Fundamentals of Linear Elasticity

Introductory Course on Multiphysics Modelling

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Outline

1 Introduction

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1. **Introduction**
2. **Equations of motion**
   - Cauchy stress tensor
   - Derivation from the Newton's second law
   - Symmetry of stress tensor

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   - Original version
   - Generalized formulation
   - Voigt-Kelvin notation
   - Thermoelastic constitutive relations

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Introduction

Two types of linearity in mechanics

1. **Kinematic linearity** – strain-displacement relations are linear. This approach is valid if the **displacements are sufficiently small** (then higher order terms may be neglected).
2. **Material linearity** – constitutive behaviour of material is described by a linear relation.

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Two types of linearity in mechanics

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In the **linear theory of elasticity**:

- both types of linearity exist,
- therefore, all the **governing equations are linear** with respect to the unknown fields,
- all these fields are therefore described with respect to the (initial) **undeformed configuration** (and one cannot distinguished between the Euler and Lagrange descriptions),
- (as in all linear theories) the **superposition principle** holds which can be extremely useful.

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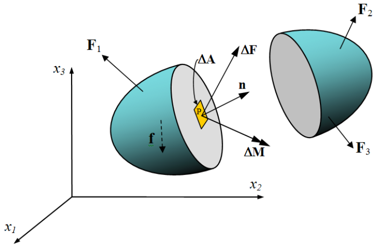
Equations of motion

Cauchy stress tensor



Traction (or **stress vector**), $\boldsymbol{t}\ \left[\frac{\mathrm{N}}{\mathrm{m}^2}\right]$

$$\boldsymbol{t} = \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{F}}{\Delta A} = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}A}$$

Here, $\Delta \boldsymbol{F}$ is the vector of resultant force acting of the (infinitesimal) area ΔA.

Equations of motion

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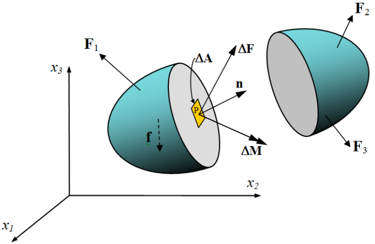

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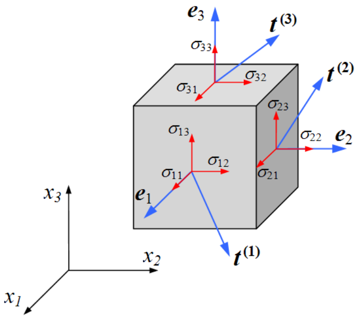
Cauchy's formula and tensor

$$\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n} \quad \text{or} \quad t_j = \sigma_{ij}\, n_i$$

Here, $\boldsymbol{n}$ is the unit normal vector and $\boldsymbol{\sigma}$ $\left[\frac{\mathrm{N}}{\mathrm{m}^2}\right]$ is the **Cauchy stress tensor**:

$$\boldsymbol{\sigma} \sim [\sigma_{ij}] = \begin{bmatrix} \boldsymbol{t}^{(1)} \\ \boldsymbol{t}^{(2)} \\ \boldsymbol{t}^{(3)} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$



Equations of motion

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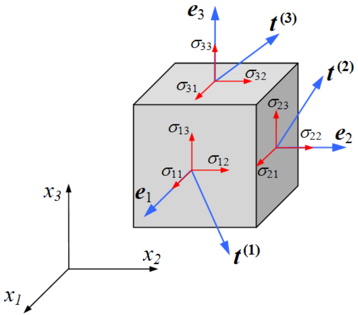

Surface tractions have three components: a direct stress normal to the surface and two shear stresses tangential to the surface.

- **Direct stresses** (normal tractions, e.g., σ_{11}) – tend to change the volume of the material (hydrostatic pressure) and are resisted by the body's bulk modulus.
- **Shear stress** (tangential tractions, e.g., σ_{12}, σ_{13}) – tend to deform the material without changing its volume, and are resisted by the body's shear modulus.

Equations of motion

Derivation from the Newton's second law

Principle of conservation of linear momentum

The time rate of **change of (linear) momentum** of particles equals the **net force** exerted on them:

$$\sum \frac{\mathrm{d}(m\boldsymbol{v})}{\mathrm{d}t} = \sum \boldsymbol{F}\,.$$

Here: m is the mass of particle, $\boldsymbol{v}$ is the particle velocity, and $\boldsymbol{F}$ is the net force acting on the particle.

