Fundamentals of Fluid Dynamics: Waves in Fluids
Introductory Course on Multiphysics Modelling

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(after: D.J. ACHESON's "Elementary Fluid Dynamics")

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Outline

1 Introduction
   ■ The notion of wave
   ■ Basic wave phenomena
   ■ Mathematical description of a traveling wave
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   - The notion of wave
   - Basic wave phenomena
   - Mathematical description of a traveling wave

2 Water waves
   - Surface waves on deep water
   - Dispersion and the group velocity
   - Capillary waves
   - Shallow-water finite-amplitude waves
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   - The speed of sound
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The notion of wave

What is a wave?

A wave is the transport of a disturbance (or energy, or piece of information) in space not associated with motion of the medium occupying this space as a whole. (Except that electromagnetic waves require no medium !!!)

- The transport is at finite speed.
- The shape or form of the disturbance is arbitrary.
- The disturbance moves with respect to the medium.
# The notion of wave

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Two general classes of wave motion are distinguished:

- **1. longitudinal waves** – the disturbance moves parallel to the direction of propagation. *Examples*: sound waves, compressional elastic waves (P-waves in geophysics);
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The notion of wave

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Two general classes of wave motion are distinguished:

1. **longitudinal waves** – the disturbance moves parallel to the direction of propagation. *Examples:* sound waves, compressional elastic waves (P-waves in geophysics);

2. **transverse waves** – the disturbance moves perpendicular to the direction of propagation. *Examples:* waves on a string or membrane, shear waves (S-waves in geophysics), water waves, electromagnetic waves.
Basic wave phenomena

reflection – change of wave direction from hitting a reflective surface,
refraction – change of wave direction from entering a new medium,
diffraktion – wave circular spreading from entering a small hole (of the wavelength-comparable size), or wave bending around small obstacles,
interference – superposition of two waves that come into contact with each other,
dispersion – wave splitting up by frequency,
rectilinear propagation – the movement of light wave in a straight line.

Standing wave

A standing wave, also known as a stationary wave, is a wave that remains in a constant position. This phenomenon can occur: when the medium is moving in the opposite direction to the wave, (in a stationary medium:) as a result of interference between two waves travelling in opposite directions.
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Mathematical description of a harmonic wave

Introduction

Water waves

Sound waves

Mathematical description of a harmonic wave

\[ T = \frac{2\pi}{\omega} \]

\[ \lambda = \frac{2\pi}{k} \]

Traveling waves

*Simple wave* or **traveling wave**, sometimes also called *progressive wave*, is a disturbance that varies both with time \( t \) and distance \( x \) in the following way:

\[
u(x, t) = A(x, t) \cos (kx - \omega t + \theta_0) = A(x, t) \sin (kx - \omega t + \theta_0 \pm \frac{\pi}{2})
\]

where \( A \) is the **amplitude**, \( \omega \) and \( k \) denote the **angular frequency** and **wavenumber**, and \( \theta_0 \) (or \( \tilde{\theta}_0 \)) is the initial **phase**.
Mathematical description of a harmonic wave

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\[ = A(x, t) \sin (k x - \omega t + \theta_0 \pm \frac{\pi}{2}) \]

- **Amplitude** \( A \) [e.g. m, Pa, V/m] — a measure of the maximum disturbance in the medium during one wave cycle (the maximum distance from the highest point of the crest to the equilibrium).

- **Phase** \( \theta = k x - \omega t + \theta_0 \) [rad], where \( \theta_0 \) is the *initial* phase (shift), often ambiguously, called the phase.
Mathematical description of a harmonic wave

\[ T = \frac{2\pi}{\omega} \]

- **Period** \( T \) [s] – the time for one complete cycle for an oscillation of a wave.
- **Frequency** \( f \) [Hz] – the number of periods per unit time.

**Frequency and angular frequency**

The **frequency** \( f \) [Hz] represents the number of periods per unit time

\[ f = \frac{1}{T}. \]

The **angular frequency** \( \omega \) [Hz] represents the frequency in terms of radians per second. It is related to the frequency by

\[ \omega = \frac{2\pi}{T} = 2\pi f. \]
Mathematical description of a harmonic wave

Wavelength $\lambda \ [\text{m}]$ – the distance between two sequential crests (or troughs).

Wavenumber and angular wavenumber

The **wavenumber** is the spatial analogue of frequency, that is, it is the measurement of the number of repeating units of a propagating wave (the number of times a wave has the same phase) per unit of space.

*Application of a Fourier transformation on data as a function of time yields a frequency spectrum; application on data as a function of position yields a wavenumber spectrum.*

The **angular wavenumber** $k \ [\frac{1}{\text{m}}]$, often misleadingly abbreviated as “wave-number”, is defined as

$$k = \frac{2\pi}{\lambda}.$$
There are two velocities that are associated with waves:

1. **Phase velocity** – the rate at which the wave propagates:

   \[ c = \frac{\omega}{k} = \lambda f. \]
Mathematical description of a harmonic wave

There are two velocities that are associated with waves:

1. **Phase velocity** – the rate at which the wave propagates:
   \[
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   \]

2. **Group velocity** – the velocity at which variations in the shape of the wave’s amplitude (known as the *modulation* or *envelope* of the wave) propagate through space:
   \[
   c_g = \frac{d\omega}{dk} .
   \]

This is (in most cases) the signal velocity of the waveform, that is, the **rate at which information or energy is transmitted** by the wave. However, if the wave is travelling through an absorptive medium, this does not always hold.
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   - Surface waves on deep water
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   - Introduction
   - Acoustic wave equation
   - The speed of sound
   - Sub- and supersonic flow
Consider two-dimensional water waves: \( u = [u(x, y, t), v(x, y, t), 0] \).

