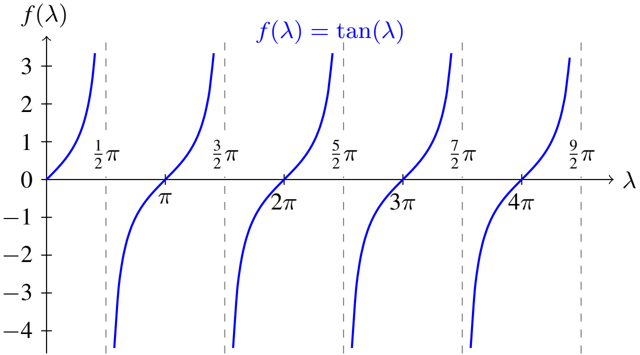
PLOT

$$a_i = \frac{2\lambda_i}{\lambda_i - \sin(\lambda_i) \, \cos(\lambda_i)} \int_0^1 f(x) \, \sin(\lambda_i \, x) \, \mathrm{d}x \,.$$

Example (results for $h = 3$)

Eigenvalues solution



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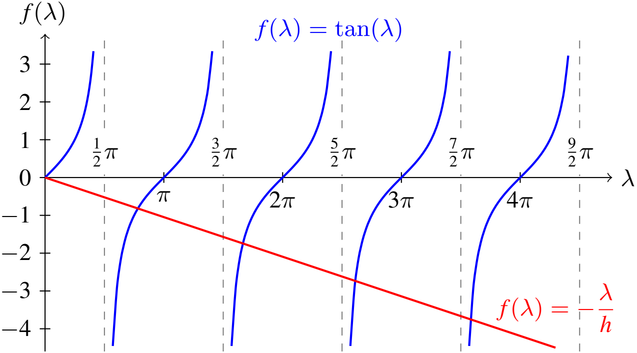

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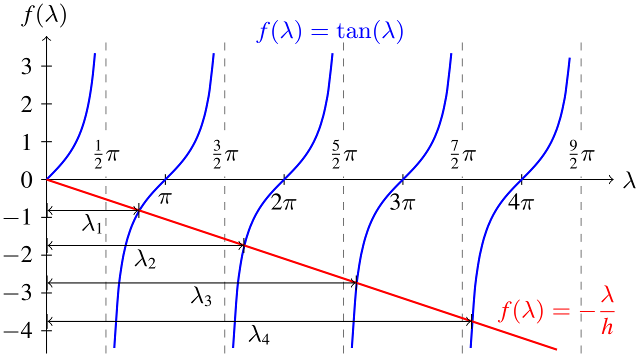
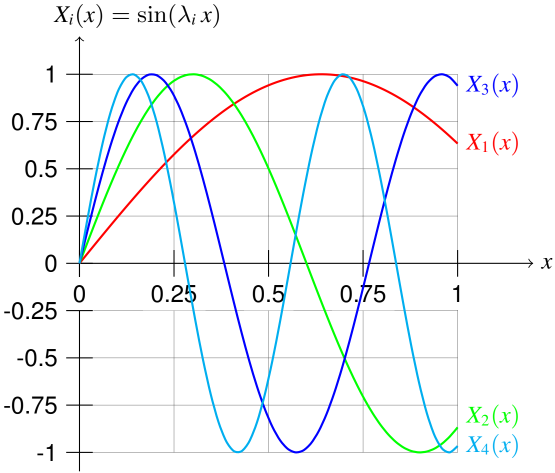


RETURN

Example (results for $h = 3$)

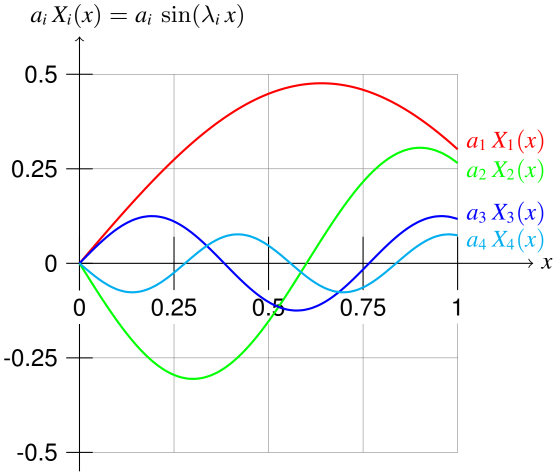
Initial shapes (i.e., $t = 0$) of four fundamental solutions



