Some heteroclinic solutions of a model of skin pattern formation

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SUMMARY

In this paper we study travelling wave solutions to a system of four non-linear partial differential equations, which arise in a tissue interaction model for skin morphogenesis. Under the ‘small-stress’ assumption we prove the existence and uniqueness (up to a translation) of solutions with the dermis and epidermis cell densities being positive, which are a perturbation of a uniform epidermal cell density. We discuss the problem of the minimal wave-speed. Copyright © 2004 John Wiley & Sons, Ltd.

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1. THE MODEL

Travelling waves are not only a frequent phenomenon in the nature, but also they are of large cognitive value, since they are easy to be generated and measured in experiments. By measuring the profile and the speed of such a wave we can make estimations of the roles of various quantities characterizing the medium under consideration.

The aim of this paper is a rigorous mathematical analysis of travelling wave solutions of a model proposed by Cruywagen and Murray [1] to study skin pattern formation and analysed in many aspects in References [1–5]. In particular, in Reference [3] the problem of travelling wave solutions to the proposed system equations was discussed. The essential
difference between that paper and the present one is that in Reference [3] Cruywagen, Maini and Murray were looking for high-speed waves, whereas we use a different set of assumptions allowing the wave to travel at the minimal speed which cannot be treated as a large quantity. It is worth mentioning that in Reference [3] a numeric analysis of the problem is performed, therefore we do not do any calculations.

According to the Cruywagen–Murray model, the skin consists of two layers, epidermis and dermis, separated by a thin basal lamina. The epidermis is modelled as a two-dimensional visco-elastic continuous medium. Under the biologically reasonable assumption that Reynolds number of the motion of the epidermis is low, the inertial terms of the equation of motion are ignored. The body force balances the elastic force, the viscous force, and the cell traction generated within the epidermis by a morphogen produced in the dermis. The force balance equation reads

\[ \nabla \cdot \left\{ \frac{E}{1 + \nu} \left[ \varepsilon - \beta_1 \nabla^2 \varepsilon + \frac{\nu}{1 - 2\nu} (\theta - \beta_2 \nabla^2 \theta) \right] I + \mu_1 \frac{\partial \varepsilon}{\partial t} + \mu_2 \frac{\partial \theta}{\partial t} I + \tau s I \right\} = \rho \mathbf{u} \] (1)

where \( \mathbf{u} = \mathbf{u}(\mathbf{x}, t) \) is the displacement at time \( t \) of a material point in the epidermis which was initially at position \( \mathbf{x} \), \( s(\mathbf{x}, t) \) is the concentration of the signal chemical produced in the dermis, \( E \) is the constant Young modulus, \( \nu \) is the constant Poisson ratio, \( \mu_1, \mu_2 \) are the constant shear and bulk viscosities, \( \beta_1, \beta_2 \) are positive constants, \( I \) is the unit \( 2 \times 2 \) matrix, and \( \tau \) is a positive parameter characterizing the strength of the traction \( s \). The epidermis is attached to the basal lamina and \( \rho \) is a positive constant measuring the strength of this attachment. Next,

\[ \varepsilon = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \] (2)

is the strain tensor, where \( T \) denotes the transpose, and

\[ \theta = \nabla \cdot \mathbf{u} \] (3)

is the dilatation.

This equation is supplemented by another two equations expressing the conservation laws of the epidermal cell density \( N(\mathbf{x}, t) \) and the dermal cell density \( n(\mathbf{x}, t) \). In the model of Cruywagen and Murray [1], the cells of the epidermis are mutually attached; therefore the only contribution to the cell density flux is convection. Therefore this equation is of the form

\[ \frac{\partial N}{\partial t} = -\nabla \cdot \left( N \frac{\partial \mathbf{u}}{\partial t} \right) \] (4)

The conservation law of the dermal cell density \( n \) is more complicated. This density changes due to random cell migration since the cells in the dermis are loosely packed, due to chemotaxis and due to mitosis, i.e. cell production. To model the random cell migration, the Fick’s law of diffusion is used. The conservation equation of the dermal cell density \( n(\mathbf{x}, t) \) reads

\[ \frac{\partial n}{\partial t} = \nabla \cdot (d(n,N)\nabla n) - \alpha \nabla \cdot (n \nabla e) + m(n) \] (5)

where \( d \) is the coefficient of diffusion, which can be a function of the cell densities \( N \) and \( n \), \( \alpha \) is chemotaxis coefficient, \( e(\mathbf{x}, t) \) is the concentration of the signal chemical produced in the epidermis and \( m(n) \) describes the mitosis.
1.1. Specific models

To close the system of equations (1), (4), and (5) we need a set of ‘equations of state’ for $s(n, N)$, $d(n, N)$, $e(n, N)$, and $m(n)$. Below we give examples of models, which can be found in the literature.

(i) **The traction** is defined when the function of the concentration $s$ of the chemical secreted in the dermis is given. Various models of this function can be found in References [1–5]. For instance

$$s(n, N) = \frac{P_s \kappa_s n}{vP_s + (v + P_s)\hat{v}N}$$

where $P_s$, $\kappa_s$, $v$, and $\hat{v}$ are some positive constants.

(ii) To model **the chemotaxis** it is sufficient to define the function of the concentration $e$ of the chemical secreted in the epidermis. For example, for the morphogen $e(n, N)$ the following formula was used in References [1–5]

$$e(n, N) = \frac{P_e \kappa_e N}{\hat{\gamma}P_e + (\hat{\gamma} + P_e)\gamma n}$$

where $P_e$, $\kappa_e$, $\gamma$, and $\hat{\gamma}$ are some positive constants.

(iii) **The diffusion** coefficient $d(n, N)$ was assumed in all the papers [1–5] to be a positive constant or even ignored.

(iv) The last quantity we have to model is **the mitosis**. In the model of Cruywagen and Murray [1] (see also References [2–5]) it is a function of the dermal cell density only. When modelling this function it was noticed that: (1) if there are no cells then the mitosis does not take place; (2) there is a limiting cell density at which the cell growth stops, in other words the number of cell cannot grow through the mitosis to infinity and (3) for the intermediate dermis cell densities $n$, i.e. for such densities that $0 < n < n_0$ the mitosis function is positive. In all papers [1–5] the logistic growth law

$$m(n) = rn(n_0 - n)$$

where $r$ is a positive constant, was used.

The paper is organized as follows. In Section 2 the non-dimensionalization of the equations describing the considered model is given. In Section 3 the travelling wave problem is formulated. In Section 4 we consider a particular, but very important for our future considerations, case when the morphogens secreted in the dermis do not induce any stress in the epidermis. In this case the considered equations reduce to one equation, which is a more general form of the celebrated Fisher equation so, as a matter of fact, we quote the results concerning the latter model. The main property of this equation is that the dermis cell density is positive if and only if the wave-speed $\sigma$ is not less than a certain minimal wave-speed $\sigma^*$. The crucial part of the paper consists of Section 5 and the supplementary Section 6. In Section 5.1 we introduce a generalization of the system of the biological model (equations (34), (35)) of Cruywagen and Murray [1], also we formulate the basic assumptions concerning our generalized system. In particular, we introduce the notion of unperturbed problem and assume its properties, which in principle imitate those of the Fisher equation. The critical points of the full, generalized system are studied in Section 5.2. The analysis of the existence and uniqueness of travelling...
waves with positive dermis cell density is contained in Sections 5.3–5.5 and in Section 6. As we use the Implicit Function Theorem, the analysis depends on the properties of the system from which the perturbation is carried on. In consequence, in Section 5.3 we consider the case when the unperturbed wave moves at the speed higher then the minimal, for the unperturbed problem, speed \( \sigma^* \). In Section 5.4 we consider the case when the speed of the unperturbed wave is equal to \( \sigma^* \). In this case the dermis cell density is positive only if the speed of the perturbed wave is a uniquely chosen function of the perturbation parameter. We show that it is a continuous function of the perturbation parameter. Finally, in Section 5.5 we consider the case when the speed of the unperturbed wave is smaller then \( \sigma^* \). In this case the dermis cell density as described by the unperturbed problem becomes a negative function for some values of its argument and we prove that the dermis cell density as described by the full (i.e. perturbed as we call it) problem has also this property. So, this function loses its biological sense. In Section 6 we prove that under an additional assumption concerning the eigenvalues at the rest point before the wave and for sufficiently small values of the perturbation parameter, the dermis cell density, as described by the perturbed problem, is positive provided that the wave speed is not smaller then wave-speed found in Section 5.4.