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For any (sub)domain Ω of a solid continuum of density $\varrho\ \left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right]$, subject to body forces (per unit volume) $\boldsymbol{f}\ \left[\frac{\mathrm{N}}{\mathrm{m}^3}\right]$ and surface forces (per unit area) $\boldsymbol{t}\ \left[\frac{\mathrm{N}}{\mathrm{m}^2}\right]$ acting on the boundary Γ, the principle of conservation of linear momentum reads:

$$\int_\Omega \varrho\,\frac{\partial^2\boldsymbol{u}}{\partial t^2}\,\mathrm{d}\Omega = \int_\Omega \boldsymbol{f}\,\mathrm{d}\Omega + \int_\Gamma \boldsymbol{t}\,\mathrm{d}\Gamma\,,$$

where $\boldsymbol{u}\ [\mathrm{m}]$ is the displacement vector.

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$$\int_\Omega \varrho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \, \mathrm{d}\Omega = \int_\Omega \boldsymbol{f} \, \mathrm{d}\Omega + \int_\Gamma \boldsymbol{t} \, \mathrm{d}\Gamma \,,$$

where $\boldsymbol{u}$ $[\mathrm{m}]$ is the displacement vector. The **Cauchy's formula** and **divergence theorem** can be used for the last term, namely

$$\int_\Gamma \boldsymbol{t} \, \mathrm{d}\Gamma = \int_\Gamma \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma = \int_\Omega \nabla \cdot \boldsymbol{\sigma} \, \mathrm{d}\Omega \,.$$

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Global and local equations of motion

$$\int_\Omega \left(\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{f} - \varrho\, \frac{\partial^2 \boldsymbol{u}}{\partial t^2}\right) \mathrm{d}\Omega = \boldsymbol{0} \quad\rightarrow\quad \nabla\cdot\boldsymbol{\sigma} + \boldsymbol{f} = \varrho\, \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \quad \text{or} \quad \sigma_{ji|j} + f_i = \varrho\, \ddot{u}_i\,.$$

The global form is true for *any* subdomain Ω, which yields the local form.

Equations of motion

Symmetry of stress tensor

Principle of conservation of angular momentum

The time rate of **change of the total moment of momentum** for a system of particles is equal to the vector **sum of the moments of external forces** acting on them:

$$\sum \frac{\mathrm{d}(m\boldsymbol{v} \times \boldsymbol{x})}{\mathrm{d}t} = \sum \boldsymbol{F} \times \boldsymbol{x}\,.$$

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For continuum, in the absence of body couples (i.e., without volume-dependent couples), the principle leads to **the symmetry of stress tensor**, that is,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}} \qquad \text{or} \qquad \sigma_{ij} = \sigma_{ji} \,.$$

Thus, only six (of nine) stress components are independent:

$$\boldsymbol{\sigma} \sim \left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ \text{sym.} & & \sigma_{33} \end{bmatrix} .$$

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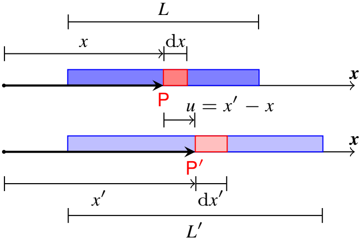
Kinematic relations

Strain measure and tensor (for small displacements)

Longitudinal strain (global and local) is defined as follows:

$$\varepsilon = \frac{L' - L}{L}\,, \qquad \varepsilon(x) = \frac{\mathrm{d}x' - \mathrm{d}x}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}\,.$$



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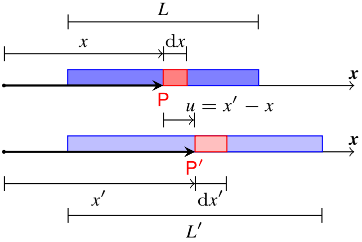

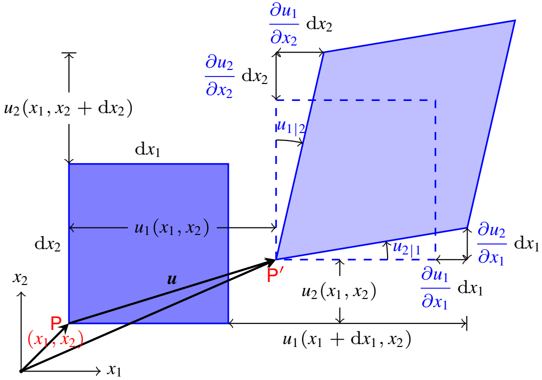

Strain tensor

$$\boldsymbol{\varepsilon} = \mathrm{sym}(\nabla \boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}}\right),$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i|j} + u_{j|i}\right)\,,$$

