Fundamentals of Fluid Dynamics: Ideal Flow Theory & Basic Aerodynamics

Introductory Course on Multiphysics Modelling

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(after: D.J. ACHESON's "*Elementary Fluid Dynamics*")

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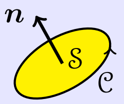
1 Introduction

1.1 Mathematical preliminaries

Theorem 1 (Divergence theorem)**.** Let the region $\mathcal{V}$ be bounded by a simple surface $\mathcal{S}$ with unit outward normal $\boldsymbol{n}$. Then:

$$\int_{\mathcal{S}} \boldsymbol{f} \cdot \boldsymbol{n} \, \mathrm{d}\mathcal{S} = \int_{\mathcal{V}} \nabla \cdot \boldsymbol{f} \, \mathrm{d}\mathcal{V} \,; \quad \text{in particular} \quad \int_{\mathcal{S}} f \, \boldsymbol{n} \, \mathrm{d}\mathcal{S} = \int_{\mathcal{V}} \nabla f \, \mathrm{d}\mathcal{V} \,. \tag{1}$$

Theorem 2 (Stokes' theorem)**.** Let $\mathcal{C}$ be a simple closed curve spanned by a surface $\mathcal{S}$ with unit normal $\boldsymbol{n}$. Then:



$$\int_{\mathcal{C}} \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{x} = \int_{\mathcal{S}} (\nabla \times \boldsymbol{f}) \cdot \boldsymbol{n} \, \mathrm{d}\mathcal{S} \,. \tag{2}$$

Green's theorem in the plane may be viewed as a special case of Stokes' theorem (with $\boldsymbol{f} = \big[u(x,y),\ v(x,y),\ 0\big]$):

$$\int_{\mathcal{C}} u \, \mathrm{d}x + v \, \mathrm{d}y = \int_{\mathcal{S}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y \,. \tag{3}$$

1.2 Basic notions and definitions

A usual way of describing a fluid flow is by means of the **flow velocity** defined at any point $\boldsymbol{x} = (x, y, z)$ and at any time t:

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t) = \big[u(\boldsymbol{x}, t),\ v(\boldsymbol{x}, t),\ w(\boldsymbol{x}, t)\big]. \tag{4}$$

Here, u, v, w are the **velocity components** in Cartesian coordinates.

Definition 1 (Steady flow)**.** A **steady flow** is one for which

$$\frac{\partial \boldsymbol{u}}{\partial t} = 0 \,, \quad \text{that is,} \quad \boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}) = \big[u(\boldsymbol{x}),\ v(\boldsymbol{x}),\ w(\boldsymbol{x})\big] \,. \tag{5}$$

Definition 2 (Two-dimensional flow)**.** A **two-dimensional flow** is of the form

$$\boldsymbol{u} = \big[u(\boldsymbol{x}, t),\ v(\boldsymbol{x}, t),\ 0\big] \quad \text{where} \quad \boldsymbol{x} = (x, y) \,. \tag{6}$$

Definition 3 (Two-dimensional steady flow)**.** A **two-dimensional steady flow** is of the form

$$\boldsymbol{u} = \big[u(\boldsymbol{x}),\ v(\boldsymbol{x}),\ 0\big] \quad \text{where} \quad \boldsymbol{x} = (x, y) \,. \tag{7}$$

Definition 4 (Streamline)**.** A **streamline** is a curve which, at any particular time t, has the same direction as $\boldsymbol{u}(\boldsymbol{x},t)$ at each point. A streamline $x = x(s)$, $y = y(s)$, $z = z(s)$ (s is a parameter) is obtained by solving at a particular time t:

$$\frac{\frac{\mathrm{d}x}{\mathrm{d}s}}{u} = \frac{\frac{\mathrm{d}y}{\mathrm{d}s}}{v} = \frac{\frac{\mathrm{d}z}{\mathrm{d}s}}{w}\,. \tag{8}$$

Remarks:

- For a steady flow the streamline pattern is the same at all times, and fluid particles travel along them.
- In an unsteady flow, streamlines and particle paths are usually quite different.