2. NON-DIMENSIONALIZATION

According to the standard procedure we rewrite all equations introduced in the preceding section in a non-dimensional form using the non-dimensionalization introduced in References [1] and [2]. Let \( L_0 \) and \( T_0 \) be the typical length and time scales, respectively, \( N_0 \) and \( n_0 \) typical cell densities. The dimensionless position, time, cell densities and the morphogen concentrations are defined as in References [1,2]

\[
t^* = \frac{t}{T_0}, \quad x^* = \frac{x}{L_0}, \quad u^* = \frac{u}{u^*}, \quad N^* = \frac{N}{N_0}, \quad n^* = \frac{n}{n_0}
\]

Using these quantities, we define

\[
\beta_1^* = \frac{1 - 2v}{1 - v} \beta_1, \quad \beta_2^* = \frac{v}{1 - v} \beta_2, \quad \rho^* = \frac{1 - 2v}{1 - v} \frac{\rho L_0^2 (1 + v)}{E},
\]

\[
\mu_1^* = \frac{1 - 2v}{1 - v} \frac{\mu_1 (1 + v)}{ET_0}, \quad \mu_2^* = \frac{1 - 2v}{1 - v} \frac{\mu_2 (1 + v)}{ET_0},
\]

\[
\tau^* = \frac{(1 - 2v)(1 + v)}{1 - v} \frac{\tau}{E}, \quad \varphi^* = \frac{T_0}{L_0^2}
\]

and

\[
s(n^*, N^*) = s(n_0 n^*, N_0 N^*), \quad e(n^*, N^*) = e(n_0 n^*, N_0 N^*)
\]

\[
d^*(n^*, N^*) = \frac{T_0}{L_0^2} d(n_0 n^*, N_0 N^*), \quad m^*(n^*) = \frac{T_0}{n_0} m(n_0 n^*)
\]
The non-dimensionalized form of the system (1), (4), (5) is
\[ \nabla \cdot \left[ 1 - \frac{2v}{1-v} \varepsilon - \beta_1 \nabla^2 \varepsilon + \left( \frac{v}{1-v} \theta - \beta_2 \nabla^2 \theta \right) I + \mu_1 \frac{\partial \varepsilon}{\partial t} + \mu_2 \frac{\partial \theta}{\partial t} I + \tau(N,n)I \right] = \rho u \] (12)
\[ \frac{\partial N}{\partial t} = -\nabla \cdot \left( N \frac{\partial u}{\partial t} \right) \] (13)
and
\[ \frac{\partial n}{\partial t} = \nabla \cdot (d(n,N)\nabla n) - \alpha \nabla \cdot (n \nabla e(n,N)) + m(n) \] (14)
where all quantities are in the dimensionless form. We hope that the adopted convention of suppressing the asterisks by the dimensionless quantities and denoting them with the same symbols, as their dimensional counterparts should not lead to confusion since from now on all quantities are treated as non-dimensional.

3. TRAVELLING WAVES

We look for solution of the system (12)–(14) in the form of travelling waves. These are solutions of the form
\[ (u,N,n)(t,x) = (u,N,n)(\xi), \quad \xi = x \cdot k - \sigma t \] (15)
where \( k \) is a constant unit vector in the direction of propagation of the wave, and \( \sigma \in \mathbb{R}^1 \) is the speed of the wave. Inserting these relations into (12)–(14) we arrive at the following systems of ordinary differential equations
\[ \frac{1}{2} \left[ \frac{1}{1-v} \frac{d^2}{d\xi^2} \left[ u + k(k \cdot u) \right] - \frac{1}{2} \beta_1 \frac{d^4}{d\xi^4} \left[ u + k(k \cdot u) \right] + \frac{v}{1-v} k \frac{d^2}{d\xi^2} (k \cdot u) - \beta_2 k \frac{d^4}{d\xi^4} (k \cdot u) \right] - \frac{1}{2} \mu_1 \sigma \frac{d^3}{d\xi^3} \left[ u + k(k \cdot u) \right] - \mu_2 \sigma k \frac{d^3}{d\xi^3} (k \cdot u) + k \frac{d}{d\xi} s(N,n) = \rho u \] (16)
\[ \sigma \frac{dN}{d\xi} = -\sigma \frac{d}{d\xi} \left[ N \frac{d}{d\xi} (k \cdot u) \right] \] (17)
and
\[ \frac{d}{d\xi} \left[ d(n,N) \frac{dn}{d\xi} \right] - \alpha \frac{d}{d\xi} \left[ n \frac{d}{d\xi} e(n,N) \right] + \sigma \frac{dn}{d\xi} + m(n) = 0 \] (18)

Definition 1

Let \( I \) be a constant unit vector perpendicular to \( k \). We denote
\[ u(\xi) = k \cdot u(\xi), \quad w(\xi) = l \cdot u(\xi) \] (19)
We have

**Proposition 1**

For any bounded solution \((u, N, n)(\xi), -\infty < \xi < \infty,\) of (16)–(18)

\[ w(\xi) \equiv 0 \quad (20) \]

**Proof**

Multiplying Equation (16) scalarly by the vector \(l\) we obtain for \(w\) the following equation:

\[
\frac{1}{2} \frac{d^4w}{d\xi^4} - \frac{1}{2} \frac{d^2w}{d\xi^2} - \frac{\mu_1}{2} \frac{d^3w}{d\xi^3} = \rho w
\]

Since the characteristic equation corresponding to the above equation

\[
\beta_1 \omega^4 + \frac{\mu_1}{2} \omega^2 - \frac{1}{2} \frac{1 - 2v}{1 - v} \omega^2 + \rho = 0
\]

does not admit purely imaginary roots, the only solution bounded on all \(\mathbb{R}\) is that given by (20). The proof is complete.