RETURN

Example (results for $h = 3$, $\alpha = 1$, and $f(x) = x^2$)

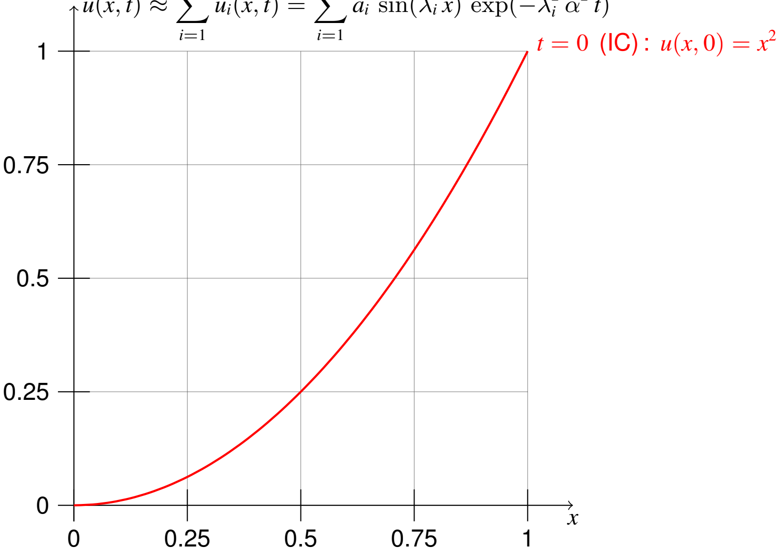
The shapes of four fundamental solutions scaled by the coefficients a_i



Example (results for $h = 3$, $\alpha = 1$, and $f(x) = x^2$)

The final solution. (Notice that $f(x) = x^2$ does not satisfy the BC at $x = 1$.)

$$u(x,t) \approx \sum_{i=1}^{16} u_i(x,t) = \sum_{i=1}^{16} a_i \, \sin(\lambda_i \, x) \, \exp(-\lambda_i^2 \, \alpha^2 \, t)$$



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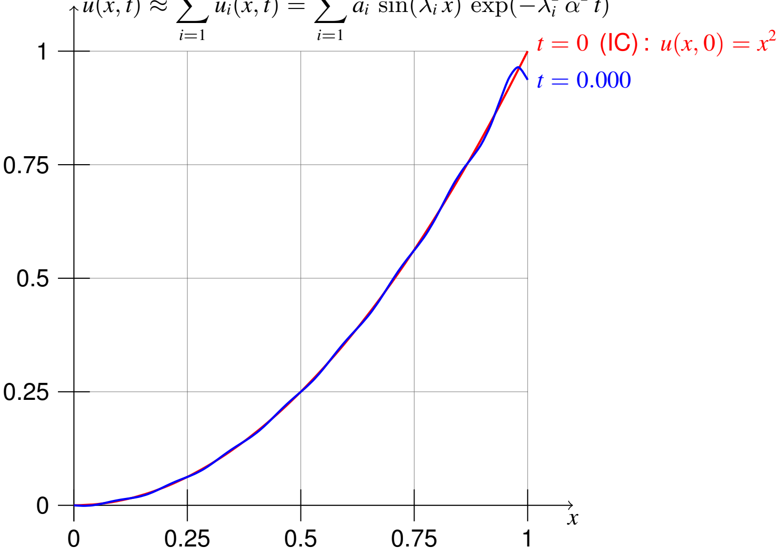