Example 1. Consider a two-dimensional flow described as follows

$$u(\boldsymbol{x},t) = u_0\,, \qquad v(\boldsymbol{x},t) = a\,t\,, \qquad w(\boldsymbol{x},t) = 0\,,$$

where u_0 and a are positive constants. Now, notice that (see Figure 1):

- in this flow streamlines are (always) straight lines,
- fluid particles follow parabolic paths:

$$x(t) = u_0\,t\,, \qquad y(t) = \tfrac{a\,t^2}{2} \qquad \to \qquad y = \tfrac{a\,x^2}{2u_0^2}\,.$$

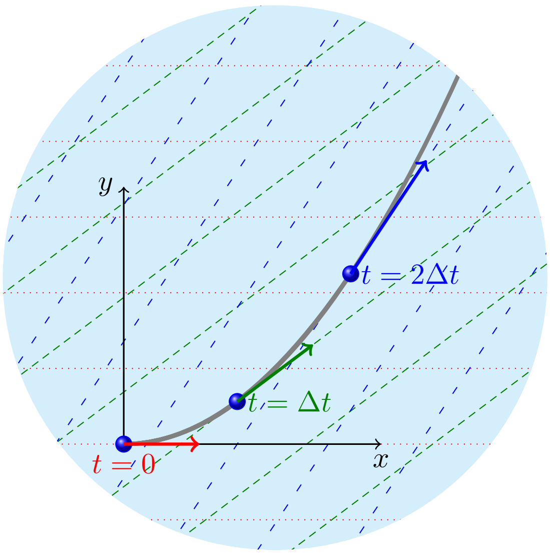


Figure 1: Streamlines and mutually proportional velocity vectors for $t = 0$ (in red), $t = \Delta t$ (in green), and $t = 2\Delta t$ (in blue), and the particle path (in gray)

1.3 Convective derivative

- Let $f(\boldsymbol{x},t)$ denote some **quantity of interest** in the fluid motion.
- Note that $\dfrac{\partial f}{\partial t}$ means the **rate of change** of f at a **fixed position** $\boldsymbol{x}$ in space.

- The **rate of change** of f "***following the fluid***" is

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t}\, f\big(\boldsymbol{x}(t), t\big) = \frac{\partial f}{\partial \boldsymbol{x}}\,\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} + \frac{\partial f}{\partial t} \tag{9}$$

where $\boldsymbol{x}(t) = \big[x(t),\ y(t),\ z(t)\big]$ is understood to change with time at the local flow velocity $\boldsymbol{u}$, so that

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \left[\frac{\mathrm{d}x}{\mathrm{d}t},\ \frac{\mathrm{d}y}{\mathrm{d}t},\ \frac{\mathrm{d}z}{\mathrm{d}t}\right] = \big[u,\ v,\ w\big]\,. \tag{10}$$

Therefore,

$$\frac{\partial f}{\partial \boldsymbol{x}}\,\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\partial f}{\partial x}\,\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\,\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\,\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}\,u + \frac{\partial f}{\partial y}\,v + \frac{\partial f}{\partial z}\,w = (\boldsymbol{u}\cdot\nabla) f\,. \tag{11}$$

Definition 5 (Convective derivative)**.** The **convective derivative** (a.k.a. the **material**, **particle**, **Lagrangian**, or **total-time derivative**) is a derivative taken with respect to a coordinate system moving with velocity $\boldsymbol{u}$ (i.e., following the fluid):

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + (\boldsymbol{u}\cdot\nabla) f\,. \tag{12}$$

By applying the convective derivative to the velocity components u, v, w in turn it follows that the **acceleration of a fluid particle** is

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\,. \tag{13}$$

Example 2. Consider fluid in uniform rotation with angular velocity Ω, so that:

$$u = -\Omega\, y\,, \qquad v = \Omega\, x\,, \qquad w = 0\,.$$

The flow is steady so $\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{0}$, but

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \left(-\,\Omega\, y\,\frac{\partial}{\partial x} + \Omega\, x\,\frac{\partial}{\partial y}\right)\big[-\,\Omega\, y,\ \Omega\, x,\ 0\big] = -\Omega^2\,\big[x,\ y,\ 0\big]\,.$$

represents the **centrifugal acceleration** $\Omega^2\sqrt{x^2+y^2}$ towards the rotation axis.

2 Ideal flow theory

2.1 Ideal fluid

Definition 6 (Ideal fluid)**.** Properties of an **ideal fluid**:

1. it is **incompressible**: no fluid element can change in volume as it moves;
2. it has **constant density**: ϱ is the same for all fluid elements and for all time t (a consequence of incompressibility);

3. it is **inviscid**, so that the force exerted across a geometrical surface element $\boldsymbol{n}\,\delta\mathcal{S}$ within the fluid is

$$p\,\boldsymbol{n}\,\delta\mathcal{S}\,,$$

where $p(\boldsymbol{x},t)$ is a scalar function of **pressure**, independent on the normal $\boldsymbol{n}$.