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2D case:

$$u_{1|1} = \varepsilon_{11} , \qquad u_{2|2} = \varepsilon_{22} , \qquad u_{1|2} + u_{2|1} = 2\varepsilon_{12} ,$$

Here, ε_{11}, ε_{22}, ε_{12} must satisfy the following **compatibility equation**:

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3D case:

$$\varepsilon_{ij|kl} + \varepsilon_{kl|ij} = \varepsilon_{lj|ki} + \varepsilon_{ki|lj} \,.$$

Of these 81 equations only 6 are different (i.e., linearly independent).

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The strain **compatibility equations are satisfied** automatically when the **strains are computed from a displacement field**.

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Constitutive equations: Hooke's Law

Original version

Original formulation of Hooke's Law (1660)

Robert Hooke (1635-1703) first presented his law in the form of a Latin anagram

$$\textit{CEIINOSSITTUV} \quad = \quad \textit{UT TENSIO, SIC VIS}$$

which translates to "*as is the extension, so is the force*" or in contemporary language "*extension is directly proportional to force*".

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The classical (1-dimensional) Hooke's Law describe the linear variation of tension with extension in an elastic spring:

$$F = k\,u \qquad \text{or} \qquad \sigma = E\,\varepsilon\,.$$

Here:

- F is the **force** acting on the spring, whereas σ is the **tension**,
- k is the **spring constant**, whereas E is the **Young's modulus**,
- u is the displacement (of the spring end), and ε is the **extension** (elongation).

Constitutive equations: Hooke's Law

Generalized formulation

Generalized Hooke's Law (GHL)

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} \qquad \text{or} \qquad \boldsymbol{\varepsilon} = \boldsymbol{S} : \boldsymbol{\sigma} \qquad \text{where} \qquad \boldsymbol{S} = \boldsymbol{C}^{-1} .$$

Here: $\boldsymbol{C}$ $\left[\mathrm{N/m^2}\right]$ is the (fourth-order) **elasticity tensor**,
$\boldsymbol{S}$ $\left[\mathrm{m^2/N}\right]$ is the **compliance tensor** (inverse of $\boldsymbol{C}$).

GHL in index notation:

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Symmetries of elastic tensor

$$C_{ijkl} = C_{klij} , \quad C_{ijkl} = C_{jikl} , \quad C_{ijkl} = C_{ijlk} .$$

- The first symmetry is valid for the so-called **hyperelastic** materials (for which the stress-strain relationship derives from a strain energy density function).
- Thus, at most, only **21** material constants out of 81 components are independent.

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Hooke's Law for an isotropic material

$$\boldsymbol{\sigma} = 2\mu\, \boldsymbol{\varepsilon} + \lambda \big(\operatorname{tr} \boldsymbol{\varepsilon}\big) \boldsymbol{I} \qquad \text{or} \qquad \sigma_{ij} = 2\mu\, \varepsilon_{ij} + \lambda\, \varepsilon_{kk}\, \delta_{ij} .$$

Here, the so-called **Lamé coefficients** are used (related to the **Young's modulus** E and **Poisson's ratio** ν):

- the **shear modulus** $\mu = \frac{E}{2(1+\nu)}$,
- the **dilatational constant** $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$.

Constitutive equations: Hooke's Law

Voigt-Kelvin notation

Rule of change of subscripts

$$11 \to 1\,,\quad 22 \to 2\,,\quad 33 \to 3\,,\quad 23 \to 4\,,\quad 13 \to 5\,,\quad 12 \to 6\,.$$

Anisotropy (**21** independent material constants)

$$\begin{Bmatrix} \sigma_1 = \sigma_{11} \\ \sigma_2 = \sigma_{22} \\ \sigma_3 = \sigma_{33} \\ \sigma_4 = \sigma_{23} \\ \sigma_5 = \sigma_{13} \\ \sigma_6 = \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{sym.} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 = \varepsilon_{11} \\ \varepsilon_2 = \varepsilon_{22} \\ \varepsilon_3 = \varepsilon_{33} \\ \varepsilon_4 = \gamma_{23} = 2\varepsilon_{23} \\ \varepsilon_5 = \gamma_{13} = 2\varepsilon_{13} \\ \varepsilon_6 = \gamma_{12} = 2\varepsilon_{12} \end{Bmatrix}$$

Notice that the elastic strain energy per unit volume equals:
$\frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\sigma_\alpha\varepsilon_\alpha$ (with summation here over $i,j = 1,2,3$ and $\alpha = 1,\ldots,6$).

