Galerkin Finite Element Model for Heat Transfer

Introductory Course on Multiphysics Modelling

TOMASZ G. ZIELIŃSKI

bluebox.ippt.pan.pl/~tzielins/

Institute of Fundamental Technological Research
of the Polish Academy of Sciences
Warsaw • Poland

Outline

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Notation remarks

- The **index notation** is used with summation over the index i.
- Consequently, the **summation rule** is also applied for the approximation expressions, that is, over the indices $r, s = 1, \dots N$ (where N is the number of degrees of freedom).
- The symbol $(\dots)_{|i}$ means a (generalized) **invariant partial differentiation** over the i-th coordinate:

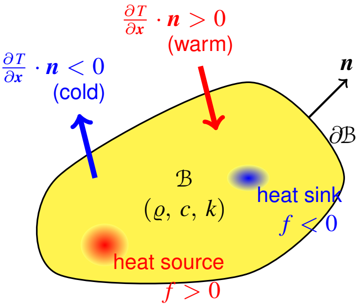
  $$(\dots)_{|i} = \frac{\partial(\dots)}{\partial x_i}\,.$$

  The invariance involves the so-called Christoffel symbols (in the case of curvilinear systems of reference).
- Symbols $\mathrm{d}V$ and $\mathrm{d}S$ are completely omitted in all the integrals presented below since it is obvious that one integrates over the specified domain or boundary. Therefore, one should understand that:

  $$\int\limits_{\mathcal{B}} (\dots) = \int\limits_{\mathcal{B}} (\dots)\,\mathrm{d}V(\boldsymbol{x})\,, \qquad \int\limits_{\partial\mathcal{B}} (\dots) = \int\limits_{\partial\mathcal{B}} (\dots)\,\mathrm{d}S(\boldsymbol{x})\,.$$

Outline

PDE for Heat Transfer Problem



- **Material data:**
  $\varrho = \varrho(\boldsymbol{x})$ – the density $\left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right]$
  $c = c(\boldsymbol{x})$ – the thermal capacity $\left[\frac{\mathrm{J}}{\mathrm{kg\cdot K}}\right]$
  $k = k(\boldsymbol{x})$ – the thermal conductivity $\left[\frac{\mathrm{W}}{\mathrm{m\cdot K}}\right]$
- **Known fields:**
  $f = f(\boldsymbol{x}, t)$ – the heat production rate $\left[\frac{\mathrm{W}}{\mathrm{m}^3}\right]$
  $u_i = u_i(\boldsymbol{x}, t)$ – the convective velocity $\left[\frac{\mathrm{m}}{\mathrm{s}}\right]$
- **The unknown field:**
  $T = T(\boldsymbol{x}, t) = ?$ – the temperature $[\mathrm{K}]$

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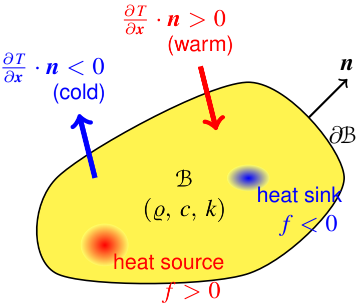

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Heat transfer equation

$\varrho\, c\, \dot{T} + q_{i|i} - f = 0$ where the heat flux vector $\left[\frac{\mathrm{W}}{\mathrm{K}}\right]$:

$$q_i = q_i(T) = \begin{cases} -k\, T_{|i} & \text{– for conduction (only),} \\ -k\, T_{|i} + \varrho\, c\, u_i\, T & \text{– for conduction and convection,} \end{cases}$$

and $\dot{T} = \frac{\partial T}{\partial t}$ is the time rate of change of temperature $\left[\frac{\mathrm{K}}{\mathrm{s}}\right]$.

Initial and boundary conditions

The initial condition (at $t = t_0$)

- $T(\boldsymbol{x}, t_0) = T_0(\boldsymbol{x})$ in $\mathcal{B}$

Prescribed field:

$T_0 = T_0(\boldsymbol{x})$ – the initial temperature $[\mathrm{K}]$

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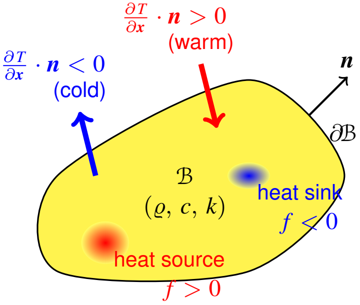
The boundary conditions (on $\partial\mathcal{B}$)

