A new result on the Nirenberg problem for expanding maps

Janusz Szczepański

Polish Academy of Sciences, Institute of Fundamental Technological Research, Świętokrzyska 21, 00-049 Warsaw, Poland

Received 10 June 1998; accepted 29 December 1998

Keywords: Nirenberg problem; Expanding map

1. Introduction

The problem of existence of solutions to nonlinear equations is one of the most important and challenging questions in analysis. In this paper we show that even very strong conditions imposed on a nonlinear operator do not assure the solvability of the corresponding nonlinear equation.

In 1974 Nirenberg [8] stated the following problem:

Let $H$ be a Hilbert space and let $f : H \to H$ be a continuous map that is expanding and whose range contains an open set. Does $f$ map $H$ onto $H$? In other words, the problem is, if these conditions imply the existence of solution to the equation $f(x) = h$ for every $h \in H$.

This question could be generalized to the case when the space considered is a Banach space $X$.

There are several partial positive answers to these problems under additional assumptions:

- $X$ is finite dimensional [1,2],
- $f = I - C$ where $C$ is compact or a contraction or more generally a $k$-set-contraction [5,9],
- $f$ is strongly monotone, i.e. there exists $s > 0$ such that [3,6]

$$\text{Re}(f(x) - f(y), x - y) \geq s||x - y||^2$$

for all $x, y \in H$.

E-mail address: jszczepa@ippt.gov.pl (J. Szczepański).
In [4] Chang and Shujie proved the surjectivity of \( f : X \rightarrow Y \) (\( Y \) is a Banach space) under the additional assumptions that \( Y \) is reflexive, \( f \) is Fréchet-differentiable and
\[
\limsup_{x \rightarrow x_0} \| f'(x) - f'(x_0) \| < 1 \quad \text{for all } x_0 \in X.
\]

Thirteen years ago Morel and Steinlein [7] gave a counterexample in the case when \( f \) acts in the Banach space \( L^1(\mathbb{N}) \).

In [10] we exhibited a continuous map acting in Hilbert space such that its range has nonempty interior, it maps spheres (centered at 0) into spheres and its trajectories exponentially diverge. The map was not onto.

In this paper, for \( H \) being the real \( L^2(\mathbb{N}) \) space, we first construct a map \( F : H \rightarrow H \) which is continuous, not onto, it maps spheres centered at 0 into spheres and is an isometry on each such sphere, and \( F(H) \) has nonempty interior. Then we construct a family of maps \( F_\varepsilon : H \rightarrow H \) such that for every \( \varepsilon > 0 \) the map \( F_\varepsilon \) satisfies the above conditions (except for the isometry condition; instead, it has the similar property that if \( \| x \| = \| y \| \) then \( \| F_\varepsilon(x) - F_\varepsilon(y) \| = c_\varepsilon \| x - y \| \), where \( c_\varepsilon \geq 2 \) is a constant) and additionally it is expanding except when both \( x \) and \( y \) are elements of a small annulus \( P_\varepsilon = \{ x \in H : 1 < \| x \| < 1 + \varepsilon \} \).

None of these maps is onto [10].

The considerations of this paper exhibit the differences between the geometry of finite-dimensional and infinite-dimensional Hilbert spaces.

2. The result

We start by constructing a map
\[
F : L^2(\mathbb{N}) \rightarrow L^2(\mathbb{N})
\]
with the following properties:
(a) \( F \) is continuous,
(b) \( F \) is not onto,
(c) \( B(0,1) \subset F(L^2(\mathbb{N})) \) where \( B(0,1) \) is the unit ball in \( L^2(\mathbb{N}) \),
(d) \( \| F(x) \| = \| x \| \) for all \( x \in L^2(\mathbb{N}) \),
(e) for every \( x, y \in L^2(\mathbb{N}) \) such that \( \| x \| = \| y \| \) we have
\[
\| F(x) - F(y) \| = \| x - y \|
\]
(f) \( F \) is an injection.