Remarks:

- All fluids are *to some extent* compressible and viscous.
- **Air**, being **highly compressible**, can behave like an **incompressible** fluid if the **flow speed** is **much smaller** than the **speed of sound**.

2.2 Incompressibility condition

The incompressible flow means that every elementary material volume of fluid remains constant throughout the flow. Mathematically, this is ensured by the incompressibility condition derived below.

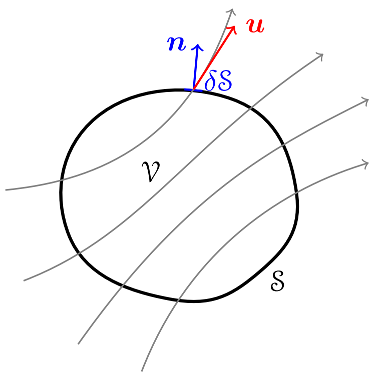


FIGURE 2: Flow thorough a closed surface

- Let $\mathcal{S}$ be a fixed closed surface drawn in the fluid, with unit outward normal $\boldsymbol{n}$, enclosing a region $\mathcal{V}$ (see Figure 2).
- Fluid enters the enclosed region $\mathcal{V}$ at some places on $\mathcal{S}$, and leaves it at others.
- The velocity component along the outward normal is $\boldsymbol{u}\cdot\boldsymbol{n}$, so the volume of fluid leaving through a small surface element $\delta\mathcal{S}$ in unit time is $\boldsymbol{u}\cdot\boldsymbol{n}\,\delta\mathcal{S}$.
- The net volume rate at which fluid is leaving $\mathcal{V}$ equals

$$\int_{\mathcal{S}} \boldsymbol{u}\cdot\boldsymbol{n}\;\mathrm{d}\mathcal{S}$$

and must be zero for an incompressible fluid.

Now, using the divergence theorem and the assumption of the "smoothness" of flow (i.e., continuous velocity gradient) gives the condition for incompressible flow.

Incompressibility condition

For all regions within an incompressible fluid

$$\int_{\mathcal{S}} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}\mathcal{S} = \int_{\mathcal{V}} \nabla \cdot \boldsymbol{u} \, \mathrm{d}\mathcal{V} = 0 \tag{14}$$

which (assuming that $\nabla \cdot \boldsymbol{u}$ is continuous) yields the following important constraint on the velocity field:

$$\boxed{\nabla \cdot \boldsymbol{u} = 0} \tag{15}$$

(everywhere in the fluid).

2.3 Euler's equations of motion

The **net force** exerted on an arbitrary fluid element is

$$-\int_{\mathcal{S}} p \, \boldsymbol{n} \, \mathrm{d}\mathcal{S} = -\int_{\mathcal{V}} \nabla p \, \mathrm{d}\mathcal{V} \tag{16}$$

(the negative sign arises because $\boldsymbol{n}$ points out of $\mathcal{S}$). Now, provided that ∇p is continuous it will be almost constant over a *small* element $\delta\mathcal{V}$. The **net force** on such a small element due to the pressure of the surrounding fluid will therefore be

$$-\nabla p \, \delta\mathcal{V} \, .$$

The **principle of linear momentum** implies that the following forces must be equal for a fluid element of volume $\delta\mathcal{V}$:

- $\big(-\nabla p + \varrho \, \boldsymbol{g}\big)\delta\mathcal{V}$ – the **total (external) net force** acting on the element in the presence of gravitational body force per unit mass $\boldsymbol{g}$ $\left[\frac{\mathrm{N}}{\mathrm{kg}} = \frac{\mathrm{m}}{\mathrm{s}^2}\right]$ (the gravity acceleration),
- $\varrho \, \delta\mathcal{V} \, \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}$ – the **inertial force**, that is, the product of the element's mass (which is conserved) and its acceleration.

This results in the **Euler's momentum equation** for an ideal fluid. Another equation of motion is the incompressibility constraint (15).