Orthotropy (**9** nonzero independent material constants)

Transversal isotropy (**5** independent out of 9 nonzero components)

Isotropy (**2** independent material constants)

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Notice that there is no interaction between the normal stresses and the shear strains.

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Orthotropy (**9** nonzero independent material constants)

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$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{sym.} & & & C_{44} & 0 \\ & & & & & \frac{C_{11}-C_{12}}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}$$

$$C_{22} = C_{11}\,, \quad C_{23} = C_{13}\,, \quad C_{55} = C_{44}\,, \quad C_{66} = \frac{C_{11} - C_{12}}{2}\,.$$

Isotropy (**2** independent material constants)

Constitutive equations: Hooke's Law

Voigt-Kelvin notation

Rule of change of subscripts

$$11 \to 1\,, \quad 22 \to 2\,, \quad 33 \to 3\,, \quad 23 \to 4\,, \quad 13 \to 5\,, \quad 12 \to 6\,.$$

Anisotropy (**21** independent material constants)

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$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda + 2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & \text{sym.} & & & \mu & 0 \\ & & & & & \mu \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}$$

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Constitutive equations: Hooke's Law

Thermoelastic constitutive relations

- **Temperature changes** in the elastic body **cause thermal expansion** of the material, even though the variation of elastic constants with temperature is neglected.

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Uncoupled thermoelasticity (theory of thermal stresses)

Usually, the above assumptions are satisfied and the thermo-mechanical problem (involving heat transfer) can be dealt as follows:

1. the heat equations are uncoupled from the (elastic) mechanical equations and are solved first,
2. the computed temperature field is used as data ("thermal loads") for the mechanical problem.

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If the thermoelastic dissipation significantly influence the thermal field the **fully coupled theory of thermo-elasticity** must be applied, where the coupled heat and mechanical equations are solved simultaneously.

Constitutive equations: Hooke's Law

Thermoelastic constitutive relations

GHL with linear thermal terms (thermal stresses)

$$\boldsymbol{\sigma} = \boldsymbol{C} : \left[\boldsymbol{\varepsilon} - \boldsymbol{\alpha}\,\Delta T\right] = \boldsymbol{C} : \boldsymbol{\varepsilon} - \underbrace{\boldsymbol{C} : \boldsymbol{\alpha}\,\Delta T}_{\text{thermal stress}} \quad \text{or} \quad \boldsymbol{\varepsilon} = \boldsymbol{S} : \boldsymbol{\sigma} + \underbrace{\boldsymbol{\alpha}\,\Delta T}_{\text{thermal strain}},$$

or in index notation

$$\sigma_{ij} = C_{ijkl}\left[\varepsilon_{kl} - \alpha_{kl}\,\Delta T\right] \qquad \text{or} \qquad \varepsilon_{ij} = S_{ijkl}\,\sigma_{kl} + \alpha_{ij}\,\Delta T\,.$$

Here, $\Delta T\ [\mathrm{K}]$ is the temperature difference (from the reference temperature of the undeformed body), whereas the tensor $\boldsymbol{\alpha}\ [\mathrm{K}^{-1}]$ groups **linear coefficients of thermal expansion**

$$\boldsymbol{\alpha} \sim \left[\alpha_{ij}\right] = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix}.$$

For **isotropic materials**: $\alpha_{11} = \alpha_{22} = \alpha_{33} \equiv \alpha$, that is, $\boldsymbol{\alpha} = \alpha\,\boldsymbol{I}$ or $\alpha_{ij} = \alpha\,\delta_{ij}$.