Let \( x = (x_1, x_2, x_3, \ldots) \in L^2(\mathbb{N}) \). Define
\[
F(x) := \begin{cases} 
  x & \text{for } \| x \| \leq 1, \\
  (x_1, x_2, \ldots, x_{n(x)-1}, x_{n(x)} \sin z_x, x_{n(x)} \cos z_x, x_{n(x)+1}, \ldots) & \text{for } 1 < \| x \| \leq 2, \\
  \sigma x & \text{for } 2 < \| x \|.
\end{cases}
\]

where \( z_x \) and \( n(x) \) are defined as follows:
\[
n(x) = \left[ \frac{1}{\| x \| - 1} \right] \quad (\lfloor \rfloor \text{ means integer part}),
\]
\[ a_x = \frac{\pi}{2} \left( \frac{1}{\|x\|_2} - \left[ \frac{1}{\|x\|_2} - 1 \right] \right) \quad \text{for} \quad 1 < \|x\|_2 \leq 2. \]

The operator \( \sigma \) is a shift operator, i.e. \( \sigma x = (0, x_1, x_2, x_3, \ldots) \).

**Remark 1.** Note that the restrictions \( F_{[0,1]} \) and \( F_{[x \in L^2(k) : 2 \leq \|x\|_2]} \) are isometries.

Now we show that (a)–(f) are satisfied. 
(a) Assume that \( x^k \to x \) where for every \( k \),
\[ x^k = (x^k_1, x^k_2, x^k_3, \ldots), \quad x = (x_1, x_2, x_3, \ldots) \in L^2(\mathbb{N}) \]
(so we have \( \|x^k\|_2 \to \|x\|_2 \)).

By Remark 1 it is enough to consider two cases:

(I) \quad \|x\| < 1 < \|x\|_2 \leq 2,

(II) \quad \|x\|_2 = 1

(IIa) By definition of \( n(x) \) and \( n(x^k) \), there is \( k_0 \) such that for every \( k > k_0 \) we have

either (IIa') \quad n(x^k) = n(x) \quad \text{or} \quad (IIb') \quad n(x^k) + 1 = n(x).

(IIa') can only occur when \( 1/(\|x\|_2 - 1) \) is an integer.

If (IIa') holds then \( x^k \to x \) and one can see that \( F(x^k) \to F(x) \). If (IIb') holds then \( x = 0 \) and \( x^k \to \pi/2. \)

Thus
\[ F(x) = (x_1, x_2, \ldots, x_{n(x)} - 1, 0, x_{n(x)} - x_{n(x) + 1}, x_{n(x) + 2}, \ldots), \]

\[ F(x^k) = (x^k_1, x^k_2, \ldots, x^k_{n(x) - 1} \sin x^k_1, x^k_{n(x) - 1} \cos x^k_1, x^k_{n(x) + 1}, x^k_{n(x) + 2}, \ldots) \]

and it is easy to check that \( F(x^k) \to F(x) \).

(IIb) If \( \|x\|_2 = 1 \) then \( n(x^k) \to \infty \) (by Remark 1 it is enough to consider the case \( \|x^k\|_2 > 1 \)). Take \( k_0 \) such that for all \( k > k_0 \),
\[ \|x - x^k\|_2 < \frac{\varepsilon}{4} \]

and
\[ \left( \sum_{i=n(x)}^{\infty} x_i^2 \right)^{1/2} < \frac{\varepsilon}{4}. \]

We have
\[ \left| \left( \sum_{i=n(x^k)}^{\infty} x_i^2 \right)^{1/2} - \left( \sum_{i=n(x^k)}^{\infty} (x_i^k)^2 \right)^{1/2} \right| \leq \left( \sum_{i=n(x^k)}^{\infty} (x_i - x_i^k)^2 \right)^{1/2} \]
\[ \leq \|x - x^k\|_2 < \frac{\varepsilon}{4}. \]

(3)