Euler's equations of motion for an ideal fluid

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\varrho}\nabla p + \boldsymbol{g}\,, \qquad \nabla\cdot\boldsymbol{u} = 0\,. \tag{17}$$

2.4 Boundary and interface-coupling conditions

Impermeable surface. The fluid cannot flow through the boundary which means that the normal velocity is constrained (no-penetration condition):

$$\boldsymbol{u}\cdot\boldsymbol{n} = \hat{u}_n\,. \tag{18}$$

Here, $\hat{u}_n$ is the prescribed normal velocity of the impermeable boundary ($\hat{u}_n = 0$ for motionless, rigid boundary).

Free surface. Pressure condition (involving surface tension effects):

$$p = p_0 \pm 2T\,\kappa_{\text{mean}}\,. \tag{19}$$

Here: p_0 is the ambient pressure,
T is the surface tension,
κ_{mean} is the local mean curvature of the free surface.

Fluid interface. The continuity of normal velocity and pressure at the interface between fluids 1 and 2:

$$\big(\boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}\big)\cdot\boldsymbol{n} = 0\,, \qquad p^{(1)} - p^{(2)} = 0 \pm 2\big(T^{(1)} + T^{(2)}\big)\kappa_{\text{mean}}\,. \tag{20}$$

3 Vorticity of flow

3.1 Bernoulli theorems

The **gravitational force**, being **conservative**, can be written as the **gradient of a potential**:

$$\boldsymbol{g} = -\nabla\chi \qquad \text{where} \qquad \chi = g\,z \quad \left[\tfrac{\mathrm{m}^2}{\mathrm{s}^2}\right], \tag{21}$$

and the momentum equation

$$\frac{\partial\boldsymbol{u}}{\partial t} + \underbrace{(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}}_{(\nabla\times\boldsymbol{u})\times\boldsymbol{u}+\nabla(\frac{1}{2}\boldsymbol{u}^2)} = -\nabla\left(\frac{p}{\varrho} + \chi\right) \tag{22}$$

can be cast into the following form

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} = -\nabla H \qquad \text{where} \qquad H = \frac{p}{\varrho} + \frac{1}{2}\boldsymbol{u}^2 + \chi\,. \tag{23}$$

For a **steady flow** ($\frac{\partial \boldsymbol{u}}{\partial t} = \mathbf{0}$): $(\nabla \times \boldsymbol{u}) \times \boldsymbol{u} = -\nabla H \quad \xrightarrow{\boldsymbol{u}\cdot} \quad \boxed{(\boldsymbol{u} \cdot \nabla)\, H = 0}$.

Theorem 3 (Bernoulli streamline theorem)**.** If an ideal fluid is in steady flow, then H is constant along a streamline.

Definition 7 (Irrotational flow)**.** A flow is **irrotational** if $\nabla \times \boldsymbol{u} = \mathbf{0}$.

For a **steady irrotational flow**: $\boxed{\nabla H = \mathbf{0}}$.

Theorem 4 (Bernoulli theorem for irrotational flow)**.** If an ideal fluid is in steady irrotational flow, then H is constant throughout the whole fluid.

3.2 Vorticity

Definition 8 (Vorticity)**. Vorticity** is a concept of central importance in fluid dynamics; it is defined as

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \qquad \left[\tfrac{1}{\mathrm{s}}\right]. \tag{24}$$

Remarks:

- For an **irrotational flow**: $\boldsymbol{\omega} = \mathbf{0}$.
- Vorticity has nothing directly to do with any *global* rotation of the fluid.

Interpretation of vorticity (in 2D)

For two-dimensional flow (when $\boldsymbol{u} = \big[u(\boldsymbol{x},t),\ v(\boldsymbol{x},t),\ 0\big]$):

$$\boldsymbol{\omega} = \big[0,\ 0,\ \omega\big] \quad \text{where} \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,. \tag{25}$$

Then, at any point of the flow field $\frac{\omega}{2}$ represents the **average angular velocity** of two short fluid line-elements (AB and BC, see Figure 3) that happen, at that instant, to be mutually perpendicular.

Therefore, the vorticity ω acts as a **measure of the local rotation**, or spin, of fluid elements.