When, in turn, we take the scalar product of Equation (16) with the vector \(k\) we obtain the following equation

\[
\frac{\beta}{2} \frac{d^4u}{d\xi^4} + \mu \frac{d^2u}{d\xi^2} - \tau \frac{d}{d\xi} s(N, n) = -\rho u \quad (21)
\]

where \(\beta = \beta_1 + \beta_2, \mu = \mu_1 + \mu_2.\) Now, from (3) and (15) we have

\[ \theta(\xi) = u' \quad (22) \]

We differentiate Equation (21) with respect to \(\xi,\) use (22) and obtain

\[
\beta \frac{d^4\theta}{d\xi^4} + \mu \frac{d^3\theta}{d\xi^3} - \frac{d^2\theta}{d\xi^2} - \tau \frac{d^2}{d\xi^2} s(N, n) = -\rho \theta \quad (23)
\]

Assuming \(\sigma \neq 0\) we can integrate once Equation (17), and using (19), (22) we get \(N = \tilde{N} - N\theta,\) where \(\tilde{N}\) is a constant of integration. Taking \(\tilde{N} = 1\) we obtain

\[ N = \frac{1}{1 + \theta} \quad (24) \]

Formula (24) enables us to eliminate the unknown function \(N\) from Equations (18) and (23), and by simply inserting it into these equations they become

\[
\beta \frac{d^4\theta}{d\xi^4} + \mu \frac{d^3\theta}{d\xi^3} - \frac{d^2\theta}{d\xi^2} - \tau \frac{d^2}{d\xi^2} S(n, \theta) = -\rho \theta \quad (25)
\]

and

\[
\frac{d}{d\xi} \left[ D(n, \theta) \frac{dn}{d\xi} \right] - \alpha \frac{d}{d\xi} \left[ n \frac{d}{d\xi} E(n, \theta) \right] + \sigma \frac{dn}{d\xi} + m(n) = 0 \quad (26)
\]

where

\[ S(n, \theta) = s \left( n, \frac{1}{1 + \theta} \right), \quad D(n, \theta) = d \left( n, \frac{1}{1 + \theta} \right), \quad E(n, \theta) = e \left( n, \frac{1}{1 + \theta} \right), \quad \theta > -1 \quad (27) \]
4. TRAVELLING WAVES IN A NO-STRESS MODEL

Setting formally $\theta \equiv 0$ in Equation (26) we obtain

$$(a(n)n' + \sigma n' + m(n)) = 0 \quad (28)$$

where the dash denotes differentiation with respect to $\xi$, and

$$a(n) = D(n, 0) - x n E'(n, 0) > 0 \quad (29)$$

The critical points of Equation (28) are $(n, n_1) = (1, 0)$ and $(n, n_1) = (0, 0)$, where $n_1 = n'$. The characteristic exponents corresponding to $(n, n_1) = (1, 0)$ are

$$\lambda_{1}^{\pm} = \frac{-\sigma \pm \sqrt{\sigma^2 - 4a(1)m'(1)}}{2a(1)}$$

since they are the roots of the quadratic equation

$$a(1)\lambda^2 + \sigma \lambda + m'(1) = 0$$

From the properties of the adopted mitosis function $m(n)$ it follows that both of them are real and satisfy

$$\text{if } \sigma > 0, \text{ then } \lambda_{1}^- < 0 < \lambda_{1}^+ \quad (30)$$

Similarly, the characteristic exponents corresponding to $(n, n_1) = (1, 0)$ are solutions of the equation

$$a(0)\lambda^2 + \sigma \lambda + m'(0) = 0 \quad (31)$$

so, obviously, they are given by

$$\lambda_{0}^{\pm} = \lambda_{0}^{\pm}(\sigma) = \frac{-\sigma \pm \sqrt{\sigma^2 - 4a(0)m'(0)}}{2a(0)} \quad (32)$$

We see from the above formula that if the wave speed $\sigma$ is such that $\sigma^2 < 4m'(0)a(0)$, then characteristic exponents are complex and the cell density $n(\xi)$ oscillates around its asymptotic state $n = 0$, consequently it has to be negative for some values of $\xi$. This is a contradiction with the biologic sense of this quantity. Hence, we have to impose a lower bound for the wave-speed in order to have realistic travelling wave solutions (cf. References [3], and [6,7] concerning the theory of the Fisher equation as well as References [8,9] for its generalization). Therefore, we assume that $\sigma^2 \geq 4m'(0)a(0)$.

Let us make the transformation

$$n \to U(n) = \int_{0}^{n} a(y) \, dy$$

As $a > 0$ for $n > 0$, then $U' > 0$ and this transformation has the inverse $U \to n(U)$. Then, Equation (28) changes to

$$\frac{d^2U}{d\xi^2} + \sigma_{r}(U) \frac{dU}{d\xi} + \Phi(U) = 0 \quad (33)$$
where $\gamma(U) = [a(n(U))]^{-1}$, $\Phi(U) = m(n(U))$. Due to the fact that $a(n)$ is strictly positive, $\gamma$ is smooth, bounded and positive. Thus, by the methods of phase plane analysis as in References [6,7] (for the Fisher case) or using the results of References [8,9] one can prove the existence of a unique heteroclinic solution to Equation (28) tending for $\zeta \to -\infty$ to $U(1)$ and for $\zeta \to \infty$ to 0. Consequently, we have

**Theorem 1**

Let the functions $e(n,N)$, $m(n)$ and $E(n,\theta)$ be such as in (7), (8) and (27), respectively, and let the diffusion coefficient be a positive constant. Then there is a positive constant $\sigma^*$ such that there exists a unique (except for translation) monotone heteroclinic solution of (28) tending for $\zeta \to -\infty$ to 1 and for $\zeta \to \infty$ to 0 if and only if $\sigma \geq \sigma^*$ and

(a) If $\sigma^* = 2\sqrt{m''_n(0)a(0)}$, then the wave corresponding to $\sigma^*$ enters the node $(0,0)$ in the direction $(dn_1/d\zeta) = -2\sqrt{m''_n(0)/a(0)}$.

(b) If $\sigma^* > 2\sqrt{m''_n(0)a(0)}$, then the wave corresponding to $\sigma^*$ enters the node $(0,0)$ in the direction $dn_1/d\zeta = \lambda_0^*(\sigma^*)$.

(c) If $\sigma > \sigma^*$, then the wave corresponding to $\sigma$ enters the node $(0,0)$ in the direction $dn_1/d\zeta = \lambda_0^*(\sigma)$.

5. EXISTENCE OF HETEROCLINIC ORBITS IN THE CASE OF WEAK TRACTION

5.1. Generalities

The specific models presented in Section 1 are just examples of the formulae closing the system of Equations (1)–(5). In our analysis we need only some analytical properties of these functions and not their specific form. Therefore, we will consider a more general system than that consisting of (25), (26), namely, we consider the following system of equations

\[
\beta \frac{d^4\theta}{d\zeta^4} + \mu \sigma \frac{d^3\theta}{d\zeta^3} - \frac{d^2\theta}{d\zeta^2} + \rho \theta + \tau P \left( \sigma, \theta, \frac{d\theta}{d\zeta}, \frac{d^2\theta}{d\zeta^2}, \frac{d^3\theta}{d\zeta^3}, n, \frac{dn}{d\zeta} \right) = 0 \tag{34}
\]

\[
\frac{d^2n}{d\zeta^2} + R \left( \sigma, \theta, \frac{d\theta}{d\zeta}, \frac{d^2\theta}{d\zeta^2}, \frac{d^3\theta}{d\zeta^3}, n, \frac{dn}{d\zeta} \right) = 0 \tag{35}
\]

where $\sigma \in (-\infty, \infty)$ is a real parameter interpreted as the wave-speed, and $\tau$ is another parameter, which has a biological sense only if $\tau \geq 0$, but in our analysis its sign does not play any role and therefore we assume that $\tau \in (-\infty, \infty)$.