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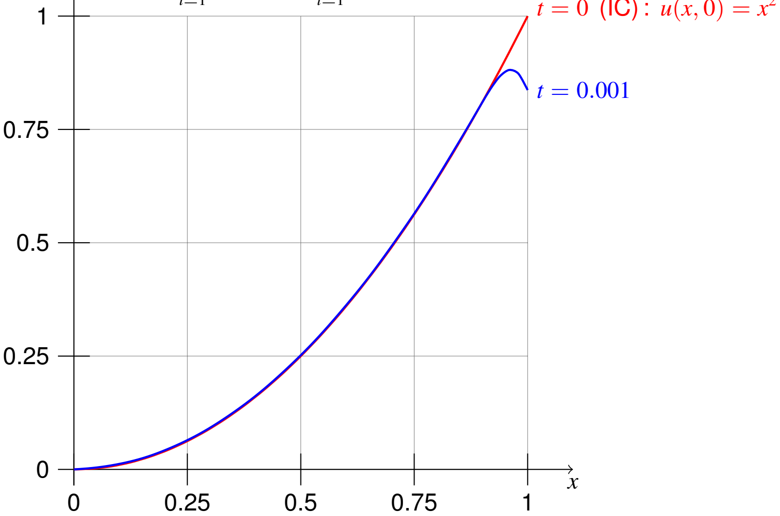

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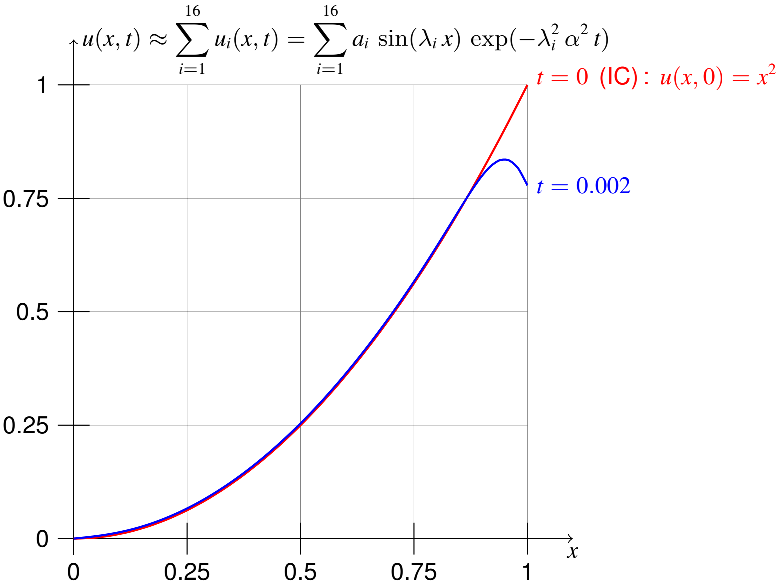

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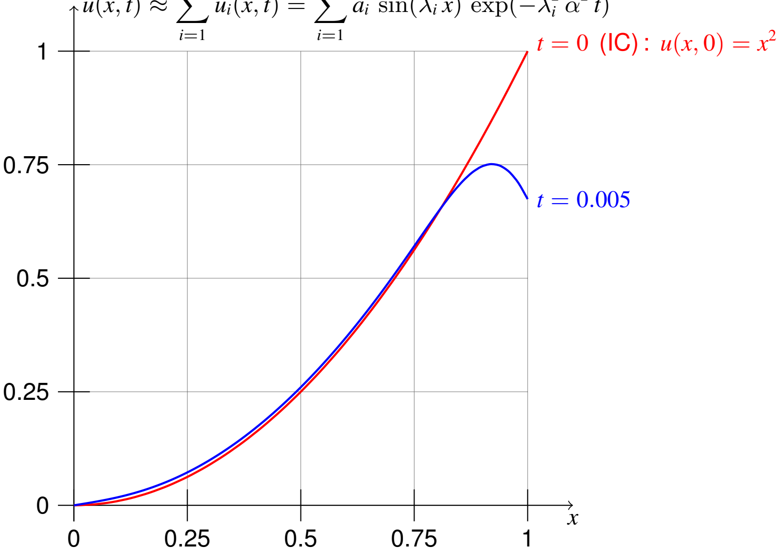