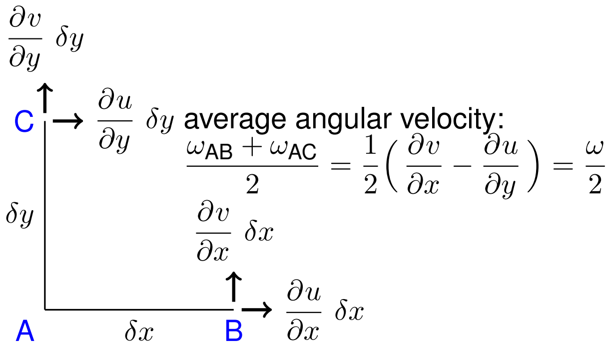


FIGURE 3: Interpretation of vorticity in 2D flow. The velocity components are relative to the fluid particle at A.

3.3 Cylindrical flows

Flow may be written in cylindrical polar coordinates (r, θ, z):

$$\boldsymbol{u} = u_r\, \boldsymbol{e}_r + u_\theta\, \boldsymbol{e}_\theta + u_z\, \boldsymbol{e}_z\,. \tag{26}$$

Consider the following steady, two-dimensional flows with $u_\theta = u_\theta(r)$ and $u_r = u_z = 0$ (here, Ω [1/s] and a [m] are constants):

Line vortex flow: $\boldsymbol{u} = \frac{\Omega\, a^2}{r} \boldsymbol{e}_\theta$ (see Figure 4(*bottom left*)) – although the fluid is clearly **rotating in a global sense** the flow is in fact **irrotational** since $\boldsymbol{\omega} = \boldsymbol{0}$ (except at $r = 0$, where neither $\boldsymbol{u}$ nor $\boldsymbol{\omega}$ is defined);

Uniformly rotating flow: $\boldsymbol{u} = \Omega\, r\, \boldsymbol{e}_\theta$ (see Figure 4(*bottom right*)) – here, $\boldsymbol{\omega} = [0,\ 0,\ 2\Omega]$ and a "vorticity meter" is carried around as if it were embedded in a rigid body.

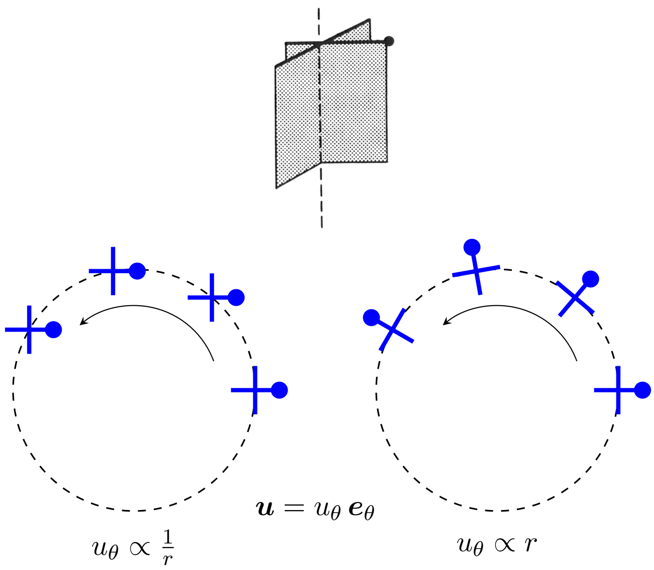


FIGURE 4: A crude "vorticity meter" (*top*) and its behaviour when immersed in a line vortex flow (*bottom left*) and a uniformly rotating flow (*bottom right*)

3.4 Rankine vortex

Definition 9 (Rankine vortex)**. Rankine vortex** is a steady, two-dimensional flow described as

$$\boldsymbol{u} = u_\theta \, \boldsymbol{e}_\theta \quad \text{with} \quad u_\theta = \begin{cases} \Omega \, r & \text{for } r \le a \text{ (uniformly rotating flow)}, \\ \frac{\Omega \, a^2}{r} & \text{for } r > a \text{ (line vortex flow)}, \end{cases} \tag{27}$$

where Ω and a are constants. Therefore,

$$\boldsymbol{\omega} = \omega \, \boldsymbol{e}_z \quad \text{with} \quad \omega = \frac{1}{r}\frac{\partial (r \, u_\theta)}{\partial r} \begin{cases} \frac{1}{r}\frac{\partial(\Omega \, r^2)}{\partial r} = 2\,\Omega & \text{for } r \le a \text{ (\textit{vortex core})}, \\ \frac{1}{r}\frac{\partial(\Omega \, a^2)}{\partial r} = 0 & \text{for } r > a \text{ (irrotational)}. \end{cases} \tag{28}$$

Figure 5 shows the distribution of u_θ and ω in such a vortex.