This system (34), (35) coincides with (25), (26) if the functions $P$ and $R$ are given by

\[
R(\sigma, \theta, \theta', \theta'', n, n') = -\frac{znE''_n(n, \theta)}{D(n, \theta) - znE''_n(n, \theta)} \theta'' + \frac{D'_n(n, \theta) - znE''_n(n, \theta) - znE''_n(n, \theta)}{D(n, \theta) - znE''_n(n, \theta)} n'^2
\]
\[ + \frac{D_p(n, \theta) - x n E'_p(n, \theta) - 2 x n E''_p(n, \theta)}{D(n, \theta) - x n E'_p(n, \theta)} n' \theta' - \frac{x n E'''_p(n, \theta)}{D(n, \theta) - x n E'_p(n, \theta)} \theta'^2 \\
+ \frac{\sigma n' + m(n)}{D(n, \theta) - x n E'_p(n, \theta)} \]

and

\[ P(\sigma, \theta, \theta', \theta'', \sigma', \sigma'') = R(\sigma, \theta, \theta', \theta'', \sigma', \sigma'') S'_n(n, \theta) - S'_n(n, \theta) \theta' - S''_n(n, \theta) n'^2 \\
- 2 S''_n(n, \theta) n' \theta' - S''_n(n, \theta) \theta'^2 \]

where the notations are the same as previously mentioned.

We consider the system (34), (35) independently of system (25), (26), but imposing assumptions on the functions \( P \) and \( R \) we are guided by the above formulae, since they are of biological meaning.

We treat the system (34), (35) as a perturbation of the reduced system, which is obtained from (34), (35) by setting formally \( \tau = 0 \). In this case Equation (34) reduces to

\[ \beta \frac{d^4 \theta}{d \tau^4} + \mu \sigma \frac{d^3 \theta}{d \tau^3} - \frac{d^2 \theta}{d \tau^2} + \rho \theta = 0 \]

The only solution of this equation bounded in \( C^4(\mathbb{R}^1) \) is \( \theta \equiv 0 \). Owing to that Equation (35) reduces to

\[ n'' + R_0(\sigma, n, n') = 0 \]

where

\[ R_0(\sigma, n, n') = R(\sigma, 0, 0, 0, 0, n, n') \]

The asymptotic states of Equation (36) are solutions of the following equation

\[ R_0(\sigma, n, 0) = R(\sigma, 0, 0, 0, 0, n, 0) = 0 \]

We take

**Assumption 1**

For any real \( \sigma \), Equation (38) has exactly two solutions, which are \( n = 0 \) and \( n = 1 \), such that

\[ R'_{0,n}(\sigma, 0, 0) > 0, \quad R'_{0,n}(\sigma, 1, 0) < 0 \]

\[ R'_{0,n'}(\sigma, 0, 0) > 0, \quad R'_{0,n'}(\sigma, 1, 0) > 0 \]

The characteristic equation of Equation (36) linearized around the critical state \((1, 0)\) is

\[ \lambda^2 + R'_{0,n}(\sigma, 1, 0) \lambda + R'_{0,n}(\sigma, 1, 0) = 0 \]

Due to Assumption 1, it has two real solutions \( \lambda_1^\pm \) such that

\[ \lambda_1^- < 0 < \lambda_1^+ \]
In turn, the characteristic equation of Equation (36) linearized around the critical state \((0,0)\) is
\[
\lambda^2 + R_{0,n}'(\sigma,0,0)\lambda + R_{0,n}(\sigma,0,0) = 0 \tag{42}
\]
Similarly to the case of Equation (31) this equation can have complex roots. To avoid such undesirable situation we take

**Assumption 2**
There is \(\sigma = \sigma_c\) such that for all \(\sigma > \sigma_c\) the following relation
\[
(R_{0,n}'(\sigma,0,0))^2 \geq 4R_{0,n}(\sigma,0,0) \tag{43}
\]
holds, and for \(\sigma = \sigma_c\) only it becomes equality.

Under Assumptions 1 and 2, Equation (42) has two real solutions \(\lambda_0^\pm = \lambda_0^\pm(\sigma)\) such that
\[
\lambda_0^- < \lambda_0^+ < 0 \tag{44}
\]
Now we formulate one of our most important assumptions. Namely, we take

**Assumption 3**
There is a positive constant \(\sigma^*\) such that there exists a unique (except for translation) monotone heteroclinic solution \(n = G(\xi)\) of (36) tending for \(\xi \to -\infty\) to 1 and for \(\xi \to \infty\) to 0 if and only if \(\sigma > \sigma^*\), and

(a) If \(\sigma^* = \sigma_c\), then the wave corresponding to \(\sigma^*\) enters the node \((0,0)\) in the direction \(dn_1/dn = -\sqrt{R_{0,n}(\sigma_c,0,0)}.\)
(b) If \(\sigma^* > \sigma_c\), then the wave corresponding to \(\sigma^*\) enters the node \((0,0)\) in the direction \(dn_1/dn = \lambda_0^-(\sigma^*).\)
(c) If \(\sigma > \sigma^*\), then the wave corresponding to \(\sigma\) enters the node \((0,0)\) in the direction \(dn_1/dn = \lambda_0^+(\sigma).\)

As it is readily seen, our Assumptions 1–3 are suggested by the results quoted in Section 4.

**Assumption 4**
Let \(\Omega\) be the neighbourhood of the set \(\bigcup_{\xi=-\infty}^{\xi=\infty} \{((0,0,0,0,0),G(\xi),G'(\xi))\}\). The function \(P\) is defined in \(\mathbb{R}^1 \times \Omega\), it is twice continuously differentiable there with respect to its arguments, and satisfies
\[
P(\theta,0',0'',0',0,0) = 0, \quad P(\theta,0,0,0,1,0) = 0 \tag{45}
\]

**Assumption 5**
Let \(\mathbf{n} = (n,n_1)\), \(\mathbf{t} = (\theta,\theta_1,\theta_2,\theta_3)\), \(|\mathbf{n}| = |n| + |n_1|, |\mathbf{t}| = |\theta| + \sum_{i=1}^{3} |\theta_i|\). The function \(R(\sigma,n,\mathbf{t})\) is defined in \(\mathbb{R}^1 \times \Omega\), where \(\Omega\) is as in Assumption 4, it is twice continuously differentiable with respect to its all arguments, and satisfies

(a) the conditions formulated in Assumptions 1 and 2;
(b) for \(\sigma \geq \sigma_c\), the following equality
\[
R(\sigma,\mathbf{t},0,0,0) = 0
\]
is fulfilled;
(c) if \((n, 0) \in \Omega\) then for \(\sigma \geq \sigma_c\) the following asymptotic estimates

\[ R'_{\sigma, 0}(\sigma, 0, 0) - R'_{\sigma, 0}(\sigma, 0, n) = O(|n| |\theta|) \quad \text{as } |n| |\theta| \to 0 \quad (46) \]

are satisfied for \(i = 0, 1, 2, 3\), where \(\theta = (0, 0, 0, 0)\),

(d) there is a constant \(C > 0\) such that for any \((\theta, n) \in \Omega\), if \(\sigma \geq \sigma_c\), then the following asymptotic estimate

\[ |R(\sigma, \theta + \chi, n + h) - R(\theta, \chi) - R'_n(\theta, \chi) \cdot h - R'_\theta(\theta, n) \cdot \chi| \leq C(\|h\|^2 + |h| |\chi| + |n| |\chi|^2) \quad (47) \]

holds true, where \(h = (h_0, h_1)\), \(\chi = (\chi_0, \chi_1, \chi_2, \chi_3)\) are so small that \((\theta + \chi, n + h) \in \Omega\).

We check easily that all these assumptions are satisfied trivially for the models (6)–(8).