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   - Initial-Boundary-Value Problem
   - Displacement formulation of elastodynamics
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Problem of linear elasticity

Initial-Boundary-Value Problem of elastodynamics

General IBVP of elastodynamics

Find **15** unknown fields: u_i (**3** displacements), ε_{ij} (**6** strains), and σ_{ij} (**6** stresses) – satisfying:

- **3** equations of motion: $\sigma_{ij|j} + f_i = \varrho\, \ddot{u}_i$,
- **6** strain-displacement relations: $\varepsilon_{ij} = \frac{1}{2}\left(u_{i|j} + u_{j|i}\right)$,
- **6** stress-strain laws: $\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$,

with the **initial conditions** (at $t = t_0$):

$$u_i(\boldsymbol{x}, t_0) = u_i^0(\boldsymbol{x}) \quad \text{and} \quad \dot{u}_i(\boldsymbol{x}, t_0) = v_i^0(\boldsymbol{x}) \quad \text{in } \Omega\,,$$

and subject to the **boundary conditions**:

$$u_i(\boldsymbol{x}, t) = \hat{u}_i(\boldsymbol{x}, t) \ \text{ on } \Gamma_u\,, \qquad \sigma_{ij}(\boldsymbol{x}, t)\, n_j = \hat{t}_i(\boldsymbol{x}, t) \ \text{ on } \Gamma_t\,,$$

$$\sigma_{ij}(\boldsymbol{x}, t)\, n_j = \hat{t}_i + h\,(\hat{u}_i - u_i) \ \text{ on } \Gamma_h\,,$$

where $\Gamma_u \cup \Gamma_t \cup \Gamma_h = \Gamma$, and $\Gamma_u \cap \Gamma_t = \emptyset$, $\Gamma_u \cap \Gamma_h = \emptyset$, $\Gamma_t \cap \Gamma_h = \emptyset$.

Problem of linear elasticity

Displacement formulation of elastodynamics

Anisotropic case:

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} = C_{ijkl}\,\frac{1}{2}\big(u_{k|l} + u_{l|k}\big) = C_{ijkl}\,u_{k|l} \quad \big(\text{since } C_{ijkl} = C_{ijlk}\big)$$

Displacement formulation of elastodynamics

$$\big(C_{ijkl}\,u_{k|l}\big)_{|j} + f_i = \varrho\,\ddot{u}_i \quad \text{or} \quad \nabla\cdot\big(\boldsymbol{C} : \nabla\boldsymbol{u}\big) + \boldsymbol{f} = \varrho\,\ddot{\boldsymbol{u}}$$

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Isotropic case:

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\delta_{ij} = \mu\big(u_{i|j} + u_{j|i}\big) + \lambda\,u_{k|k}\,\delta_{ij}$$

Navier's equations for isotropic elasticity

For homogeneous materials (i.e., when $\mu = \text{const}\,.$ and $\lambda = \text{const}\,.$):

$$\mu\,u_{i|jj} + \big(\mu + \lambda\big)u_{j|ji} + f_i = \varrho\,\ddot{u}_i \quad \text{or} \quad \mu\,\triangle\boldsymbol{u} + \big(\mu + \lambda\big)\nabla\big(\nabla\cdot\boldsymbol{u}\big) + \boldsymbol{f} = \varrho\,\ddot{\boldsymbol{u}}$$

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Boundary conditions:

(Dirichlet) $u_i = \hat{u}_i$ on $\Gamma_u\,,$

(Neumann) $t_i = \hat{t}_i$ on $\Gamma_t\,,$

(Robin) $t_i = \hat{t}_i + h\left(\hat{u}_i - u_i\right)$ on $\Gamma_h\,,$

$$t_i = \sigma_{ij}\, n_j = \begin{cases} C_{ijkl}\, u_{k|l}\, n_j & \text{– for anisotropic materials,} \\ \mu\left(u_{i|j} + u_{j|i}\right) n_j + \lambda\, u_{k|k}\, n_i & \text{– for isotropic materials.} \end{cases}$$

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Principle of virtual work

Admissible and virtual displacements

Definition (Admissible displacements)

Admissible displacements (or configuration) of a mechanical system are any displacements (configuration) that satisfy the *geometric constraints* of the system. The **geometric constraints** are:

- geometric (essential) boundary conditions,
- kinematic relations (strain-displacement equations and compatibility equations).

Of all (kinematically) admissible configurations only one corresponds to the equilibrium configuration under the applied loads (it is the one that also satisfies Newton's second law).

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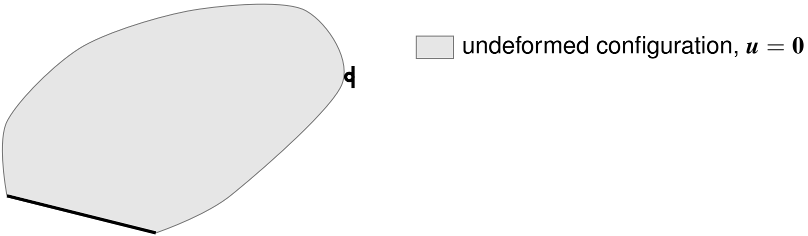
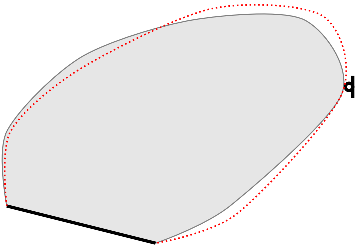
Definition (Virtual displacements)

Virtual displacements are any displacements that describe small (infinitesimal) *variations* of the true configurations. They satisfy the *homogeneous* form of the specified *geometric boundary conditions*.