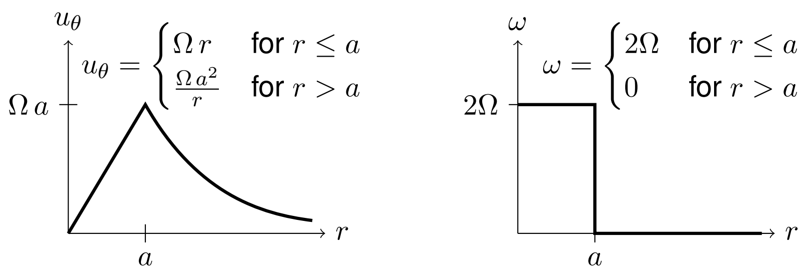


FIGURE 5: Distribution of azimuthal velocity u_θ and vorticity ω in a Rankine vortex

- The **Rankine vortex** serves as a simple **idealized model** for a real vortex.
- **Real vortices** are typically characterized by fairly **small vortex *cores*** (a is small) in which, by definition, the **vorticity** is **concentrated**.
- **Outside the core** the flow is **irrotational**.

3.5 Vorticity equation

Take on the curl of the (Euler's) momentum equation:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} = -\nabla H \qquad \xrightarrow{\nabla\times} \qquad \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \boldsymbol{0} \tag{29}$$

which eventually becomes

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\,\boldsymbol{u} + \underbrace{\boldsymbol{\omega}\,\nabla \cdot \boldsymbol{u}}_{\boldsymbol{0}} - \underbrace{\boldsymbol{u}\,\nabla \cdot \boldsymbol{\omega}}_{\boldsymbol{0}} = \boldsymbol{0}\,. \tag{30}$$

Here, the fourth term vanishes because the fluid is incompressible while the fifth term vanishes because $\nabla \cdot \nabla \times = 0$. All that leads to the following equation.

Vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\, \boldsymbol{u}\,, \quad \text{or} \quad \frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \nabla)\, \boldsymbol{u}\,. \tag{31}$$

The vorticity equation is extremely valuable: as a matter of fact, it involves only $\boldsymbol{u}$ since the pressure has been eliminated and $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$.

In the **two-dimensional flow** ($\boldsymbol{u} = \big[u(\boldsymbol{x},t),\ v(\boldsymbol{x},t),\ 0\big]$, $\boldsymbol{\omega} = \big[0,\ 0,\ \omega(\boldsymbol{x},t)\big]$) of an ideal fluid subject to a conservative body force $\boldsymbol{g}$ the **vorticity** ω of each individual fluid element is **conserved**:

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = 0\,. \tag{32}$$

In the **steady, two-dimensional flow** of an ideal fluid subject to a conservative body force $\boldsymbol{g}$ the **vorticity** ω is **constant along a streamline**:

$$(\boldsymbol{u} \cdot \nabla)\, \omega = 0\,. \tag{33}$$

4 Basic aerodynamics

4.1 Steady flow past a fixed wing

Steady flow past a wing at **small angle of attack** (incidence) is typically **irrotational**. This results from equation (33) and Figure 6 as discussed below.

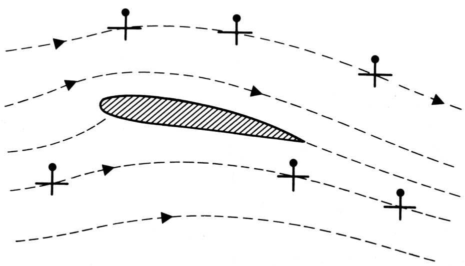


FIGURE 6: Irrotational flow past a fixed wing at small angle of attack.

- There are **no regions of closed streamlines** in the flow (Figure 6); all the streamlines can be traced back to $x \to -\infty$.
- The **vorticity is constant along each streamline**, cf. Equation (33), and hence equal on each one to whatever it is on that particular streamline at $x \to -\infty$.
- As the flow is uniform at $x \to -\infty$, the **vorticity is zero** on all streamlines there; hence, it is zero throughout the flow field **around the wing**.