So, we have
\[ \left( \sum_{i=n(x^k)}^{\infty} (x_i^k)^2 \right)^{1/2} < \left( \sum_{i=n(x^k)}^{\infty} x_i^2 \right)^{1/2} + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}. \]

(4)
Finally, from (1), (2) and (4) we get for $k > k_0$,
\[
\|F(x) - F(x^k)\|_2 \leq \left( \sum_{i=1}^{n(x^k)-1} (x_i - x^k_i)^2 \right)^{1/2} + \left( \sum_{i=n(x^k)}^\infty x_i^2 \right)^{1/2} + \left( \sum_{i=n(X)}^\infty (x^k_i)^2 \right)^{1/2} < \varepsilon
\]
and thus $F(x^k) \to F(x)$, which finishes the proof of (a).

Properties (b), (c) and (d) follow immediately from the definition of $F$.

(e) If $\|x\|_2 = \|y\|_2$ then $n(x) = n(y)$, $x_\infty = y_\infty$ and one can see that $\|F(x) - F(y)\|_2 = \|x - y\|_2$.

Property (f) follows from (d) and (e).

Now we show the following

**Theorem.** There exists a family of maps $F_\varepsilon : L^2(\mathbb{N}) \to L^2(\mathbb{N})$ satisfying the following conditions for every $\varepsilon > 0$:

(a) $F_\varepsilon$ is continuous,

(b) $F_\varepsilon$ is not onto,

(c) $B(0,1) \subset F_\varepsilon(L^2(\mathbb{N}))$ where $B(0,1)$ is the unit ball in $L^2(\mathbb{N})$,

(d) $\|F_\varepsilon(x)\|_2 = c_\varepsilon \|x\|_2$ for all $x \in L^2(\mathbb{N})$ where $c_\varepsilon \geq 2$ is a constant,

(e) for all $x, y \in L^2(\mathbb{N})$ such that $\|F_\varepsilon(x) - F_\varepsilon(y)\|_2 < \|x - y\|_2$ we have
\[
\|F_\varepsilon(x) - F_\varepsilon(y)\|_2 = c_\varepsilon \|x - y\|_2
\]

(f) $F_\varepsilon$ is an injection,

(g) for every $x, y \in L^2(\mathbb{N})$ satisfying the condition
\[
(\Pi_\varepsilon) \quad x \notin P_\varepsilon \text{ or } y \notin P_\varepsilon \quad \text{where } P_\varepsilon = \{ x \in L^2(\mathbb{N}) : 1 < \|x\|_2 < 1 + \varepsilon \}
\]

the expanding property holds, i.e.
\[
\|F_\varepsilon(x) - F_\varepsilon(y)\|_2 \geq \|x - y\|_2
\]

**Proof.** Define $F_\varepsilon(x) := c_\varepsilon F(x)$ for $x \in L^2(\mathbb{N})$ where $c_\varepsilon \geq 2$ is a constant to be chosen later. By the conditions (a)–(f) for $F$, one can see that for every $c_\varepsilon \geq 2$ the conditions (a)–(f) are satisfied.

Thus, we must prove (g). To do this we show that for fixed $\varepsilon > 0$ there exists a constant $M_\varepsilon > 0$ such that for all $x, y$ satisfying $(\Pi_\varepsilon)$ we have
\[
\|F(x) - F(y)\|_2 \geq M_\varepsilon \|x - y\|_2
\]

Then we put $c_\varepsilon := \max \{ 2, 1/M_\varepsilon \}$.

The proof will be by contradiction.