Suppose that the flow is irrotational: \( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \).

Therefore, there exists a velocity potential \( \phi(x, y, t) \) so that
\[
    u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}.
\]

The fluid is incompressible, so by the virtue of the incompressibility condition, \( \nabla \cdot \mathbf{u} = 0 \), the velocity potential \( \phi \) will satisfy Laplace’s equation
\[
    \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.
\]
Surface waves on deep water

- Consider **two-dimensional** water waves: \( u = [u(x, y, t), v(x, y, t), 0] \).
- Suppose that the flow is **irrotational**: \( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \).
- Therefore, there exists a **velocity potential** \( \phi(x, y, t) \) so that
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  \]
- The fluid is **incompressible**, so \( \phi \) will satisfy **Laplace’s equation**
  \[
  \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.
  \]

**Free surface**

The fluid motion arises from a deformation of the water surface. The equation of this free surface is denoted by \( y = \eta(x, t) \).
Surface waves on deep water

Kinematic condition at the free surface

Fluid particles on the surface must remain on the surface.

The kinematic condition entails that \( F(x, y, t) = y - \eta(x, t) \) remains constant (in fact, zero) for any particular particle on the free surface which means that

\[
\frac{DF}{Dt} = \frac{\partial F}{\partial t} + (u \cdot \nabla) F = 0 \quad \text{on} \quad y = \eta(x, t),
\]

and this is equivalent to

\[
\frac{\partial \eta}{\partial t} + u \cdot \frac{\partial \eta}{\partial x} = v \quad \text{on} \quad y = \eta(x, t).
\]
Surface waves on deep water

Pressure condition at the free surface:

The fluid is **inviscid** (by assumption), so the condition at the free surface is simply that the pressure there is equal to the atmospheric pressure $p_0$:

$$p = p_0 \quad \text{on} \quad y = \eta(x, t).$$
Surface waves on deep water

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The fluid is **inviscid** (by assumption), so the condition at the free surface is simply that the pressure there is equal to the atmospheric pressure $p_0$:

$$p = p_0 \text{ on } y = \eta(x,t).$$

Bernoulli’s equation for unsteady irrotational flow

If the flow is irrotational (so $u = \nabla \phi$ and $\nabla \times u = 0$), then, by integrating (over the space domain) the **Euler’s momentum equation**:

$$\frac{\partial \nabla \phi}{\partial t} = -\nabla \left( \frac{p}{\rho} + \frac{1}{2}u^2 + \chi \right),$$

the **Bernoulli’s equation** is obtained

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}u^2 + \chi = G(t).$$

Here, $\chi$ is the gravity potential (in the present context $\chi = g y$ where $g$ is the gravity acceleration) and $G(t)$ is an arbitrary function of time alone (a constant of integration).
Surface waves on deep water

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the **Bernoulli’s equation** is obtained

\[ \frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2} u^2 + g y = G(t). \]

Here, $G(t)$ is an arbitrary function of time alone (a constant of integration).

Now, by choosing $G(t)$ in a convenient manner, $G(t) = \frac{p_0}{\rho}$, the **pressure condition** may be written as:

\[ \frac{\partial \phi}{\partial t} + \frac{1}{2} (u^2 + v^2) + g \eta = 0 \quad \text{on} \quad y = \eta(x, t). \]
Surface waves on deep water
Small amplitude waves: the linearized surface conditions

Small-amplitude waves
The free surface displacement \( \eta(x, t) \) and the fluid velocities \( u, v \) are small.

- Linearization of the **kinematic condition**

\[
v = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x}
\]

\( \text{small} \quad \rightarrow \quad v(x, \eta, t) = \frac{\partial \eta}{\partial t} \)

\[
\text{Taylor series} \quad v(x, 0, t) + \eta \frac{\partial v}{\partial y} (x, 0, t) + \cdots = \frac{\partial \eta}{\partial t}
\]

\( \text{small} \quad \rightarrow \quad v(x, 0, t) = \frac{\partial \eta}{\partial t} \)

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\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \quad \text{on } y = 0.
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- Linearization of the **pressure condition**

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\]
Surface waves on deep water
Dispersion relation and travelling wave solution

A sinusoidal travelling wave solution

The **free surface** is of the form

\[ \eta = A \cos(kx - \omega t), \]

where \( A \) is the **amplitude** of the surface displacement, \( \omega \) is the **circular frequency**, and \( k \) is the **circular wavenumber**.
Introduction Water waves Sound waves

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where \( A \) is the **amplitude** of the surface displacement, \( \omega \) is the **circular frequency**, and \( k \) is the **circular wavenumber**.