**Definition 2**
A solution of Equations (34), (35) defined for all \(\xi \in \mathbb{R}^1\) whose derivatives vanish at \(\pm \infty\) and such that \((n(\xi), \theta(\xi))\), \(-\infty < \xi < + \infty\), tends to different constant vectors as \(\xi \to \pm \infty\) is called heteroclinic. The vectors to which they tend are called their asymptotic states.

### 5.2. Critical points

The system of two equations (34), (35) is equivalent to the following system of six first order ordinary equations

\[
\begin{align*}
\theta' &= \theta_1 \\
\theta'_1 &= \theta_2 \\
\theta'_2 &= \theta_3 \\
\theta'_3 &= -\frac{1}{p} \left[ \mu \sigma \theta_3 - \theta_2 + \rho \theta + \tau P(\sigma, \theta, \theta_1, \theta_2, \theta_3, n, n_1) \right] \\
n' &= n_1 \\
n'_1 &= -R(\sigma, \theta, \theta_1, \theta_2, \theta_3, n, n_1) 
\end{align*}
\quad (48)
\]

The asymptotic states of Definition 2 nullify the right hand sides of the above equations, i.e. they have to coincide with the critical states of the considered system. We check quickly that this system, due to Assumptions 1–4, has only two critical points, they are \(Z = (0, 0, 0, 0, 0, 0)\) and \(J = (0, 0, 0, 0, 1, 0)\). Therefore we impose the following limit conditions.

\[
\begin{align*}
\lim_{\xi \to \pm \infty} \theta^{(i)}(\xi) &= 0, \quad i = 0, 1, 2, 3, 4 \\
\lim_{\xi \to -\infty} n(\xi) &= 1, \quad \lim_{\xi \to +\infty} n(\xi) = 0, \quad \lim_{\xi \to \pm \infty} n^{(i)}(\xi) = 0, \quad i = 1, 2 
\end{align*}
\quad (49)
\]
Let \( n_0 = 0 \) or \( n_0 = 1 \), we denote \( c = (0, 0, 0, 0, n_0, 0) \), so that \( c = Z \) or \( c = J \). We linearize the system (48) around the critical state \( c \) and obtain the following system of equations

\[
\begin{pmatrix}
\theta' \\
\theta'_1 \\
\theta'_2 \\
\theta'_3 \\
n' \\
n'_1
\end{pmatrix} = M
\begin{pmatrix}
\theta \\
\theta_1 \\
\theta_2 \\
\theta_3 \\
n \\
n_1
\end{pmatrix}
\] (50)

where

\[
M = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\beta & -m_{41} & \beta & -m_{42} & \beta & -m_{43} \\
\beta & -m_{44} & \beta & -m_{45} & \beta & -m_{46} \\
0 & 0 & 0 & 0 & 0 & 1 \\
-m_{61} & -m_{62} & -m_{63} & -m_{64} & -m_{65} & -m_{66}
\end{pmatrix}
\] (51)

with

\[
m_{41} = \rho + \tau P'_{0}(c), \quad m_{42} = \tau P'_{01}(c), \quad m_{43} = -1 + \tau P'_{1}(c), \quad m_{44} = \mu \sigma + \tau P'_{02}(c), \quad m_{45} = \tau P'_{n1}(c) \]
\[
m_{46} = \tau P'_{n1}(c) \]
\[
m_{61} = R'_{0}(c), \quad m_{62} = R'_{01}(c), \quad m_{63} = R'_{02}(c), \quad m_{64} = R'_{n1}(c), \quad m_{65} = R'_{n}(c), \quad m_{66} = R'_{n1}(c)
\] (52)

The eigenvalues of the matrix \( M \) are solutions of the sixth degree equation

\[
(\beta \lambda^4 - m_{44} \lambda^3 - m_{43} \lambda^2 - m_{42} \lambda - m_{41})(\lambda^2 - m_{66} \lambda - m_{65}) = (m_{64} \lambda^3 + m_{63} \lambda^2 + m_{62} \lambda + m_{61})(m_{46} \lambda + m_{45})
\] (53)

If \( c = Z \), then Equation (53) reduces to

\[
[\beta \lambda^4 + \mu \sigma \lambda^3 - \lambda^2 + \rho][\lambda^2 + R'_{n1}(\sigma, Z) \lambda + R'_{n}(\sigma, Z)] = 0
\] (54)

owing to Assumptions 4 and 5. Hence \( \lambda \) is either a solution of

\[
W_4(\lambda) \equiv \beta \lambda^4 + \mu \sigma \lambda^3 - \lambda^2 + \rho = 0
\] (55)

or it is a solution of Equation (42).
We take

**Assumption 6**

The algebraic equations (42) and (55) do not have common roots.

Equation (52) was thoroughly analyzed in Reference [3]. The result of the analysis is

**Proposition 2 (Cruywagen et al. [3])**

Equation (55) has

(i) either four real roots; two of them are negative, the other two are positive,
(ii) or it has two real negative roots and two roots are complex; the complex roots have positive real parts,
(iii) or else, this equation has four complex roots; two of them have negative real parts and the real parts of the other two are positive.

Hence the solutions of Equation (55) admit the following ordering

\[ \text{Re } \lambda_1 \leq \text{Re } \lambda_2 < 0 < \text{Re } \lambda_3 \leq \text{Re } \lambda_4 \] (56)

The eigenvectors of \( M \) corresponding to these eigenvalues can be chosen in the form

\[ r_i = (1, \lambda_i, \lambda_i^2, \lambda_i^3, 0, 0), \quad i = 1, 2, 3, 4 \] (57)

The eigenvectors of \( M \) corresponding to \( \lambda_0^\pm \), i.e. to these eigenvalues which satisfy Equation (42) are of the form

\[ r^\pm = (\xi^\pm, \xi^\pm, -1, -1, \xi^\pm, 1, \xi^\pm) \] (58)

where

\[ \xi^\pm = \tau \frac{P_m'(Z) + P_n'(Z) \lambda_0^\pm}{W_d(\lambda_0^\pm)} \]

Unfortunately, in the case of the singular point \( c = J = (0, 0, 0, 0, 1, 0) \) we are unable to solve Equation (55). Everything we can do is to perform an asymptotic analysis of it. We have

**Proposition 3**

In the case when \( c = J \), if each root of Equation (55) is single and if this equation has no roots common with Equation (40), then for \( \tau \) sufficiently small Equation (55) has three solutions with negative real part and three its solutions have positive real part.

**Proof**

By means of the implicit function theorem, the following expressions can be proved to be asymptotic expansions of the roots of Equation (55) as \( \tau \to 0 \)

\[ \lambda_i^{(1)} = \lambda_i + O(\tau), \quad i = 1, 2, 3, 4 \]

\[ \lambda^\pm_1 = \lambda^\pm_1 + O(\tau) \]

where \( \lambda_i, i = 1, 2, 3, 4 \), are solutions of (53), and \( \lambda_i^\pm \) are solutions of Equation (40). If \( \tau \) is sufficiently small, then the signs of the real parts of \( \lambda_i^{(1)}, i = 1, 2, 3, 4 \), coincide with those of \( \lambda_i, i = 1, 2, 3, 4 \), and the signs of \( \lambda_1^\pm \) are the same as those of \( \lambda_1 \). But from Assumption 3 it follows that \( \lambda^- < 0 < \lambda^+ \). Hence, the proof is complete. \( \square \)
System (34), (35) is an autonomous one. If it is satisfied by \((\theta(\xi), n(\xi))\), then, for any \(\xi_0\), 
\((\theta(\xi + \xi_0), n(\xi + \xi_0))\) is also its solution. To get rid of this ambiguity, \(n\) will be subjected to an additional condition:

\[
n(0) = \frac{1}{2} [n(-\infty) + n(\infty)] \tag{59}
\]

We introduce spaces of functions.