Figure 7 shows typical measured pressure distribution on the upper and lower surfaces of a fixed wing in steady flow. One should notice what follows:

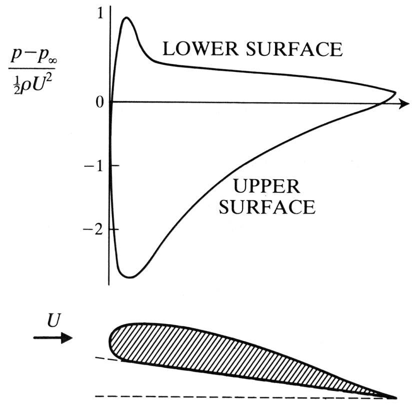


FIGURE 7: Typical pressure distribution on a wing in steady flow

- the pressures on the **upper surface** are substantially **lower** than the free-stream value p_∞;
- the pressures on the **lower surface** are a little **higher** than p_∞;
- in fact, the wing gets most of its **lift** from a **suction effect** on its **upper surface**.

Why the pressures above the wing are less than those below?

- The flow is irrotational and the **Bernoulli theorem** states that $\varrho H = p + \frac{1}{2}\varrho \boldsymbol{u}^2$ is **constant throughout 2D irrotational flows.**
- Explaining the pressure differences, and hence the lift on the wing, thus reduces to explaining why the **flow speeds above the wing are greater than those below**.
- An explanation is in terms of the **concept of circulation.**

4.2 Fluid circulation round a wing

Definition 10 (Circulation)**.** The **circulation** Γ round some closed curve $\mathcal{C}$ lying in the fluid region is defined as

$$\Gamma = \int_{\mathcal{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} \quad \left[\frac{\mathrm{m}^2}{\mathrm{s}}\right]. \tag{34}$$

If $\mathcal{S}$ is the region enclosed by the curve $\mathcal{C}$ then the **Stokes' theorem** gives

$$\Gamma = \int_{\mathcal{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \int_{\mathcal{S}} (\nabla \times \boldsymbol{u}) \cdot \boldsymbol{n} \, \mathrm{d}\mathcal{S}\,, \tag{35}$$

or in the two-dimensional context

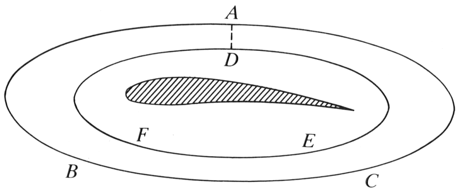
$$\Gamma = \int_{\mathcal{C}} u \, \mathrm{d}x + v \, \mathrm{d}y = \int_{\mathcal{S}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y \,. \tag{36}$$

Notice that in the surface integrals vorticity terms appear.

Corollaries. The following statements can be formulated about the fluid circulation round a wing:

- $\Gamma = 0$ if the closed curve $\mathcal{C}$ is spanned by a surface $\mathcal{S}$ which lies *wholly* in the region of irrotational flow, that is, $\Gamma = 0$ **for any closed curve not enclosing the wing**.
- This cannot be stated **for any closed curve that does enclose the wing**. What can be stated is that **such circuits have the same value of** Γ.

  Proof: Consider two *arbitrary* closed curves, ABCA and DFED, enclosing a wing.



$$\begin{aligned} 0 &= \Gamma_{ABCADEFDA} \\ &= \Gamma_{ABCA} + \Gamma_{AD} + \Gamma_{DEFD} + \Gamma_{DA} \\ &= \Gamma_{ABCA} + \Gamma_{AD} - \Gamma_{DFED} - \Gamma_{AD} \qquad \longrightarrow \qquad \Gamma_{ABCA} = \Gamma_{DFED} \end{aligned}$$

QED

- Therefore:

  circulation round a wing is permissible
  in a steady irrotational flow.

- However, two questions still arise:
  1. Why there should be any circulation?
  2. Why it should be negative, corresponding to larger flow speeds above the wing than below?

  The answers are given by the **Kutta–Joukowski condition**.

4.3 Kutta–Joukowski theorem and condition

- Consider a steady, irrotational flow of fluid round a wing.
- According to ideal flow theory, the drag on the wing (the force per unit length of wing parallel to the oncoming stream) is zero.

- What is the lift of the wing (i.e., the force per unit length of wing perpendicular to the stream) is stated by the following theorem.

Theorem 5 (Kutta–Joukowski lift theorem)**.** Let ϱ be the fluid density and U the flow speed at infinity. Then, the **lift of the wing** is

$$F_y = -\varrho\, U\, \Gamma \qquad \left[\tfrac{\mathrm{N}}{\mathrm{m}}\right], \tag{37}$$

where Γ is the fluid circulation around the wing.