Assume that there are no such $M_\varepsilon$. Then there are sequences $(x^k)$ and $(y^k)$, with each pair $x^k, y^k$ satisfying condition $(\Pi_\varepsilon)$, such that
\[
\frac{\|F(x^k) - F(y^k)\|_2}{\|x^k - y^k\|_2} \to 0
\]
First, by (d) and the triangle inequality,
\[ \|F(x^k) - F(y^k)\| \geq \|F(x^k)\| - \|F(y^k)\| = \|x^k\| - \|y^k\| \]
and consequently
\[ \frac{\|F(x^k) - F(y^k)\|}{\|x^k - y^k\|_2} \geq \frac{\|x^k\| - \|y^k\|_2}{\|x^k\|_2 + \|y^k\|_2}. \] (9)

If one of the sequences \((\|x^k\|_2), (\|y^k\|_2)\) is bounded and the other tends to infinity then the right-hand side of (9) tends to 1, contrary to (7).

By Remark 1, if either \(\|x^k\|_2 \leq 1\) and \(\|y^k\|_2 \leq 1\), or \(2 \leq \|x^k\|_2\) and \(2 \leq \|y^k\|_2\), then \(\|F(x^k) - F(y^k)\|_2 = \|x^k - y^k\|_2\). Thus, we must assume that both sequences are bounded. Assume that \(\|x^k\|_2 + \|y^k\|_2 < c_1\).

First, notice that we need only consider the case \(\|x^k\|_2 - \|y^k\|_2 \to 0\) since otherwise there are subsequences of \((\|x^k\|_2)\) and \((\|y^k\|_2)\) (denoted again by \((\|x^k\|_2)\) and \((\|y^k\|_2)\)) and a constant \(c_2 > 0\) such that \(\|x^k\|_2 - \|y^k\|_2 > c_2\) and consequently by (9) we have
\[ \frac{\|F(x^k) - F(y^k)\|}{\|x^k - y^k\|_2} \geq \frac{c_2}{c_1} > 0, \] (10)
contrary to (7).

Thus it remains to consider the case when \(\|x^k\|_2 - \|y^k\|_2 \to 0, 1 < \|x^k\|_2 \leq 2\) and \(1 < \|y^k\|_2 \leq 2\).

From condition (II.) and the fact that \(\|x^k\|_2 - \|y^k\|_2 \to 0\) we conclude that there are subsequences of \((\|x^k\|_2)\) and \((\|y^k\|_2)\) (denoted again by \((\|x^k\|_2)\) and \((\|y^k\|_2)\)) and some number \(1 + \epsilon \leq p \leq 2\) such that \(\|x^k\|_2 \to p\) and \(\|y^k\|_2 \to p\) and consequently, from the definition of \(F\), for every sufficiently large \(k\),

either \((I_{\text{II}})\) \(n(x^k) = n(y^k)\) or \((\text{II}_n)\) \(n(x^k) + 1 = n(y^k)\).

(I_{\text{II}}) In this case we need the following notation:
\[ x^k_o = (0, 0, \ldots, 0, x^k_{n(x^k)}), 0, 0, \ldots), \]
\[ y^k_o = (0, 0, \ldots, 0, y^k_{n(x^k)}), 0, 0, \ldots), \]
\[ x^k_p = (x^k_1, x^k_2, \ldots, x^k_{n(x^k)-1}, 0, x^k_{n(x^k)+1}, \ldots), \]
\[ y^k_p = (y^k_1, y^k_2, \ldots, y^k_{n(x^k)-1}, 0, y^k_{n(x^k)+1}, \ldots), \]
\[ x^k_r = (0, \ldots, 0, x^k_{n(x^k)} \sin x^k_r, x^k_{n(x^k)} \cos x^k_r, 0, \ldots), \]
\[ y^k_r = (0, \ldots, 0, y^k_{n(x^k)} \sin y^k_r, y^k_{n(x^k)} \cos y^k_r, 0, \ldots). \]

We have \(x^k_o + x^k_p = x^k\) and \(y^k_o + y^k_p = y^k\).