- The corresponding **velocity potential** is

\[ \phi = q(y) \sin(kx - \omega t). \]

- It satisfies the Laplace’s equation, \( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \)

- Therefore, \( q(y) \) must satisfy \( q'' - k^2 q = 0 \), the general solution of which is

\[ q = C \exp(ky) + D \exp(-ky). \]

- For **deep water waves** \( D = 0 \) (if \( k > 0 \) which may be assumed without loss of generality) in order that the velocity be bounded as \( y \to -\infty \).
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Now, the (linearized) free surface conditions yield what follows:

1. the kinematic condition \( \left( \frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \text{ on } y = 0 \right): \)

\[ Ck = A\omega \quad \Rightarrow \quad \phi = \frac{A\omega}{k} \exp(ky) \sin(kx - \omega t), \]

2. the pressure condition \( \left( \frac{\partial \phi}{\partial t} + g\eta = 0 \text{ on } y = 0 \right): \)

\[ -C\omega + gA = 0 \quad \Rightarrow \quad \omega^2 = gk. \] (dispersion relation!)
Surface waves on deep water

Particle paths

The **fluid velocity** components:

\[ u = A \omega \exp(ky) \cos(kx - \omega t), \quad v = A \omega \exp(ky) \sin(kx - \omega t). \]
Surface waves on deep water

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**Particle paths**

Any particle departs only a **small amount** \((X, Y)\) from its mean **position** \((x, y)\). Therefore, its position as a function of time may be found by integrating \( u = \frac{dX}{dt} \) and \( v = \frac{dY}{dt} \); whence:

\[ X(t) = -A \exp(ky) \sin(kx - \omega t), \quad Y(t) = A \exp(ky) \cos(kx - \omega t). \]
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- Particle paths are circular.
- The radius of the path circles, \(A \exp(ky)\), decrease exponentially with depth. So do the fluid velocities.
- Virtually all the energy of a surface water wave is contained within half a wavelength below the surface.
Surface waves on deep water

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Surface waves on deep water
Effects of finite depth

Effects of finite depth

If the fluid is bonded below by a rigid plane \( y = -h \), so that

\[
v = \frac{\partial \phi}{\partial y} = 0 \quad \text{at} \quad y = -h,
\]

the dispersion relation and the phase speed are as follows:

\[
\omega^2 = g k \tanh(k h), \quad c^2 = \frac{g}{k} \tanh(k h).
\]
Surface waves on deep water

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the dispersion relation and the phase speed are as follows:

$$\omega^2 = g k \tanh(kh), \quad c^2 = \frac{g}{k} \tanh(kh).$$

There are two limit cases:

1. $h \gg \lambda$ (infinite depth): $kh = 2\pi \frac{h}{\lambda}$ is large and $\tanh(kh) \approx 1$, so $c^2 = \frac{g}{k}$. In practice, this is a good approximation if $h > \frac{1}{3}\lambda$.

2. $h \ll \lambda/2\pi$ (shallow water): $kh \ll 1$ and $\tanh(kh) \approx kh$, so $c^2 = gh$, which means that $c$ is independent of $k$ in this limit.

Thus, the gravity waves in shallow water are non-dispersive.
Surface waves on deep water

Effects of finite depth

If the fluid is bonded below by a rigid plane $y = -h$, the dispersion relation and the phase speed are as follows:

$$\omega^2 = g k \tanh(kh), \quad c^2 = \frac{g}{k} \tanh(kh).$$
Dispersion and the group velocity

Dispersion of waves

Dispersion of waves is the phenomenon that the **phase velocity of a wave depends on its frequency**.
Dispersion and the group velocity

Dispersion of waves

Dispersion of waves is the phenomenon that the phase velocity of a wave depends on its frequency.

There are generally two sources of dispersion:

1. the **material dispersion** comes from a frequency-dependent response of a material to waves
2. the **waveguide dispersion** occurs when the speed of a wave in a waveguide depends on its frequency for geometric reasons, independent of any frequency-dependence of the materials from which it is constructed.
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Dispersion relation

\[
\omega = \omega(k) = c(k) k, \quad c = c(k) = \frac{\omega(k)}{k}.
\]

If \(\omega(k)\) is a linear function of \(k\) then \(c\) is constant and the medium is non-dispersive.
Dispersion and the group velocity

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If \( \omega(k) \) is a linear function of \( k \) then \( c \) is constant and the medium is non-dispersive.

- **deep water waves:** \( \omega = \sqrt{g k}, \quad c = \sqrt{\frac{g}{k}}. \)
- **finite depth waves:** \( \omega = \sqrt{g k \tanh(k h)}, \quad c = \sqrt{\frac{g}{k} \tanh(k h)}. \)
- **shallow water waves:** \( \omega = \sqrt{g h k}, \quad c = \sqrt{g h} \rightarrow \text{non-dispersive!} \)
Dispersion and the group velocity

**Dispersion relation**

\[
\omega = \omega(k) = c(k) k, \quad c = c(k) = \frac{\omega(k)}{k}.
\]

If \(\omega(k)\) is a **linear** function of \(k\) then \(c\) is **constant** and the medium is **non-dispersive**.

- **deep water waves:** \(\omega = \sqrt{gk}\), \(c = \sqrt{\frac{g}{k}}\).
- **finite depth waves:** \(\omega = \sqrt{g k \tanh(k h)}\), \(c = \sqrt{\frac{g}{k} \tanh(k h)}\).
- **shallow water waves:** \(\omega = \sqrt{g h k}\), \(c = \sqrt{g h} \rightarrow \text{non-dispersive!}\)

**Group and phase velocity**

\[
c_g = \frac{d\omega}{dk}, \quad c = \frac{\omega}{k}.
\]

- In **dispersive systems** both velocities are different and frequency-dependent (i.e., wavenumber-dependent): \(c_g = c_g(k)\) and \(c = c(k)\).
- In **non-dispersive systems** they are equal and constant: \(c_g = c\).
Dispersion and the group velocity

Important properties of the group velocity:

1. At this velocity the isolated wave packet travels as a whole.

Discussion for a wave packet: for \( k \) in the neighbourhood of \( k_0 \)

\[
\omega(k) \approx \omega(k) + (k - k_0) c_g,
\]

where \( c_g = \frac{d\omega}{dk} \bigg|_{k=k_0} \),

and \( \omega(k) = 0 \) outside the neighbourhood; the Fourier integral equals

\[
\eta(x, t) = \text{Re} \left[ \int_{-\infty}^{\infty} a(k) \exp \left( i (k x - \omega t) \right) dk \right] \quad \leftarrow \text{(for a general disturbance)}
\]

\[
\approx \text{Re} \left[ \exp \left( i (k_0 x - \omega(k_0) t) \right) \int_{-\infty}^{\infty} a(k) \exp \left( i (k - k_0) (x - c_g t) \right) dk \right] .
\]
Dispersion and the group velocity

Important properties of the group velocity:

1. At this velocity the isolated wave packet travels as a whole.
2. The energy is transported at the group velocity (by waves of a given wavelength).
Dispersion and the group velocity

Important properties of the group velocity:

1. At this velocity the isolated wave packet travels as a whole.
2. The energy is transported at the group velocity.
3. One must travel at the group velocity to see the waves of the same wavelength.

A slowly varying wavetrain can be written as

\[ \eta(x, t) = \text{Re} \left[ A(x, t) \exp\left( i \theta(x, t) \right) \right], \]

where the phase function \( \theta(x, t) \) describes the oscillatory aspect of the wave, while \( A(x, t) \) describes the gradual modulation of its amplitude. The local wavenumber and frequency are defined by

\[ k = \frac{\partial \theta}{\partial x}, \quad \omega = -\frac{\partial \theta}{\partial t}. \]

For purely sinusoidal wave \( \theta = k x - \omega t \), where \( k \) and \( \omega \) are constants.
Dispersion and the group velocity

Important properties of the group velocity:
1. At this velocity the isolated wave packet travels as a whole.
2. The energy is transported at the group velocity.
3. One must travel at the group velocity to see the waves of the same wavelength.

The local wavenumber and frequency are defined by

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For purely sinusoidal wave \( \theta = k x - \omega t \), where \( k \) and \( \omega \) are constants. In general, \( k \) and \( \omega \) are functions of \( x \) and \( t \). It follows immediately that

\[ \frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \quad \rightarrow \quad \frac{\partial k}{\partial t} + \frac{d\omega}{dk} \frac{\partial k}{\partial x} = \frac{\partial k}{\partial t} + c_g(k) \frac{\partial k}{\partial x} = 0 \]

which means that \( k(x,t) \) is constant for an observer moving with the velocity \( c_g(k) \).
Capillary waves

Surface tension

A **surface tension force** $T \left[ \frac{N}{m} \right]$ is a force per unit length, directed tangentially to the surface, acting on a line drawn parallel to the wavecrests.
Capillary waves

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A **surface tension force** $T \left[ \frac{N}{m} \right]$ is a force per unit length, directed tangentially to the surface, acting on a line drawn parallel to the wavecrests.

- The **vertical component** of surface tension force equals $T \frac{\partial \eta}{\partial s}$, where $s$ denotes the distance along the surface.
- For **small wave amplitudes** $\delta s \approx \delta x$, and then $T \frac{\partial \eta}{\partial s} \approx T \frac{\partial \eta}{\partial x}$.
Capillary waves

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A **surface tension force** $T \left[ \frac{N}{m} \right]$ is a force per unit length, directed tangentially to the surface, acting on a line drawn parallel to the wavecrests.

- The **vertical component** of surface tension force equals $T \frac{\partial \eta}{\partial s}$, where $s$ denotes the distance along the surface.
- For **small wave amplitudes** $\delta s \approx \delta x$, and then $T \frac{\partial \eta}{\partial s} \approx T \frac{\partial \eta}{\partial x}$.
- A small portion of surface of length $\delta x$ will experience surface tension at both ends, so the **net upward force** on it will be

$$T \frac{\partial \eta}{\partial x} \bigg|_{x+\delta x} - T \frac{\partial \eta}{\partial x} \bigg|_x = T \frac{\partial^2 \eta}{\partial x^2} \delta x$$

- Therefore, an upward force **per unit area of surface** is $T \frac{\partial^2 \eta}{\partial x^2}$.
Capillary waves

Local equilibrium at the free surface

The net upward force per unit area of surface, \( T \frac{\partial^2 \eta}{\partial x^2} \), must be balanced by the difference between the atmospheric pressure \( p_0 \) and the pressure \( p \) in the fluid just below the surface:

\[
p_0 - p = T \frac{\partial^2 \eta}{\partial x^2} \quad \text{on} \quad y = \eta(x, t).
\]
Capillary waves

Local equilibrium at the free surface

The net upward force per unit area of surface, $T \frac{\partial^2 \eta}{\partial x^2}$, must be balanced by the difference between the atmospheric pressure $p_0$ and the pressure $p$ in the fluid just below the surface:

$$p_0 - p = T \frac{\partial^2 \eta}{\partial x^2} \quad \text{on} \quad y = \eta(x, t).$$

This **pressure condition** at the free surface takes into consideration the **effects of surface tension**. The kinematic condition remains the same: fluid particles cannot leave the surface.