- the Dirichlet type:
  $T(\boldsymbol{x}, t) = \hat{T}(\boldsymbol{x}, t)$ on $\partial\mathcal{B}_T$
- the Neumann type:
  $-q_i\big(T(\boldsymbol{x}, t)\big)\, n_i = \hat{q}(\boldsymbol{x}, t)$ on $\partial\mathcal{B}_q$

Prescribed fields:

$\hat{T} = \hat{T}(\boldsymbol{x}, t)$ – the temperature $[\mathrm{K}]$
$\hat{q} = \hat{q}(\boldsymbol{x}, t)$ – the inward heat flux $\left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$



Initial-Boundary-Value Problem

IBVP of the heat transfer

Find $T = T(\boldsymbol{x}, t)$ for $\boldsymbol{x} \in \mathcal{B}$ and $t \in [t_0, t_1]$ satisfying the **equation of heat transfer** by conduction (a), or by conduction and convection (b):

$$\varrho\, c\, \dot{T} + q_{i|i} - f = 0 \quad \text{where} \quad q_i = q_i(T) = \begin{cases} -k\, T_{|i} & \leftarrow \text{(a)} \\ -k\, T_{|i} + \varrho\, c\, u_i\, T & \leftarrow \text{(b)} \end{cases}$$

with the **initial condition** (at $t = t_0$):

$$T(\boldsymbol{x}, t_0) = T_0(\boldsymbol{x}) \quad \text{in } \mathcal{B}\,,$$

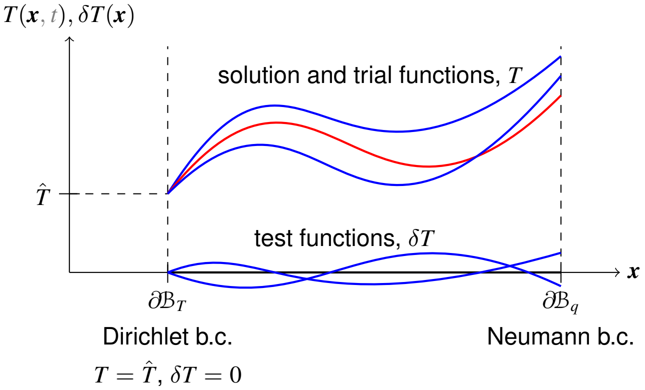
and subject to the **boundary conditions**:

$$T(\boldsymbol{x}, t) = \hat{T}(\boldsymbol{x}, t) \;\text{ on } \partial\mathcal{B}_T\,, \qquad -q_i\big(T(\boldsymbol{x}, t)\big)\, n_i = \hat{q}(\boldsymbol{x}, t) \;\text{ on } \partial\mathcal{B}_q\,,$$

where $\partial\mathcal{B}_T \cup \partial\mathcal{B}_q = \partial\mathcal{B}$ and $\partial\mathcal{B}_T \cap \partial\mathcal{B}_q = \emptyset$.

Outline

Test functions



Test function $\delta T(\boldsymbol{x})$ is an arbitrary (but sufficiently regular) function defined in $\mathcal{B}$, which meets the *admissibility condition*:

$$\delta T = 0 \quad \text{on } \partial\mathcal{B}_T \, .$$

Notice that **test functions are always time-independent.**

Weighted formulation and weak variational form

Weighted integral formulation

$$\int_{\mathcal{B}} \left(\varrho\, c\, \dot{T} + q_{i|i} - f\right)\delta T = 0 \quad \text{(for } \textbf{every } \delta T\text{)}$$

Weighted formulation and weak variational form

Weighted integral formulation

$$\int_{\mathcal{B}} \varrho\, c\, \dot{T}\, \delta T + \int_{\mathcal{B}} q_{i|i}\, \delta T - \int_{\mathcal{B}} f\, \delta T = 0 \quad \text{(for **every** } \delta T\text{)}$$

The term $q_{i|i}$ introduces the second derivative of T: $q_{i|i} = -k\, T_{|ii} + \dots$.
However, the heat PDE needs to be satisfied in the integral sense.
Therefore, the requirements for T can be **weaken** as follows.