**Definition 3**

Let \(i\) be a fixed positive integer, and let \(p, q\) be given, fixed real numbers. Let \(B_{i}^{p,q}\) denote the subspace of \(C^i(\mathbb{R}^1)\) consisting of functions \(y\) for which the expressions

\[
y_n = \max \left\{ \sup_{\xi \leq 0} e^{-p\xi} |y^{(n)}(\xi)|, \sup_{\xi \geq 0} e^{-q\xi} |y^{(n)}(\xi)| \right\}, \quad n \leq i
\]

where the symbol \(y^{(n)}\) denotes the \(n\)th derivative of \(y\), are finite.

\(B_{i}^{p,q}\) is a Banach space with the norm \(\|y\|_{B_{i}^{p,q}} = \sum_{n=0}^{i} y_n\). With the symbol \(B_{i,0}^{p,q}\) we denote the subspace of \(B_{i}^{p,q}\) consisting of functions \(y\) such that \(y(0) = 0\).

5.3. **The case when the wave-speed exceeds the minimal speed**

We consider first the case of the waves moving at the speed \(\sigma\) larger than the minimal speed \(\sigma^*\), where \(\sigma^*\) was introduced in Assumption 3.

The main result of this section is

**Theorem 2**

Let Assumptions 1–5 hold and let \(\sigma > \sigma^*\) be an arbitrary but fixed real number. Then for every \(\sigma > \sigma^* > \sigma_c\) there is \(\tau_1(\sigma)\) such that \(\inf_{\sigma \in [\sigma^*, \sigma]} \tau_1(\sigma) > 0\) and that for all \(|\tau| < \tau_1(\sigma)\):

1. The limit value problem (34)–(35), (49) has a unique solution \((\theta, n) \in \mathbb{R}^1 \times B_{\min(\text{Re} \lambda_{3, \lambda_1^+}) - \delta, \max(\text{Re} \lambda_{3, \lambda_0^+})} B_{0, \lambda_0^+}^{2,0}\), where \(\lambda_0^+\) is the larger of the two solutions of Equation (42), \(\lambda_1^+\) is the positive root of Equation (40), and \(\delta\) is a positive sufficiently small real number;
2. these functions are continuously differentiable with respect to \(\tau\) and such that \((\theta(\xi, \tau), n(\xi, \tau)) \to (0, G(\xi))\) in the norm of \(B_{\min(\text{Re} \lambda_{3, \lambda_1^+}) - \delta, \max(\text{Re} \lambda_{3, \lambda_0^+})} \times B_{0, \lambda_0^+}^{2,0}\), where \(G(\xi)\) is the solution of Equation (36);
3. if \(\sigma > \sigma^*\), then there is \(0 < \tau_2(\sigma) \leq \tau_1(\sigma)\) such that the dermis cell density function \(n(\xi)\) is positive for any \(\xi \in \mathbb{R}^1\) and \(|\tau| < \tau_2(\sigma)\).

This theorem will be proved in a sequence of lemmas. Firstly, since the subset of \(C^4(\mathbb{R}^1) \times C^2(\mathbb{R}^1)\) consisting of such functions \((\theta, n)\) which satisfy the limit conditions (49) does not form a linear vector space, we decompose \(n\) in the following way:

\[
n(\xi) = G(\xi) + h(\xi) \tag{60}
\]

where \(G(\xi)\) is a function introduced in Assumption 3(b) or, respectively, 3(c). With this decomposition the problem consisting of determining the solutions to Equations (34), (35)
subject to the limit conditions (49) becomes equivalent to the following one:

find solutions of the differential equations

\[ \beta \frac{d^4 \theta}{d \zeta^4} + \mu \sigma \frac{d^3 \theta}{d \zeta^3} - \frac{d^2 \theta}{d \zeta^2} + \rho \theta = -\tau P\left( \sigma, \theta, \frac{d \theta}{d \zeta}, \frac{d^2 \theta}{d \zeta^2}, \frac{d^3 \theta}{d \zeta^3}, h, \frac{dh}{d \zeta} \right) \]  

(61)

\[ \frac{d^2}{d \zeta^2} h + R_0'(\sigma, G, \frac{dG}{d \zeta}) \frac{dh}{d \zeta} + R_0(\sigma, G, \frac{dG}{d \zeta}) h + \sum_{i=0}^{3} r_i(\sigma, G, \frac{dG}{d \zeta}) \theta^{(i)} \]  

(62)

subject to the limit value conditions

\[ \lim_{\xi \to \pm \infty} \theta^{(i)}(\xi) = 0, \quad i = 1, 2, 3, 4, \quad \lim_{\xi \to \pm \infty} n(\xi) = 0, \quad \lim_{\xi \to \pm \infty} n^{(i)}(\xi) = 0, \quad i = 1, 2 \]  

(63)

where

\[ P_1(\sigma, 0, \theta', 0'', h, h') = P(\sigma, 0, \theta', 0'', h, (G + h)'') \]  

\[ R_1(\sigma, 0, \theta', 0'', h, h') = R(\sigma, 0, \theta', 0'', h, (G + h)'') - R_0(\sigma, G, G') \]  

\[ - R_0'(\sigma, G, G') h' - R_0(\sigma, G, G') h - \sum_{i=0}^{3} w_i(\sigma, G, G') \theta^{(i)} \]  

(64)

We begin our study from the following

Lemma 1

If the function \( g \in B^p_{p,q} \), where \( p, q \) are real numbers such that \( p > 0, \quad p \neq \text{Re} \lambda_3 \) and \( q < 0, \quad q \neq \text{Re} \lambda_2 \), where \( \lambda_i, \quad i = 1, 2, 3, 4 \), are the solutions of Equation (55) ordered as in (56), then the equation

\[ \beta \frac{d^4 \theta}{d \zeta^4} + \mu \sigma \frac{d^3 \theta}{d \zeta^3} - \frac{d^2 \theta}{d \zeta^2} + \rho \theta = g \]  

(65)

has in \( B^4_{\text{min}(p, \text{Re} \lambda_3), \text{max}(q, \text{Re} \lambda_2)} \) a uniquely determined solution given by the formula

\[ \theta(\xi) = K(g) = \frac{\exp(\lambda_1 \xi)}{\beta(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)} \int_{-\infty}^{\xi} g(\eta) \exp(-\lambda_1 \eta) d\eta \]

\[ - \frac{\exp(\lambda_2 \xi)}{\beta(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)} \int_{-\infty}^{\xi} g(\eta) \exp(-\lambda_2 \eta) d\eta \]
Furthermore, the operator $K$ is bounded as acting from $B^0_{p,q}$ into $B^4_{\min(p, \text{Re} \lambda_3), \max(q, \text{Re} \lambda_2)}$. So

$$\theta = K(g) \in B^4_{\min(p, \text{Re} \lambda_3), \max(q, \text{Re} \lambda_2)} \quad (67)$$