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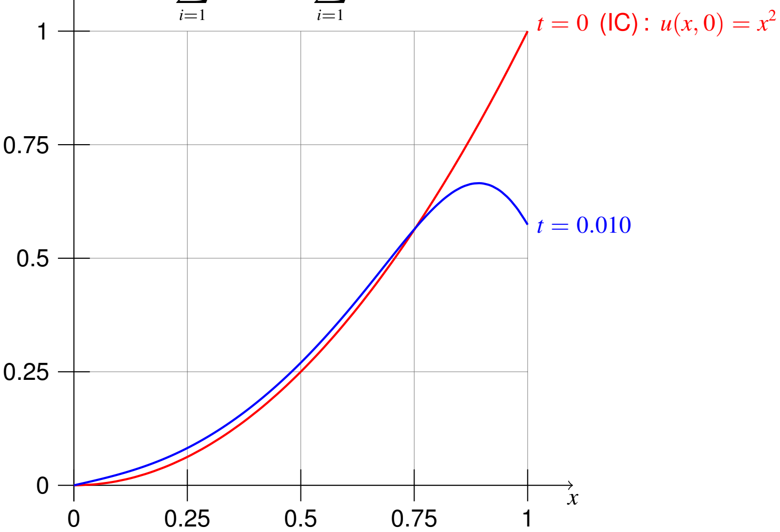

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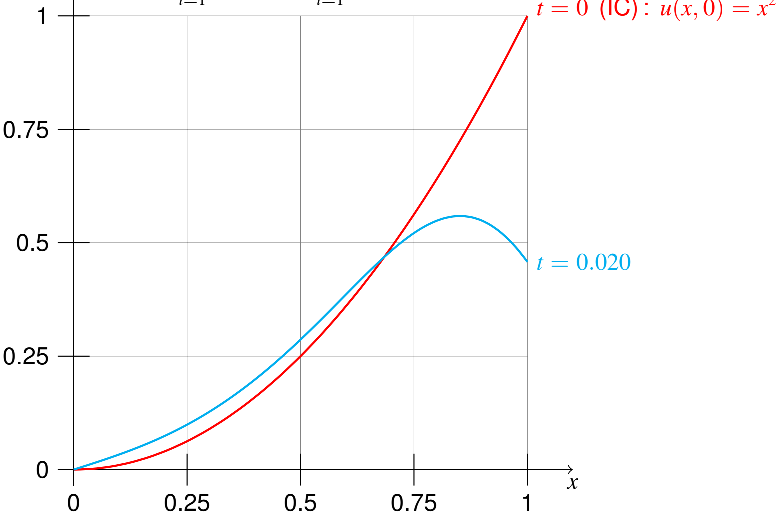

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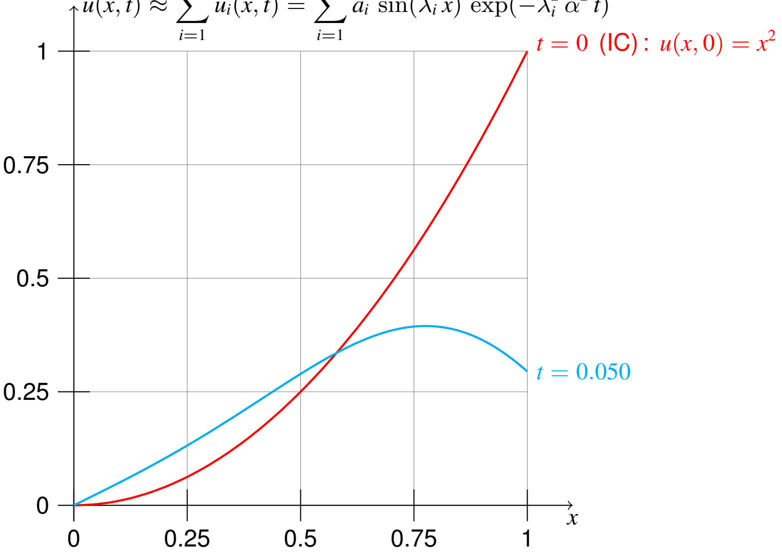