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- Integrating by parts (using the divergence theorem)

$$\int_{\mathcal{B}} q_{i|i}\, \delta T = \int_{\mathcal{B}} (q_i\, \delta T)_{|i} - \int_{\mathcal{B}} q_i\, \delta T_{|i} = \int_{\partial\mathcal{B}} q_i\, \delta T\, n_i - \int_{\mathcal{B}} q_i\, \delta T_{|i}$$

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- Using the Neumann b.c and the property of test function

$$\int_{\partial\mathcal{B}} q_i\, n_i\, \delta T = \int_{\partial\mathcal{B}_q} \underbrace{q_i\, n_i}_{-\hat{q}}\, \delta T + \int_{\partial\mathcal{B}_T} q_i\, n_i\, \underbrace{\delta T}_{0} = -\int_{\partial\mathcal{B}_q} \hat{q}\, \delta T$$

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After integrating by parts and using the Neumann boundary condition

$$\int_{\mathcal{B}} q_{i|i}\, \delta T = -\int_{\partial\mathcal{B}_q} \hat{q}\, \delta T - \int_{\mathcal{B}} q_i\, \delta T_{|i}$$

Weak variational form

$$\int_{\mathcal{B}} \varrho\, c\, \dot{T}\, \delta T - \int_{\mathcal{B}} q_i\, \delta T_{|i} - \int_{\partial\mathcal{B}_q} \hat{q}\, \delta T - \int_{\mathcal{B}} f\, \delta T = 0 \quad \text{(for **every** } \delta T)$$

Weighted formulation and weak variational form

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Weak variational form

$$\int_{\mathcal{B}} \varrho\, c\, \dot{T}\, \delta T - \int_{\mathcal{B}} q_i\, \delta T_{|i} - \int_{\partial\mathcal{B}_q} \hat{q}\, \delta T - \int_{\mathcal{B}} f\, \delta T = 0 \quad \text{(for **every** } \delta T\text{)}$$

Now, only the first order spatial-differentiability of T is required.

In this formulation the Neumann boundary condition is already met (it has been used in a *natural* way). Therefore, the only additional requirements are the Dirichlet boundary condition and the initial condition.

Outline

Approximation functions and space

The spatial approximation of solution in the domain $\mathcal{B}$ is accomplished by a linear combination of (global) **shape functions**, $\phi_s = \phi_s(\boldsymbol{x})$,

$$T(\boldsymbol{x},t) = \theta_s(t)\,\phi_s(\boldsymbol{x}) \qquad (s = 1,\dots N;\ \text{ summation over } s)$$

where $\theta_s(t)$ $[\mathrm{K}]$ are (time-dependent) coefficients – the **degrees of freedom** (N is the total number of degrees of freedom). Consistent result is obtained now for the time rate of temperature

$$\dot{T}(\boldsymbol{x},t) = \dot{\theta}_s(t)\,\phi_s(\boldsymbol{x}) \quad \text{where} \quad \dot{\theta}_s(t) = \frac{\mathrm{d}\theta_s(t)}{\mathrm{d}t} \ \ \left[\tfrac{\mathrm{K}}{\mathrm{s}}\right].$$

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Distinctive feature of the Galerkin method:

The same shape functions are used to approximate the solution as well as the test function, namely

$$\delta T(\boldsymbol{x}) = \delta\theta_r\,\phi_r(\boldsymbol{x}) \qquad (r = 1,\dots N;\ \text{ summation over } r)\,.$$

Transient heat transfer (system of ODEs)

To reduce the “regularity” requirements for solution the approximations

$$T = \theta_s \, \phi_s \quad \left(\dot{T} = \dot{\theta}_s \, \phi_s \right) , \qquad \delta T = \delta\theta_r \, \phi_r$$

are used for the **weak variational form** of the heat transfer problem

$$\int_{\mathcal{B}} \varrho \, c \, \dot{T} \, \delta T - \int_{\mathcal{B}} q_i \, \delta T_{|i} - \int_{\partial\mathcal{B}_q} \hat{q} \, \delta T - \int_{\mathcal{B}} f \, \delta T = 0 \, .$$

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1. $$\int_{\mathcal{B}} \varrho \, c \, \dot{T} \, \delta T = \dot{\theta}_s \, \delta\theta_r \int_{\mathcal{B}} \varrho \, c \, \phi_s \, \phi_r = \dot{\theta}_s \, \delta\theta_r \, M_{rs}$$

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2. $$\int_{\mathcal{B}} q_i\,\delta T_{|i} = \int_{\mathcal{B}} \left(-k\,T_{|i} + \varrho\,c\,u_i\,T\right)\delta T_{|i} = \theta_s\,\delta\theta_r \int_{\mathcal{B}} \left(-k\,\phi_{s|i} + \varrho\,c\,u_i\,\phi_s\right)\phi_{r|i} = \theta_s\,\delta\theta_r\,K_{rs}$$