Since \(n(x^k) = n(y^k)\) and \(0 < x^k_r < \pi/2, 0 < y^k_r < \pi/2\), we have \(\|x^k_r - y^k_r\|_2 \geq \frac{1}{2}\|x^k_o - y^k_o\|_2\) and consequently
\[ \frac{\|F(x^k) - F(y^k)\|}{\|x^k - y^k\|_2} = \frac{\|x^k_o - y^k_o\|_2 + \|x^k_r - y^k_r\|_2}{\|x^k_o - y^k_o\|_2 + \|x^k_r - y^k_r\|_2} \geq \frac{1}{2}, \] (11)
contrary to (7).
(II

\_\_1) We introduce the following notation:
\[
\begin{align*}
x_k^i &= (0, 0, \ldots, 0, x_{m(x^i)}^k, x_{m(x^i)+1}^k, 0, 0, \ldots), \\
y_k^i &= (0, 0, \ldots, 0, y_{m(x^i)}^k, y_{m(x^i)+1}^k, 0, 0, \ldots), \\
x_k^i &= (x_1^k, x_2^k, \ldots, x_{m(x^i)}^k - 1, 0, 0, x_{m(x^i)+2}^k, x_{m(x^i)+3}^k, \ldots), \\
y_k^i &= (y_1^k, y_2^k, \ldots, y_{m(x^i)}^k - 1, 0, 0, y_{m(x^i)+2}^k, y_{m(x^i)+3}^k, \ldots), \\
x_k^i &= (0, 0, \ldots, 0, x_{m(x^i)}^k \sin z_{x^i}, x_{m(x^i)+1}^k \cos z_{x^i}, 0, 0, \ldots), \\
y_k^i &= (0, 0, \ldots, 0, y_{m(x^i)}^k, y_{m(x^i)+1}^k \sin z_{y^i}, 0, 0, \ldots).
\end{align*}
\]
We have \(x_k^i + y_k^i = x^k\) and \(y_k^i + y_k^i = y^k\). Then
\[
\frac{\|F(x^k) - F(y^k)\|_2^2}{\|x^k - y^k\|_2^2} = 1 + \frac{\|x_k^i - y_k^i\|_2^2}{\|x_k^i - y_k^i\|_2^2} \frac{\|x_k^i - y_k^i\|_2^2}{\|x_k^i - y_k^i\|_2^2} \quad (12)
\]
Fix an arbitrary number \(0 < c_D < \infty\).

The idea is to consider two possibilities:

(II

\_\_1) \(\|x_k^i - y_k^i\|_2^2 < c_D \|x_k^i - y_k^i\|_2^2\)

and

(II

\_\_2) \(\|x_k^i - y_k^i\|_2^2 \geq c_D \|x_k^i - y_k^i\|_2^2\).

In the case (II

\_\_1) the right-hand side in (12) is greater than \(1/(1 + c_D)\), which contradicts (7).

In case (II

\_\_2), we show that there exists a constant \(c_3 > 0\) such that, for \(k\) large enough
\[
\|x_k^i - y_k^i\|_2 \geq c_3 \|x_k^i - y_k^i\|_2. \quad (13)
\]

During the proof we will apply the well-known equivalence of norms in a finite-dimensional space: for every \(z = (z_1, z_2) \in \mathbb{R}^2\) we have
\[
\frac{1}{\sqrt{2}} (|z_1| + |z_2|) \leq (z_1^2 + z_2^2)^{1/2} \leq |z_1| + |z_2|. \quad (E)
\]
To prove (13) we first show that there exist constants \(c_4 > 0, c_5 > 0\) such that for every sufficiently large \(k\)
\[
\begin{align*}
|x_{m(x^i)}^k \sin z_{x^i} - y_{m(x^i)}^k| &\geq c_4 |x_{m(x^i)}^k - y_{m(x^i)}^k| - c_5 \|x_k^i - y_k^i\|_1, \quad (14) \\
|x_{m(x^i)+1}^k - y_{m(x^i)+1}^k \cos z_{y^i}| &\geq c_4 |x_{m(x^i)+1}^k - y_{m(x^i)+1}^k| - c_5 \|x_k^i - y_k^i\|_1. \quad (15)
\end{align*}
\]
We only show (14) because (15) can be proved in the same way (by adding and subtracting inside the absolute value in the left-hand side of (15) the term \(x_{m(x^i)+1}^k \cos z_{y^i}\)). We have
\[
|x_{m(x^i)}^k \sin z_{x^i} - y_{m(x^i)}^k| = |(x_{m(x^i)}^k - y_{m(x^i)}^k) \sin z_{x^i} + (\sin z_{x^i} - 1) y_{m(x^i)}^k| \geq |\sin z_{x^i} |(x_{m(x^i)}^k - y_{m(x^i)}^k)| - |\sin z_{x^i} - 1| y_{m(x^i)}^k|, \quad (16)
\]

