Exercise 4.1

Displacement formulation of linear elastodynamics: strong and weak forms, Galerkin FE model

General IBVP of linear elastodynamics

Find $u_i = u_i(\boldsymbol{x}, t) =?$, $\varepsilon_{ij} = \varepsilon_{ij}(\boldsymbol{x}, t) =?$, $\sigma_{ij} = \sigma_{ij}(\boldsymbol{x}, t) =?$ satisfying in Ω:

$$\sigma_{ij|j} + f_i = \varrho\, \ddot{u}_i\,, \qquad \varepsilon_{ij} = \frac{1}{2}\left(u_{i|j} + u_{j|i}\right), \qquad \sigma_{ij} = \begin{cases} C_{ijkl}\, \varepsilon_{kl} & \text{(anisotropy)}\,, \\ 2\mu\, \varepsilon_{ij} + \lambda\, \varepsilon_{kk}\, \delta_{ij} & \text{(isotropy)}\,, \end{cases}$$

with the initial conditions (at $t = t_0$):

$$u_i(\boldsymbol{x}, t_0) = u_i^0(\boldsymbol{x}) \quad \text{and} \quad \dot{u}_i(\boldsymbol{x}, t_0) = v_i^0(\boldsymbol{x}) \quad \text{in } \Omega\,,$$

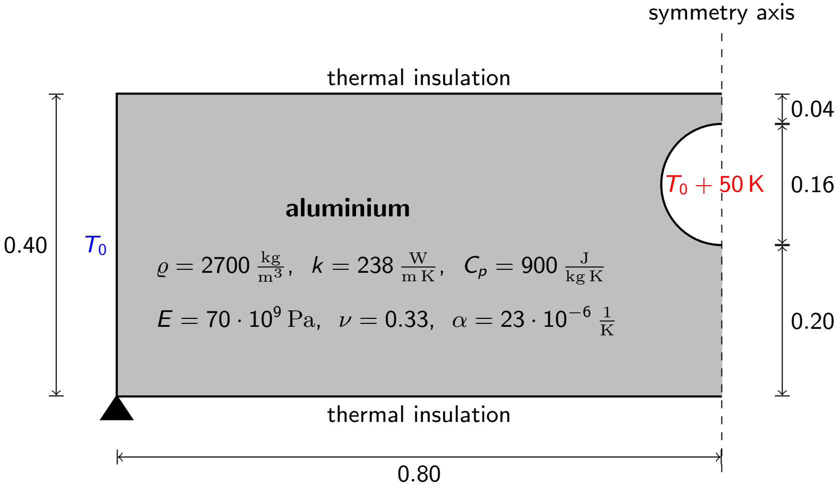
and subject to the boundary conditions on $\Gamma = \Gamma_u \cup \Gamma_t$ $(\Gamma_u \cap \Gamma_t = \emptyset)$:

$$u_i(\boldsymbol{x}, t) = \hat{u}_i(\boldsymbol{x}, t) \ \text{ on } \Gamma_u\,, \qquad \sigma_{ij}(\boldsymbol{x}, t)\, n_j = \hat{t}_i(\boldsymbol{x}, t) \ \text{ on } \Gamma_t\,.$$

1. Derive the displacement formulation of elasticity (Navier's equations)
2. Derive the weak variational form of the displacement formulation
3. Derive the Galerkin FE model (i.e., $\mathbf{M}\, \ddot{\mathbf{q}}(t) + \mathbf{K}\, \mathbf{q}(t) = \mathbf{Q}(t)$)

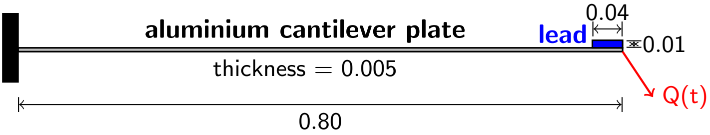
Exercise 4.2

A simple thermo-elastic problem



Exercise 4.3

Using the Weak Form PDE Interface in COMSOL Multiphysics

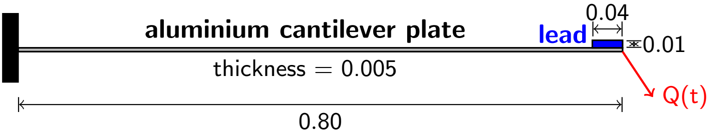


$$\varrho_{\mathsf{Al}} = 2700\,\frac{\mathrm{kg}}{\mathrm{m}^3}\,, \qquad E_{\mathsf{Al}} = 70 \cdot 10^9\,\mathrm{Pa}\,, \qquad \nu_{\mathsf{Al}} = 0.33\,,$$

$$\varrho_{\mathsf{Pb}} = 11340\,\frac{\mathrm{kg}}{\mathrm{m}^3}\,, \qquad E_{\mathsf{Pb}} = 16 \cdot 10^9\,\mathrm{Pa}\,, \qquad \nu_{\mathsf{Pb}} = 0.44\,.$$

Exercise 4.3

Using the Weak Form PDE Interface in COMSOL Multiphysics



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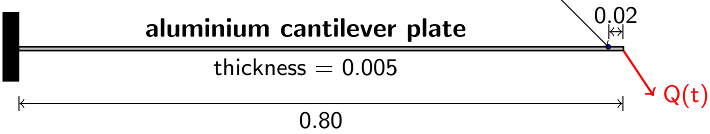
concentrated mass weak term

$$m_{\mathsf{Pb}} = \varrho_{\mathsf{Pb}} \cdot 0.04\,[\mathrm{m}] \cdot 0.01\,[\mathrm{m}]$$



Exercise 4.3

Using the Weak Form PDE Interface in COMSOL Multiphysics

Weak form for harmonic linear elasticity

$$\omega^2 \int_{\Omega} \varrho\, u_i\, \delta u_i - \int_{\Omega} \sigma_{ij}\, \delta u_{i|j} + \int_{\Omega} f_i\, \delta u_i + \int_{\Gamma_t} \hat{t}_i\, \delta u_i = 0$$

Here: $\sigma_{ij} = \sigma_{ij}(\boldsymbol{u}) = \begin{cases} C_{ijkl}\, u_{k|l} & \text{– for anisotropic material} \\ \mu\left(u_{i|j} + u_{j|i}\right) + \lambda\, u_{k|k}\, \delta_{ij} & \text{– for isotropic material} \end{cases}$

Weak form for harmonic linear elasticity

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- Constants:

```
omega = 2*pi*f
mu = E/2/(1+nu)     lam = nu*E/(1+nu)/(1-2*nu)
```

- Domain expressions (for an isotropic material):

```
s11 = 2*mu*u1x + lam*(u1x+u2y)
s22 = 2*mu*u2y + lam*(u1x+u2y)
s12 = mu*(u1y+u2x)
```

Weak form for harmonic linear elasticity

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- Domain integrand (in Ω):

```
omega^2*rho*( u1*test(u1) + u2*test(u2) )
- s11*test(u1x) - s12*test(u1y+u2x) - s22*test(u2y)
f1*test(u1) + f2*test(u2)
```

- Neumann boundary integrand (on Γ_t):

```
t1*test(u1) + t2*test(u2)
```

- Concentrated mass weak term:

```
omega^2*mass*( u1*test(u1) + u2*test(u2) )
```