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1 $$\int_{\mathcal{B}} \varrho\,c\,\dot{T}\,\delta T = \dot{\theta}_s\,\delta\theta_r \int_{\mathcal{B}} \varrho\,c\,\phi_s\,\phi_r = \dot{\theta}_s\,\delta\theta_r\,M_{rs}$$

2 $$\int_{\mathcal{B}} q_i\,\delta T_{|i} = \int_{\mathcal{B}} \left(-k\,T_{|i} + \varrho\,c\,u_i\,T\right)\delta T_{|i} = \theta_s\,\delta\theta_r \int_{\mathcal{B}} \left(-k\,\phi_{s|i} + \varrho\,c\,u_i\,\phi_s\right)\phi_{r|i} = \theta_s\,\delta\theta_r\,K_{rs}$$

3 $$\int_{\partial\mathcal{B}_q} \hat{q}\,\delta T = \delta\theta_r \int_{\partial\mathcal{B}_q} \hat{q}\,\phi_r = \delta\theta_r\,Q_r$$

Transient heat transfer (system of ODEs)

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3 $$\int_{\partial\mathcal{B}_q} \hat{q} \, \delta T = \delta\theta_r \int_{\partial\mathcal{B}_q} \hat{q} \, \phi_r = \delta\theta_r \, Q_r$$

4 $$\int_{\mathcal{B}} f \, \delta T = \delta\theta_r \int_{\mathcal{B}} f \, \phi_r = \delta\theta_r \, F_r$$

Transient heat transfer (system of ODEs)

Matrix formulation of the heat transfer problem

$$\big[M_{rs}\,\dot{\theta}_s - K_{rs}\,\theta_s - (Q_r + F_r)\big]\delta\theta_r = 0 \quad \text{for every } \delta\theta_r.$$

This produces the following system of first-order **ordinary differential equations** (for $\theta_s = \theta_s(t) = ?$):

$$\boxed{M_{rs}\,\dot{\theta}_s - K_{rs}\,\theta_s = (Q_r + F_r)} \qquad (r, s = 1, \ldots N).$$

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- $M_{rs} = \int\limits_{\mathcal{B}} \varrho\, c\, \phi_s\, \phi_r$ – the thermal capacity matrix $\left[\frac{\mathrm{J}}{\mathrm{K}}\right]$,
- $K_{rs} = \int\limits_{\mathcal{B}} \Big(-k\,\phi_{s|i} + \varrho\, c\, u_i\, \phi_s\Big)\,\phi_{r|i}$ – the heat transfer matrix $\left[\frac{\mathrm{W}}{\mathrm{K}}\right]$,
- $Q_r = \int\limits_{\partial\mathcal{B}_q} \hat{q}\,\phi_r$ – the inward heat flow vector $[\mathrm{W}]$,
- $F_r = \int\limits_{\mathcal{B}} f\,\phi_r$ – the heat production vector $[\mathrm{W}]$.

Stationary heat transfer (algebraic equations)

$$T = T(\boldsymbol{x})\,, \qquad f = f(\boldsymbol{x})\,, \qquad u_i = u_i(\boldsymbol{x}) \qquad (\text{for } \boldsymbol{x} \in \mathcal{B})\,.$$

BVP of stationary heat flow: Find $T = T(\boldsymbol{x})$ satisfying (in $\mathcal{B}$)

$$q_{i|i} - f = 0 \quad \text{where:}$$

$$q_i = q_i(T) = \begin{cases} -k\,T_{|i} & \text{(no convection)},\\ -k\,T_{|i} + \varrho\,c\,u_i\,T & \text{(with convection)}, \end{cases}$$

with boundary conditions:

$$T = \hat{T} \quad \text{on } \partial\mathcal{B}_T \text{ (Dirichlet)}, \qquad -\,q_i(T)\,n_i = \hat{q} \quad \text{on } \partial\mathcal{B}_q \text{ (Neumann)}.$$

- The weak variational form lacks the rate integrand

$$-\int_{\mathcal{B}} q_i\,\delta T_{|i} - \int_{\partial\mathcal{B}_q} \hat{q}\,\delta T - \int_{\mathcal{B}} f\,\delta T = 0\,.$$

- The approximations $T(\boldsymbol{x}) = \theta_s\,\phi_s(\boldsymbol{x})$, $\delta T(\boldsymbol{x}) = \delta\theta_r\,\phi_r(\boldsymbol{x})$ lead to the following **system of linear algebraic equations** (for $\theta_s = ?$):

$$\boxed{-K_{rs}\,\theta_s = (Q_r + F_r)} \qquad (r, s = 1, \dots N).$$