\[
\begin{align*}
&= |(x_{m(x^i)}^k - y_{m(x^i)}^k) \sin z_{x^i} + (\sin z_{x^i} - 1) y_{m(x^i)}^k| \\
&\geq |\sin z_{x^i} |(x_{m(x^i)}^k - y_{m(x^i)}^k)| - |\sin z_{x^i} - 1| y_{m(x^i)}^k|.
\end{align*}
\]
But
\[ | \sin \varepsilon - 1 | = | \sin \varepsilon - \sin \frac{\pi}{2} | = | \cos \varepsilon | | \frac{\pi}{2} - \varepsilon | \]

where
\[ \varepsilon < \phi < \varepsilon < \frac{\pi}{2} \]

By condition (II\( g_1 \)), we deduce that
\[ (\| x \|, (\| y \|) \) tend to \( p \ (1 + \varepsilon < p < 2 \) and \( 1/(p - 1) \) is an integer) from two different sides so
\[ \frac{1}{\| y \|_2 - 1} = \frac{1}{p - 1} > \frac{1}{\| x \|_2 - 1} \]
\[ > \frac{1}{\| x \|_2 - 1} = \frac{1}{p - 1} - 1, \]

and we have
\[ | \frac{\pi}{2} - \varepsilon | = \frac{\pi}{2} - \frac{\pi}{2} \left( \frac{1}{\| x \|_2 - 1} - \left[ \frac{1}{\| x \|_2 - 1} \right] \right) \]
\[ < \frac{\pi}{2} \left( \frac{1}{\| y \|_2 - 1} - \frac{1}{\| y \|_2 - 1} \right) \]
\[ = \frac{\pi}{2} \left( \frac{\| y \|_2 - \| y \|_2}{(\| y \|_2 - 1)(\| y \|_2 - 1)} \right). \]

From this and (***) we see that there exists a constant \( C_0 > 0 \) such that
\[ \frac{\pi}{2} - \varepsilon \leq C_0 \| x \|_2 - \| y \|_2. \]

Now from (18) and (*) we conclude
\[ | \sin \varepsilon \| x \|_2 - \| y \|_2 | = | \varepsilon \| x \|_2 - \| y \|_2 | \geq \frac{1}{\| x \|_2 - 1} \left[ | \sin \varepsilon \| x \|_2 - \| y \|_2 | - 2C_0 | \cos \varepsilon \| x \|_2 - \| y \|_2 | \right] \]

(note that \( | y \|_2 \leq 2 \).

Now observe that from (II\( g_1 \)) it follows that
\[ \| x \|_2 - \| y \|_2 \leq 2 \]

From the form of \( x^k_\phi, y^k_\phi \), (E) and (20) we have
\[ \sqrt{\frac{1 + c_D}{c_D}} \| x^k_\phi - y^k_\phi \|_2 \geq \| x^k - y^k \|_2. \]
\[
\begin{align*}
&= \sqrt{\frac{1+c_D}{c_D}} \|x_0^k - y_0^k\|_2 \\
&\geq \|x^k - y^k\|_2 \geq \|x^k\|_2 - \|y^k\|_2,
\end{align*}
\]