Obviously, of a great importance for the lift force is the fact that $\Gamma \neq 0$. In the case of a **wing with a sharp trailing edge** this can be explained as follows: a good reason for non-zero circulation Γ is that otherwise there would be a singularity (infinity) in the velocity field (see Figure 8). This is stated by the **Kutta–Joukowski condition**.

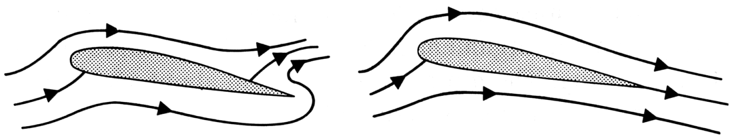


FIGURE 8: Irrotational flows past a fixed wing.

Kutta–Joukowski condition (hypothesis)

- The circulation is such that **the flow leaves the trailing edge smoothly**, or, equivalently, that **the flow speed at the trailing edge is finite**.
- The **flow speed is finite** at the trailing edge **only for one value** of the circulation around the wing: the *critical value* Γ_{K}. This particular flow will correspond to the steady flow that is actually observed.

The critical value Γ_{K} depends on the flow speed at infinity U, and on the size, shape, and orientation of the wing.

Thin, symmetrical wings

For a **thin and symmetrical wing** of length L, making an angle α with the oncoming stream, the **critical value** of circulation is

$$\Gamma_{\mathsf{K}} \approx -\pi\, U\, L\, \sin\alpha\,. \tag{38}$$

Using this formula for the lift theorem gives the following result

$$F_y \approx \pi\, \varrho\, U^2\, L\, \sin\alpha\,, \tag{39}$$

which is in excellent accord with experiment provided that the **angle of attack** α **is small**, that is only a few degrees, depending on the shape of the wing (see Figure 9).

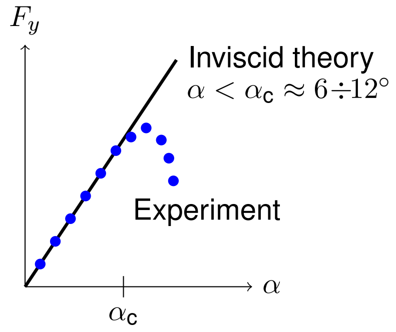


FIGURE 9: Lift on a symmetric aerofoil

4.4 Concluding remarks

- Kutta–Joukowski hypothesis provides a rational explanation for the circulation round a wing in steady flight.
- It says nothing about the dynamical process by which that circulation is *generated* when a wing starts from a state of rest.

> ***Starting vortex***
>
> The circulation is generated by the so-called **starting vortex** (see Figure 10), which is **a concentration of vorticity which forms at the trailing edge** of a wing as it accelerates from rest in a fluid. It leaves the wing (which now has an equal but opposite 'bound vortex' round it), and rapidly decays through the action of viscosity.

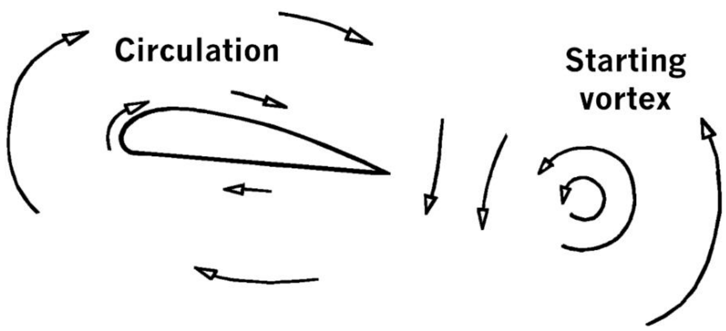


FIGURE 10: A starting vortex.

Question: *Is a starting vortex theoretically explicable?*

Answer: *Not on the basis of ideal flow theory.*

Legitimate conclusions on the basis of ideal flow theory:

- If the wing and fluid are initially at rest, the vorticity is initially zero for each fluid element.
- It remains zero since the vorticity is conserved for each fluid element.
- Therefore, there should be no starting vortex.

An explanation of the starting vortex

- **Ideal flow theory** accounts well for the **steady flow** past a wing.
- The explanation of how that flow became established involves **viscous effects** in a crucial way.
- But air, in some sense, is hardly viscous at all! Yet, viscous effects are sufficiently subtle that **shedding of the vortex**, which is an essentially viscous process, **occurs no matter how small the viscosity** of the fluid happened to be.