Linearized free surface conditions (with surface tension effects)

For small amplitude waves:

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}, \quad \frac{\partial \phi}{\partial t} + g \eta = \frac{T}{\varrho} \frac{\partial^2 \eta}{\partial x^2} \quad \text{on} \quad y = 0.$$
Introduction

Water waves

Sound waves

Capillary waves

Linearized free surface conditions (with surface tension effects)

For small amplitude waves:

\[
\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}, \quad \frac{\partial \phi}{\partial t} + g \eta = \frac{T}{\varrho} \frac{\partial^2 \eta}{\partial x^2} \quad \text{on } y = 0.
\]

A sinusoidal travelling wave solution \( \eta = A \cos(kx - \omega t) \) leads now to a new dispersion relation

\[
\omega^2 = gk + \frac{T k^3}{\varrho}.
\]

As a consequence, the phase and group velocities include now the surface tension effect:

\[
c = \frac{\omega}{k} = \sqrt{\frac{g}{k} + \frac{T k^3}{\varrho}}, \quad c_g = \frac{d\omega}{dk} = \frac{g + 3T k^2 / \varrho}{2 \sqrt{g k + T k^3 / \varrho}}.
\]
# Capillary waves

## Surface tension importance parameter

The relative importance of surface tension and gravitational forces in a fluid is measured by the following parameter

\[ \beta = \frac{T k^2}{\varrho g} . \]

(The so-called *Bond number* \(= \frac{\varrho g L^2}{T}\); it equals \(\frac{4\pi^2}{\beta}\) if \(L = \lambda\).)

Now, the dispersion relation, as well as the phase and group velocities can be written as

\[ \omega^2 = g k (1 + \beta) , \quad c = \sqrt{\frac{g}{k}} (1 + \beta) , \quad c_g = \frac{g (1 + 3\beta)}{2 \sqrt{g k (1 + \beta)}} . \]
Capillary waves

Surface tension importance parameter

\[ \beta = \frac{T k^2}{\varrho g} \]

\[ \omega^2 = g k (1 + \beta), \quad c = \sqrt{\frac{g}{k}(1 + \beta)}, \quad c_g = \frac{g(1 + 3\beta)}{2 \sqrt{g k (1 + \beta)}}. \]

Depending on the parameter $\beta$, two extreme cases are distinguished:

1. $\beta \ll 1$: the effects of surface tension are negligible – the waves are **gravity waves** for which

\[ \omega^2 = g k, \quad c = \sqrt{\frac{g}{k}} = \sqrt{\frac{g \lambda}{2\pi}}, \quad c_g = \frac{c}{2}. \]


Capillary waves

Surface tension importance parameter

\[ \beta = \frac{T k^2}{\rho g} \]

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Depending on the parameter \( \beta \), two extreme cases are distinguished:

1. \( \beta \ll 1 \): the effects of surface tension are negligible – the waves are gravity waves for which

   \[ \omega^2 = g k, \quad c = \sqrt{\frac{g}{k}} = \sqrt{\frac{g \lambda}{2\pi}}, \quad c_g = \frac{c}{2}. \]

2. \( \beta \gg 1 \): the waves are essentially capillary waves for which

   \[ \omega^2 = g k \beta = \frac{T k^3}{\rho}, \quad c = \sqrt{\frac{g}{k} \beta} = \sqrt{\frac{T k}{\rho}} = \sqrt{\frac{2\pi T}{\rho \lambda}}, \quad c_g = \frac{g 3\beta}{2 \sqrt{g k \beta}} = \frac{3}{2} c. \]
Capillary waves vs. gravity waves

CAPILLARY WAVES:
- short waves travel faster,

GRAVITY WAVES:
- long waves travel faster,
Capillary waves vs. gravity waves

**CAPILLARY WAVES:**
- short waves travel faster,
- the group velocity exceeds the phase velocity, $c_g > c$,

**GRAVITY WAVES:**
- long waves travel faster,
- the group velocity is less than the phase velocity, $c_g < c$, 

The capillary effects predominate when raindrops fall on a pond, and as short waves travel faster the wavelength decreases with radius at any particular time.

The effects of gravity predominate when a large stone is dropped into a pond, and as long waves travel faster the wavelength increases with radius at any particular time.
Capillary waves vs. gravity waves

**CAPILLARY WAVES:**
- **short waves** travel faster,
- the group velocity exceeds the phase velocity, $c_g > c$,
- the **crests move backward** through a wave packet as it moves along as a whole.

**GRAVITY WAVES:**
- **long waves** travel faster,
- the group velocity is less than the phase velocity, $c_g < c$,
- the **wavecrests move faster** than a wave packet.
Capillary waves vs. gravity waves

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The **capillary effects** predominate when **raindrops** fall on a pond, and as short waves travel faster the **wavelength decreases with radius** at any particular time.

The **effects of gravity** predominate when a **large stone** is dropped into a pond, and as long waves travel faster the **wavelength increases with radius** at any particular time.
Capillary-gravity waves

For $\beta \approx 1$ both effects (the surface tension and gravity) are significant and the waves are **capillary-gravity waves**.
Capillary waves vs. gravity waves

Example: Uniform flow past a submerged obstacle

1. $U < c_{\text{min}}$ – there are no steady waves generated by the obstacle;
2. $U > c_{\text{min}}$ – there are two values of $\lambda$ ($\lambda_1 > \lambda_2$) for which $c = U$:
   - $\lambda_1$ – the larger value represents a gravity wave:
     - the corresponding group velocity is less than $c$,
     - the energy of this relatively long-wavelength disturbance is carried downstream of the obstacle.
   - $\lambda_2$ – the smaller value represents a capillary wave:
     - the corresponding group velocity is greater than $c$,
     - the energy of this relatively short-wavelength disturbance is carried upstream of the obstacle, where it is rather quickly dissipated by viscous effects, on account of the short wavelength (in fact, each wave-crest is at rest, but relative to still water it is travelling upstream with speed $U$).
Shallow-water finite-amplitude waves

Assumptions:

- The **amplitudes of waves are finite**, that is, *not* (infinitesimally) small compared with the depth; therefore, the **linearized theory does not apply**.