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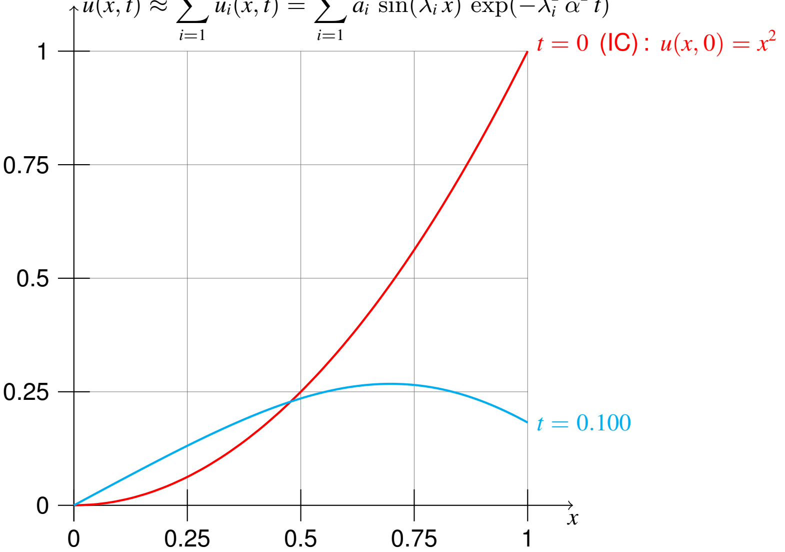

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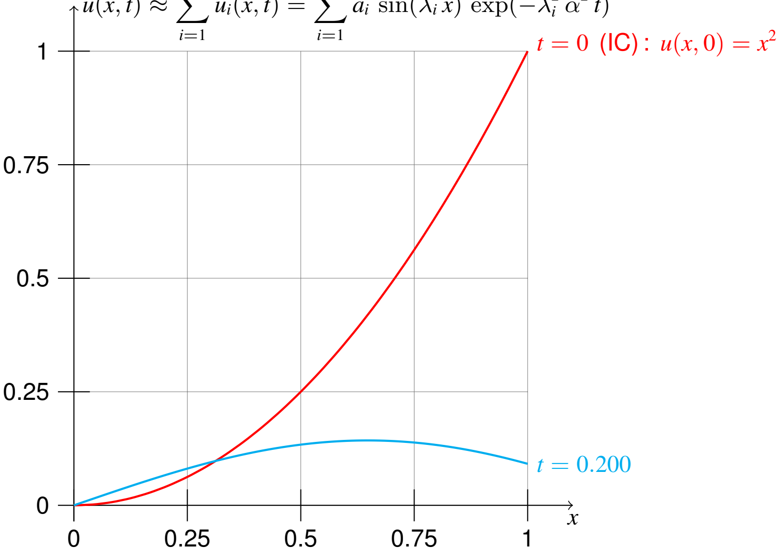

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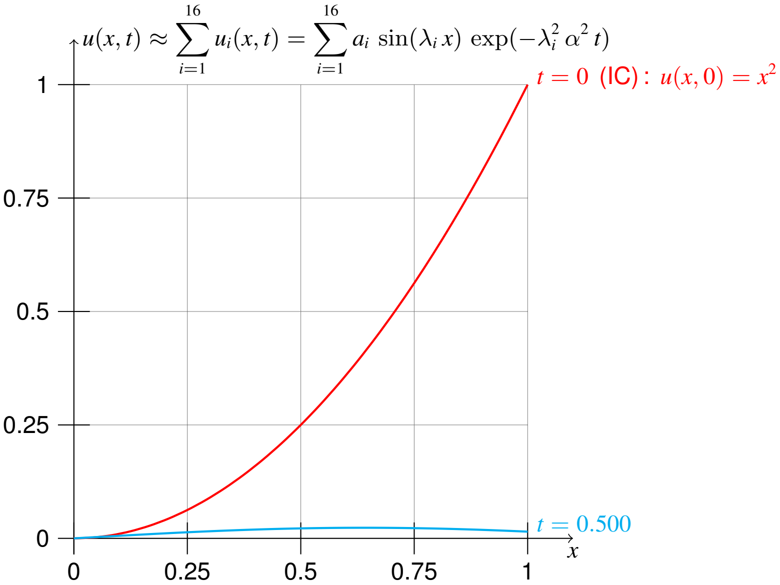

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