Principle of virtual work



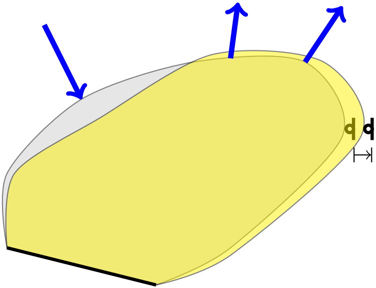
Principle of virtual work



undeformed configuration, $\boldsymbol{u} = \boldsymbol{0}$

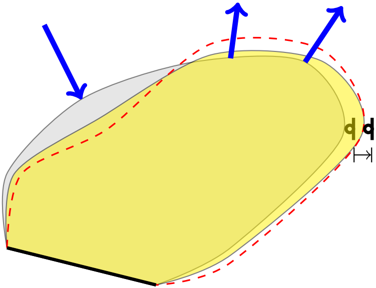
virtual displacements, $\delta \boldsymbol{u}$

Principle of virtual work



- undeformed configuration, $\boldsymbol{u} = \mathbf{0}$
- external forces (loads)
- geometric constraint
- actual configuration, $\boldsymbol{u} = \bar{\boldsymbol{u}}$

Principle of virtual work



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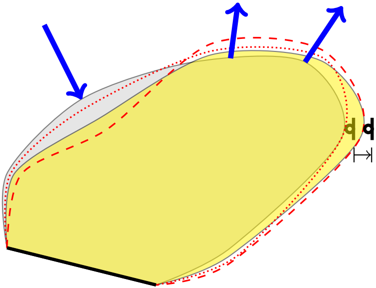

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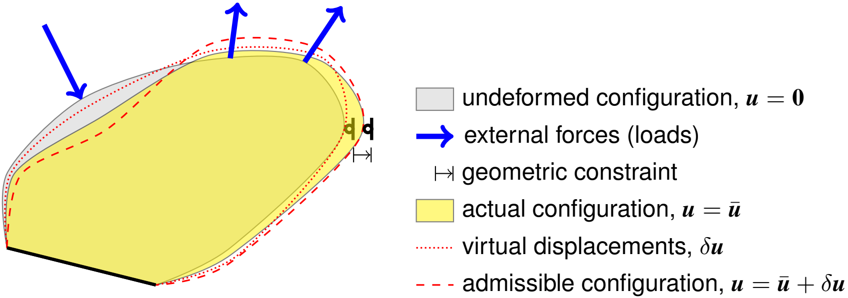
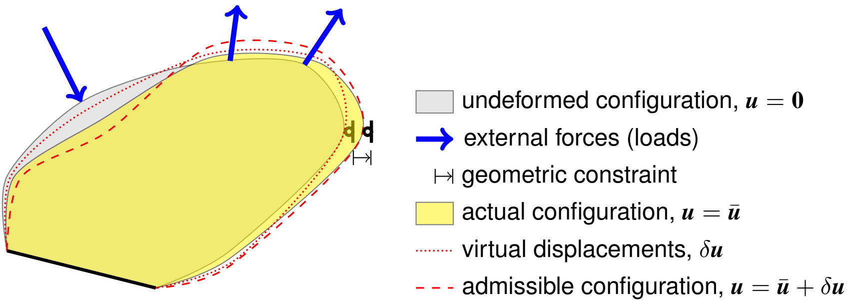


Definition (Virtual work)

Virtual work is the work done by the actual forces through the virtual displacement of the actual configuration. The virtual work in a deformable body consists of two parts:

1. the **internal virtual work** done by internal forces (stresses),
2. the **external virtual work** done by external forces (i.e., loads).

Principle of virtual work



Theorem (Principle of virtual work)

A continuous body is in equilibrium if and only if the virtual work of all forces, internal and external, acting on the body is zero in a virtual displacement:

$$\delta W = \delta W_{int} + \delta W_{ext} = 0.$$