- A typical value $h_0$ of depth $h(x, t)$ is much smaller than a typical horizontal length scale $L$ of the wave, that is: $h_0 \ll L$. This is the basis of the so-called **shallow-water approximation**.

The full (nonlinear) 2-D equations are:

$$
\frac{Du}{Dt} = -\frac{1}{\varrho} \frac{\partial p}{\partial x}, \quad \frac{Dv}{Dt} = -\frac{1}{\varrho} \frac{\partial p}{\partial y} - g, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.
$$
Shallow-water finite-amplitude waves

\[
\frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad \frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - g, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.
\]

In the shallow-water approximation (when \( h_0 \ll L \)) the vertical component of acceleration can be neglected in comparison with the gravitational acceleration:

\[
\frac{Dv}{Dt} \ll g \quad \rightarrow \quad 0 = -\frac{1}{\rho} \frac{\partial p}{\partial y} - g \quad \rightarrow \quad \frac{\partial p}{\partial y} = \rho g.
\]

Integrating and applying the condition \( p = p_0 \) at \( y = h(x, t) \) gives

\[
p(x, y, t) = p_0 - \rho g \left[ y - h(x, t) \right].
\]
Shallow-water finite-amplitude waves

\[ \frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad \frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - g, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \]

▶ In the shallow-water approximation (when \( h_0 \ll L \)) the vertical component of acceleration can be neglected and then

\[ p(x, y, t) = p_0 - \rho g [y - h(x, t)]. \]

This is used for the equation for the horizontal component of acceleration:

\[ \frac{Du}{Dt} = -g \frac{\partial h}{\partial x} \quad \frac{\partial u}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial h}{\partial x} \]

where \( u = u(x, t) \) and \( h = h(x, t) \).
Shallow-water finite-amplitude waves

\[ \frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad \frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - g, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \]

A second equation linking \( u \) and \( h \) may be obtained as follows:

\[ \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \quad \rightarrow \quad v(x, y, t) = -\frac{\partial u(x, t)}{\partial x} y + f(x, t) \quad \text{\( v = 0 \) at \( y = 0 \)} \quad \rightarrow \quad v = -\frac{\partial u}{\partial x} y, \]

and using the **kinematic condition at the free surface** — fluid particles on the surface must remain on it, so the vertical component of velocity \( v \) equals the rate of change of the depth \( h \) when moving with the horizontal velocity \( u \):

\[ v = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} \quad \text{at} \quad y = h(x, t) \quad \rightarrow \quad \left( \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} \right) = 0. \]
Shallow-water finite-amplitude waves

**Shallow-water equations**

Nonlinear equations for the horizontal component of velocity $u = u(x, t)$ and the depth $h = h(x, t)$ of finite-amplitude waves on shallow water:

$$
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = 0, \quad \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0.
$$

(The vertical component of velocity is $v(x, y, t) = -\frac{\partial u}{\partial x} y$.)

On introducing the new variable $c(x, t) = \sqrt{gh}$ and then adding and subtracting the two equations the form suited to treatment by the *method of characteristics* is obtained

$$
\left[ \frac{\partial}{\partial t} + (u + c) \frac{\partial}{\partial x} \right] (u + 2c) = 0, \quad \left[ \frac{\partial}{\partial t} + (u - c) \frac{\partial}{\partial x} \right] (u - 2c) = 0.
$$

General property: $u \pm 2c$ is constant along ‘positive’/’negative’ characteristic curves defined by $\frac{dx}{dt} = u \pm c$. 

Within the framework of the theory of finite-amplitude waves on shallow water the following problems can be solved: 

- the dam-break flow, 
- the formation of a bore, 
- the hydraulic jump.
Shallow-water finite-amplitude waves

Shallow-water equations

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = 0, \quad \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0.
\]

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\]

Let \( x = x(s), t = t(s) \) be a characteristic curve defined parametrically (\( s \) is the parameter) in the \( x-t \) plane and starting at some point \((x_0, t_0)\). In fact, two such (families of) characteristic curves are defined such that:

\[
\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = u \pm c.
\]

This (with +) is used for the first and (with −) for the second equation:

\[
\left[ \frac{dt}{ds} \frac{\partial}{\partial t} + \frac{dx}{ds} \frac{\partial}{\partial x} \right] (u \pm 2c) = 0 \quad \text{the chain rule} \quad \frac{d}{ds} (u \pm 2c) = 0.
\]
Shallow-water finite-amplitude waves

Shallow-water equations

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = 0, \quad \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0. \]

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Let \( x = x(s) = x(t), \ t = t(s) = s \) be a characteristic curve such that:

\[ \frac{dx}{dt} = u \pm c . \]

\[ \left[ \frac{\partial}{\partial t} + \frac{dx}{dt} \frac{\partial}{\partial x} \right] (u \pm 2c) = 0 \quad \text{the chain rule} \quad \frac{d}{dt} (u \pm 2c) = 0. \]