Now from (16), (19), and (21) we conclude that
\[
|x_{m(x^*)}^k \sin \varphi_{x^*} - y_{m(x^*)}^k| \\
\geq |\sin \varphi_{x^*}(x_{m(x^*)}^k - y_{m(x^*)}^k)| - 2|\cos \varphi_{x^*}|c_6 \sqrt{\frac{1+c_D}{c_D}} \|x_0^k - y_0^k\|_1.
\]

The same considerations lead to the inequality
\[
|x_{m(x^*)}^k + 1 - y_{m(x^*)}^k \cos \varphi_{y^*}| \\
\geq |\cos \varphi_{y^*}(x_{m(x^*)}^k - y_{m(x^*)}^k) + 1 - 2|\sin \varphi_{y^*}|c_6 \sqrt{\frac{1+c_D}{c_D}} \|x_0^k - y_0^k\|_1.
\]

Now observe that by definition of \(x_{y^*}, \varphi_{y^*}\) and property (***) we have \(x_{y^*} \rightarrow \pi/2\) and \(x_{y^*} \rightarrow 0\). From (**) we have \(x_{y^*} < \phi_{y^*} < \pi/2\), and similarly \(0 < \phi_{y^*} < x_{y^*}\). Thus we can assume that for every sufficiently large \(k\) sin \(x_{y^*} > c_4\), cos \(x_{y^*} > c_4\), cos \(x_{y^*} < c_5\), sin \(\phi_{y^*} < c_5\) where \(c_4\) can be taken as close to 1 as we wish and \(c_5\) can be taken arbitrarily close to 0. Now we define \(c_5 := 2c_5 c_6 \sqrt{(1+c_D)/c_D}\).

Thus, inequalities (14) and (15) are proven.

From the above we see that we can take \(c_4\) and \(c_5\) so that \(c_4 - 2c_5 > 0\). Since
\[
|x_{m(x^*)}^k - y_{m(x^*)}^k| + |x_{m(x^*)}^k + 1 - y_{m(x^*)}^k| = \|x_0^k - y_0^k\|_1,
\]

using (E) and adding (14) and (15) we get
\[
\sqrt{2}\|x_0^k - y_0^k\|_2
\]

and finally again by (E),
\[
\|x_0^k - y_0^k\|_2 \geq \frac{1}{\sqrt{2}}(c_4 - 2c_5)\|x_0^k - y_0^k\|_1 \geq \frac{1}{\sqrt{2}}(c_4 - 2c_5)\|x_0^k - y_0^k\|_2.
\]

We can put \(c_3 := \frac{1}{\sqrt{2}}(c_4 - 2c_5) > 0\). Thus, inequality (13) is proven.

Now, from (13) we have
\[
\frac{\|F(x^k) - F(y^k)\|_2^2}{\|x^k - y^k\|_2^2} \geq \min\{c_3, 1\},
\]

which contradicts (7). This finishes the proof of the Theorem.
Observe that the condition (II′) is necessary since

**Remark 2.** For \( x^k = (0, 0, \ldots, 0, 1 + 1/k, 0, 0, \ldots) \) (where \( x^k = 1 + 1/k \)) and \( y^k = (0, 0, \ldots, 0, 1, 0, 0, \ldots) \) (where \( y^k_{k+1} = 1 \)) we have

\[
\frac{\|F(x^k) - F(y^k)\|_2}{\|x^k - y^k\|_2} \to 0. \tag{27}
\]

It is easy to see that

** Remark 3.** The trajectories of each map \( F_\varepsilon \) exponentially diverge.

It was shown in [10] that there is no map with properties (a₁)–(g₁) in the finite-dimensional case.

Thus, we made a considerable progress in relation to the paper [10] since each map \( F_\varepsilon \) has the same properties as the map constructed in [10] but additionally it is expanding on each sphere (condition e₁) and it is expanding except the case when both \( x \) and \( y \) belong to the small annulus \( P_\varepsilon \).

**Acknowledgements**

I would like to acknowledge the financial support of the Polish Committee for Scientific Research (KBN), grant 7T07B03414.

**References**