**Proof**

It is obvious that under the assumptions imposed on the function $g$, all integrals in (66) are convergent and this formula defines the unique bounded solution of Equation (65) satisfying the limit conditions (49). Also we show easily that under the same conditions the following hold true:

$$e^{\lambda_3 \xi} \int_{-\infty}^{\xi} e^{-\lambda_3 \zeta} g(\zeta) \, d\zeta \in B^1_{p, \max(q, \text{Re} \lambda_3)} \quad (\text{Re} \lambda < 0, \ q \neq \text{Re} \lambda) \quad (68)$$

$$e^{\lambda_4 \xi} \int_{\xi}^{\infty} e^{-\lambda_4 \zeta} g(\zeta) \, d\zeta \in B^1_{\min(p, \text{Re} \lambda_3), q} \quad (\text{Re} \lambda > 0, \ p \neq \text{Re} \lambda) \quad (69)$$

The thesis (66) of the lemma is an immediate consequence of (68), (69). The proof is complete. \hfill \Box

**Lemma 2**

If the function $f \in B^0_{a,b}$, where $a, b$ are positive real numbers, and if $a > 0$, $a \neq \lambda_1^+$, and $b < 0$, $b \neq \lambda_0^-$, where $\lambda_0^-$ is the larger of the two solutions of Equation (42) and $\lambda_1^+$ is the positive root of Equation (40), then the solution equation

$$h'' + R'_{0,n}(\sigma, G, G')h' + R'_{0,n}(\sigma, G, G')h = f \quad (70)$$

satisfying the limit value conditions (49) and such that $h(0) = 0$ is unique in $B^2_{\min(a, \lambda_1^+), \max(b, \lambda_0^-)}$ and has the form

$$h(\xi) = L(f) = -z_1(\xi) \int_{0}^{\xi} \left[ z_2(\zeta) - \frac{z_2(0)}{z_1(0)} z_1(\zeta) \right] b(\xi) f(\zeta) \, d\zeta$$

$$+ \left[ z_2(\xi) - \frac{z_2(0)}{z_1(0)} z_1(\xi) \right] \int_{-\infty}^{\xi} z_1(\zeta) b(\xi) f(\zeta) \, d\zeta \quad (71)$$

where

$$b(\xi) = \exp \left[ \int_{0}^{\xi} R'_{0,n'}(\sigma, G(\zeta), G'(\zeta)) \, d\zeta \right]$$
and

\[ z_1 = \frac{dG}{d\zeta} \]

\[ z_2(\hat{\zeta}) = \begin{cases} 
\frac{dG}{d\zeta} \int_0^\zeta \left[ b(\zeta) \left( \frac{dG(\zeta)}{d\zeta} \right)^2 \right]^{-1} d\zeta, & \text{for } \sigma = \sigma^* \\
\frac{dG}{d\zeta} \int_\zeta^\infty \left[ b(\zeta) \left( \frac{dG(\zeta)}{d\zeta} \right)^2 \right]^{-1} d\zeta, & \text{for } \sigma > \sigma^* 
\end{cases} \]  

(72)

In addition

\[ h \in B_{\min(a, \lambda_1^1), \max(b, \lambda_0^+)}^{2,0} \]  

(73)

\textbf{Proof}

Firstly, we show that all integrals in (71) and (72) are convergent. To this end we use the following easy to derive asymptotic formulae

\[ b(\zeta) \in B_{-(\lambda_1^-, \lambda_1^+), -(\lambda_0^+, \lambda_0^-)}^2 \]  

(74)

and

\[ z_1(\hat{\zeta}) \in \begin{cases} 
B_{\lambda_1^1, -\lambda_0^-}^2 & \text{for } \sigma = \sigma^* \\
B_{\lambda_1^-, \lambda_0^-}^2 & \text{for } \sigma > \sigma^* 
\end{cases} \]

(75)

Using them in (72) we see that the function \( z_2(\hat{\zeta}) \) is a well defined function on \((-\infty, \infty)\) and that

\[ z_2(\hat{\zeta}) \in \begin{cases} 
B_{\lambda_1^-, -\lambda_0^-}^2 & \text{for } \sigma = \sigma^* \\
B_{\lambda_1^1, -\lambda_0^-}^2 & \text{for } \sigma > \sigma^* 
\end{cases} \]  

(76)

In turn, (74), (75) along with the conditions imposed on the function \( f \) imply that the right hand side of (71) is well defined.

To show that (71) defines the desired solution of (70) let us notice that \( z_1(\hat{\zeta}) \) and \( z_2(\hat{\zeta}) \) are linearly independent solutions of the homogeneous version of Equation (70). Hence, the right hand side of (71) defines uniquely the solution of the considered equation.

Using these asymptotic expressions we can prove that

\[ z_1(\hat{\zeta}) \int_0^\zeta z_2(\hat{\zeta}) b(\zeta) f(\zeta) d\zeta \in \begin{cases} 
B_{\min(a, \lambda_1^1), \max(b, \lambda_0^+)}^{1,1} & \text{for } \sigma = \sigma^* \\
B_{\min(a, \lambda_1^1), \max(b, \lambda_0^+)}^{1,1} & \text{for } \sigma > \sigma^* 
\end{cases} \]  

(76)
and

\[ z_2(\xi) \int_{-\infty}^{\xi} z_1(\zeta) b(\zeta) f(\zeta) \, d\zeta \in \begin{cases} B_{a,\max(b,\lambda_0^+)}^1 & \text{for } \sigma = \sigma^* \\ B_{a,\max(b,\lambda_0^-)}^1 & \text{for } \sigma > \sigma^* \end{cases} \]

provided that the function \( f \) satisfies the imposed conditions. The thesis (72) of the lemma is a direct consequence of (71) and (76), (77). The proof is complete. \( \square \)

**Proof of Theorem 2**

We can write the system (61), (62) in the following form

\[ \theta = \Phi_1(\tau, \theta, h), \quad h = \Phi_2(\tau, \theta, h) \]

where we have defined

\[ \Phi_1(\tau, \theta, h) = -\tau K(P_1(\sigma, \theta', \theta''', h, h')) \]

\[ \Phi_2(\tau, \theta, h) = \tau^3 \sum_{i=0}^{3} L \left( r_i(\sigma, G, G') \frac{d}{d\zeta} K(P_1(\sigma, \theta', \theta''', h, h')) \right) - K(R_1(\sigma, \theta', \theta''', h, h')) \]

where the operators \( K \) and \( L \) are defined by (66) and (71), respectively.

We use the fixed-point theorem to prove the existence and uniqueness of solutions of the system (78) in the space \( B_{a,\max(\text{Re } \lambda_1, \lambda_0^+)}^1 \times B_{a,\max(\text{Re } \lambda_2, \lambda_0^-)}^1 \). However, we omit the proof because it goes along the same lines as that of the Implicit Function Theorem and can be found, for example, in the classical textbook by Schwartz [10]. Of course, the set of \( |\tau|'s \) for which the solution exists, depends on \( \sigma \) and for any fixed \( \sigma \) is bounded from above by a number \( \tau_1(\sigma) \). The infimum of the set \( \{ \tau: \tau = \tau_1(\sigma), \sigma \in [\sigma^*, \bar{\sigma}] \} \) is positive, what follows from the fact that the norms of the operators \( \Phi_1(0, \theta, h) \) and \( \Phi_2(0, \theta, h) \) as given by (66) and (71), respectively, are bounded functions of \( \sigma \in [\sigma^*, \bar{\sigma}] \).