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**Shallow-water finite-amplitude waves**

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\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = 0, \quad \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0.
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- the dam-break flow,
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Outline

1 Introduction
   ■ The notion of wave
   ■ Basic wave phenomena
   ■ Mathematical description of a traveling wave

2 Water waves
   ■ Surface waves on deep water
   ■ Dispersion and the group velocity
   ■ Capillary waves
   ■ Shallow-water finite-amplitude waves

3 Sound waves
   ■ Introduction
   ■ Acoustic wave equation
   ■ The speed of sound
   ■ Sub- and supersonic flow
Sound waves: introduction

**Sound waves** propagate due to the **compressibility** of a medium ($\nabla \cdot \mathbf{u} \neq 0$). Depending on frequency one can distinguish:

- **infrasound waves** – below 20 Hz,
- **acoustic waves** – from 20 Hz to 20 kHz,
- **ultrasound waves** – above 20 kHz.

**Acoustics** deals with vibrations and waves in compressible continua in the **audible frequency range**, that is, from 20 Hz (16 Hz) to 20 000 Hz.

Types of waves in compressible continua:

- an **inviscid compressible fluid** – (only) longitudinal waves,
- an infinite **isotropic solid** – longitudinal and shear waves,
- an **anisotropic solid** – wave propagation is more complex.
Acoustic wave equation

**Assumptions:**
- Gravitational forces can be neglected so that the equilibrium (undisturbed-state) pressure and density take on uniform values, \( p_0 \) and \( \rho_0 \), throughout the fluid.
- Dissipative effects, that is viscosity and heat conduction, are neglected.
- The medium (fluid) is homogeneous, isotropic, and perfectly elastic.
Acoustic wave equation

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- Gravitational forces can be neglected so that the equilibrium (undisturbed-state) pressure and density take on uniform values, $p_0$ and $\varrho_0$, throughout the fluid.
- Dissipative effects, that is viscosity and heat conduction, are neglected.
- The medium (fluid) is homogeneous, isotropic, and perfectly elastic.

**Small-amplitudes assumption**

Particle velocity is small, and there are only very small perturbations (fluctuations) to the equilibrium pressure and density:

\[ u - \text{small}, \quad p = p_0 + \tilde{p} \quad (\tilde{p} - \text{small}), \quad \varrho = \varrho_0 + \tilde{\varrho} \quad (\tilde{\varrho} - \text{small}). \]

The pressure fluctuations field $\tilde{p}$ is called the **acoustic pressure**.
Acoustic wave equation

**Small-amplitudes assumption**

Particle velocity is small, and there are only very small perturbations (fluctuations) to the equilibrium pressure and density:

\[ u - \text{small}, \quad p = p_0 + \tilde{p} \quad (\tilde{p} - \text{small}), \quad \rho = \rho_0 + \tilde{\rho} \quad (\tilde{\rho} - \text{small}). \]

The pressure fluctuations field \( \tilde{p} \) is called the **acoustic pressure**.

**Momentum equation** (Euler’s equation):

\[ \rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p \quad \text{linearization} \quad \rho_0 \frac{\partial u}{\partial t} = -\nabla p. \]

Notice that \( \nabla p = \nabla (p_0 + \tilde{p}) = \nabla \tilde{p} \).

**Continuity equation:**

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \quad \text{linearization} \quad \frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \nabla \cdot u = 0. \]
Acoustic wave equation

**Momentum equation** (Euler’s equation):

\[
\varrho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p \quad \xrightarrow{\text{linearization}} \quad \varrho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p .
\]

Notice that \( \nabla p = \nabla (p_0 + \tilde{p}) = \nabla \tilde{p} \).

**Continuity equation:**

\[
\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{u}) = 0 \quad \xrightarrow{\text{linearization}} \quad \frac{\partial \tilde{\varrho}}{\partial t} + \varrho_0 \nabla \cdot \mathbf{u} = 0 .
\]

Using divergence operation for the linearized momentum equation and time-differentiation for the linearized continuity equation yields:

\[
\frac{\partial^2 \tilde{\varrho}}{\partial t^2} - \Delta p = 0 .
\]
Acoustic wave equation

**Momentum equation** (Euler’s equation):

\[
\varrho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p \quad \text{linearization} \quad \varrho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p.
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Notice that \( \nabla p = \nabla (p_0 + \tilde{p}) = \nabla \tilde{p} \).

**Continuity equation:**

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\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{u}) = 0 \quad \text{linearization} \quad \frac{\partial \tilde{\varrho}}{\partial t} + \varrho_0 \nabla \cdot \mathbf{u} = 0.
\]

Using divergence operation for the linearized momentum equation and time-differentiation for the linearized continuity equation yields:

\[
\frac{\partial^2 \tilde{\varrho}}{\partial t^2} - \triangle p = 0.
\]

**Constitutive relation:**

\[
p = p(\tilde{\varrho}) \quad \rightarrow \quad \frac{\partial p}{\partial t} = \frac{\partial p}{\partial \tilde{\varrho}} \frac{\partial \tilde{\varrho}}{\partial t} \quad \rightarrow \quad \frac{\partial^2 \tilde{\varrho}}{\partial t^2} = \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} \quad \text{where} \quad c_0^2 = \frac{\partial p}{\partial \tilde{\varrho}}.
\]
Acoustic wave equation

Using divergence operation for the linearized momentum equation and time-differentiation for the linearized continuity equation yields:

\[
\frac{\partial^2 \tilde{\rho}}{\partial t^2} - \Delta p = 0.
\]

Constitutive relation:

\[
p = p(\tilde{\rho}) \quad \rightarrow \quad \frac{\partial p}{\partial t} = \frac{\partial p}{\partial \tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial t} \quad \rightarrow \quad \frac{\partial^2 \tilde{\rho}}{\partial t^2} = \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2}
\]

where \( c_0^2 = \frac{\partial p}{\partial \tilde{\rho}} \).

Wave equation for the pressure field

\[
\left( \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} - \Delta p \right) = 0
\]

is the **acoustic wave velocity** (or the **speed of sound**). Notice that the acoustic pressure \( \tilde{p} \) can be used here instead of \( p \). Moreover, the wave equation for the density-fluctuation field \( \tilde{\rho} \) (or for the compression field \( \tilde{\rho}/\rho_0 \)), for the velocity potential \( \phi \), and for the velocity field \( u \) can be derived analogously.
The speed of sound

**Inviscid isotropic elastic liquid.** The pressure in an inviscid liquid depends on the volume dilatation \( \text{tr} \, \varepsilon \):

\[
p = -K \, \text{tr} \, \varepsilon,
\]

where \( K \) is the bulk modulus. Now,

\[
\frac{\partial p}{\partial t} = -K \, \text{tr} \frac{\partial \varepsilon}{\partial t} = -K \nabla \cdot \mathbf{u}
\]

which means that the speed of sound \( c_0 = \sqrt{\partial p / \partial \tilde{\rho}} \) is given by the well-known formula:

\[
c_0 = \sqrt{\frac{K}{\rho_0}}.
\]
The speed of sound

Inviscid isotropic elastic liquid. The speed of sound is given by the well-known formula:

\[
c_0 = \sqrt{\frac{K}{\rho_0}}.
\]

Perfect gas. The determination of speed of sound in a perfect gas is complicated and requires the use of thermodynamic considerations. The final result is

\[
c_0 = \sqrt{\gamma \frac{p_0}{\rho_0}} = \sqrt{\gamma RT_0},
\]

where \(\gamma\) denotes the ratio of specific heats (\(\gamma = 1.4\) for air), \(R\) is the universal gas constant, and \(T_0\) is the (isothermal) temperature.

▶ For air at 20°C and normal atmospheric pressure: \(c_0 = 343\ \text{m/s}\).
Sub- and supersonic flow

A steady, unseparated, compressible flow past a thin airfoil may be written in the form

\[ u = U + \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \]

where the velocity potential \( \phi \) for the small disturbance to the uniform flow \( U \) satisfies

\[ (1 - M^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \]

where \( M = \frac{U}{c_0} \)

is the Mach number defined as the ratio of the speed of free stream to the speed of sound.
Sub- and supersonic flow

A steady, unseparated, **compressible flow** past a thin airfoil may be written in the form

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\]

is the **Mach number** defined as the ratio of the speed of free stream to the speed of sound.

▶ If \( M^2 \ll 1 \) that gives the Laplace equation which is the result that arises for **incompressible theory** (i.e., using \( \nabla \cdot \mathbf{u} = 0 \)).
Sub- and supersonic flow

A steady, unseparated, **compressible flow** past a thin airfoil may be written in the form

$$u = U + \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y},$$

where the **velocity potential** $\phi$ for the small disturbance to the uniform flow $U$ satisfies

$$(1 - M^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

where $M = \frac{U}{c_0}$ is the **Mach number** defined as the ratio of the speed of free stream to the speed of sound.

► If $M^2 \ll 1$ that gives the Laplace equation which is the result that arises for **incompressible theory** (i.e., using $\nabla \cdot u = 0$).

► Otherwise, three cases can be distinguished:

1. $M < 1$ – the **subsonic flow**
2. $M > 1$ – the **supersonic flow**
3. $M \approx 1$ – the **sound barrier**
$M < 1$ – the subsonic flow:

- there is some disturbance to the oncoming flow at all distances from the wing (even though it is very small when the distance is large);
- the drag is zero (inviscid theory) and the lift $= \frac{\text{lift}_{\text{incompressible}}}{\sqrt{1-M^2}}$.
Sub- and supersonic flow

1. $M < 1$ – the subsonic flow
2. $M > 1$ – the supersonic flow:
   - there is no disturbance to the oncoming stream except between the Mach lines extending from the ends of the airfoil and making the angle $\alpha = \arcsin \left( \frac{1}{M} \right)$ with the uniform stream;
   - the drag is not zero – it arises because of the sound wave energy which the wing radiates to infinity between the Mach lines.
Sub- and supersonic flow

1. $M < 1$ – the subsonic flow:
   - there is some disturbance to the oncoming flow at all distances from the wing (even though it is very small when the distance is large);
   - the drag is zero (inviscid theory) and the lift $= \frac{\text{lift}_{\text{incompressible}}}{\sqrt{1-M^2}}$.

2. $M > 1$ – the supersonic flow:
   - there is no disturbance to the oncoming stream except between the Mach lines extending from the ends of the airfoil and making the angle $\alpha = \arcsin \left( \frac{1}{M} \right)$ with the uniform stream;
   - the drag is not zero – it arises because of the sound wave energy which the wing radiates to infinity between the Mach lines.

3. $M \approx 1$ – the sound barrier:
   - sub- and supersonic theory is not valid;
   - nonetheless, it indicates that the wing is subject to a destructive effect of exceptionally large aerodynamic forces.