It remains to prove (3). It follows from (2) that the function \( h(\xi, \tau) \) can be represented as

\[ h(\xi, \tau) = \tau h(0, \tau) \]

where \( h(\xi, \tau) \) is a bounded and continuous function of both variables. Now, it follows from Assumption 3 that there is a positive constant, say \( g_0 \) such that \( G(\xi) \geq g_0 e^{\xi} \) for \( \xi \geq \xi_0 \), if \( \xi_0 \) is sufficiently large. Next, from parts (1) and (2) of the present theorem we conclude that there is a positive constant, say \( h_0 \), such \( h(\xi, \tau) \geq -|\tau|h_0 e^{\xi} \) for \( \xi \geq \xi_0 \). Therefore, \( n(\xi, \tau) \geq e^{\xi}(g_0 - |\tau|h_0) \) if \( \xi \geq \xi_0 \). Hence, if \( |\tau| < g_0/h_0 \) then the skin cell density is positive at least for sufficiently large values of \( \xi \). Let us consider now the case of \( \xi \leq \xi_0 \). Since \( G(\xi) \) is a positive and strictly decreasing function then \( G(\xi_0) > G(\xi_0) > 0 \) provided that \( \xi \leq \xi_0 \), next, let \( h_1(\xi, \tau) = h(\xi, \tau)/|\tau| \). Hence, for \( \xi \leq \xi_0 \) we obtain the following estimate \( n(\xi, \tau) \geq G(\xi_0) + |\tau|h_1 \). The right hand side is a positive constant provided that \( |\tau| \) sufficiently small. We can identify the upper bound for \( |\tau| \) with \( \tau_2(\sigma) \). The proof is complete. \( \square \)

5.4. The case of the minimal wave speed

Now we show that in the case when the unperturbed wave described by Equation (36) moves at the minimal speed \( \sigma^* \), then it is possible to find a wave solution of the system (48) such
that the dermis cell density \( n(\xi) \) decays to 0 faster than \( \exp(\lambda^+_0(\sigma^* \xi)) \) as \( \xi \to \infty \). To this end we have to reformulate the problem in order to kill the contribution of this term in (71), which tends to zero as \( \xi \to \infty \) slower than \( \exp(\lambda^-_0 \xi) \). To this end we treat the wave speed \( \sigma \) as an additional unknown, which will be determined from the demand that the solution of the full-perturbed problem belongs to a suitably chosen Banach space.

In this subsection we change a little bit the notation. All the changes are collected in the following

**Definition 4**

In this subsection the following simplified notation is used:

1. With the symbol \( G(\xi) \) we denote the solution of Equation (36) for \( \sigma = \sigma^* \) such that \( G(-\infty) = 1, G(\infty) = 0 \).
2. The symbols \( \lambda_i, i = 1, 2, 3, 4 \) denote the roots of Equation (53) evaluated for \( \sigma = \sigma^* \).
3. Similarly, the symbols \( \lambda^\pm_i \) denote the roots of Equation (40) evaluated for \( \sigma = \sigma^* \), whereas \( \lambda^\pm_0 \) are the roots of Equation (42) for \( \sigma = \sigma^* \).

Also we take the following

**Assumption 7**

1. The minimal wave-speed \( \sigma^* \) introduced in Assumption 3 is such that \( (R'_0, \sigma^*, 0, 0))^2 > 4R'_0(\sigma^*, 0, 0) \); 2. for any \( \xi \in (-\infty, \infty) \), the inequality \( R'_0(\sigma^*, G, G') > 0 \) holds.

Assumption 7.1, when referred to Equation (28), means that in Theorem 1 we rule out the case (a). With this definition in mind, we set similarly to the previous case

\[
n(\xi) = G(\xi) + h(\xi)
\]

We can write the system (34), (35) in the following form

\[
Q_1(\tau, \sigma, \theta, h) = 0, \quad Q_2(\tau, \sigma, \theta, h) = 0
\]

where we have defined

\[
Q_1(\tau, \sigma, \theta, h) = \beta \frac{d^4 \theta}{d \xi^4} + \mu \frac{d^3 \theta}{d \xi^3} - \frac{d^2 \theta}{d \xi^2} + \rho \theta + \tau P\left( \sigma, \theta, \frac{d \theta}{d \xi}, \frac{d^2 \theta}{d \xi^2}, \frac{d^3 \theta}{d \xi^3}, G + h, \frac{d(G + h)}{d \xi} \right)
\]

(83)

\[
Q_2(\tau, \sigma, \theta, h) = \frac{d^2 h}{d \xi^2} + R\left( \sigma, \theta, \frac{d \theta}{d \xi}, \frac{d^2 \theta}{d \xi^2}, \frac{d^3 \theta}{d \xi^3}, G + h, \frac{d(G + h)}{d \xi} \right) - R_0\left( \sigma^*, G, \frac{dG}{d \xi} \right)
\]

(84)

It is clear that

**Lemma 3**

The mapping \( Q(\tau, \sigma, \theta, h) = [Q_1(\tau, \sigma, \theta, h), Q_2(\tau, \sigma, \theta, h)] \) is a continuous mapping from \( R^1 \times (R^1 \times B_0^{p - \delta, 0 + \delta} \times B_0^{2, 0}_{p - \delta, 0 + \delta}) \) into \( B_0^{p - \delta, q + \delta} \times B_0^{0, 0}_{p - \delta, 0 + \delta} \), where \( p = \min(\text{Re}\lambda_3, \lambda^+_1), q = \) \( \text{Re}\lambda_3, \lambda^+_1 \).
max(Re\(\lambda_2\), \(\lambda_0\)), with \(\delta\) being positive sufficiently small real constant. Moreover, it is continuously Fréchet differentiable with respect to \((\sigma, \theta, h)\).

Let us notice that for \(\tau = 0\), the equation \(Q(\tau, \sigma, 0, h) = 0\) is satisfied by the triple \((\sigma = \sigma^*, \theta = 0, h = 0)\). So everything we have to prove is to show that the Fréchet derivative \(DQ(\tau, \sigma, 0, h)(\tilde{\sigma}, \tilde{\theta}, \tilde{h})\) of \(Q(\tau, \sigma, 0, h)\) with respect to \((\sigma, \theta, h)\) evaluated at the point \((\tau = 0, \sigma = \sigma^*, \theta = 0, h = 0)\) is an invertible continuous operator acting between suitable Banach spaces. More precisely, we have to show that the system

\[
DQ(0, \sigma^*, 0, 0)(\tilde{\sigma}, \tilde{\theta}, \tilde{h}) = (g, f) \tag{85}
\]

has for all \((g, f) \in B^0_{p-\delta, q+\delta} \times B^0_{p-\delta, \lambda_0+\delta}\) uniquely determined solution \((\tilde{\sigma}, \tilde{\theta}, \tilde{h}) \in \mathbb{R}^1 \times B^0_{p-\delta, q+\delta} \times B^0_{p-\delta, \lambda_0+\delta}\) depending continuously on \((g, f)\) with \(p, q\) such as in Lemma 3.

**Lemma 4**

Suppose that Assumptions 1–7 hold. Then for all \((g, f) \in B^0_{p-\delta, q+\delta} \times B^0_{p-\delta, \lambda_0+\delta}\) the system (85) has a uniquely determined solution \((\tilde{\sigma}, \tilde{\theta}, \tilde{h}) \in \mathbb{R}^1 \times B^4_{p-\delta, q+\delta} \times B^2_{p-\delta, \lambda_0+\delta}\), with \(p, q\) defined in Lemma 3 and \(\delta\) being a positive sufficiently small real number. Thus the operator \(DQ(0, \sigma^*, 0, 0)\) has a bounded inverse.

**Proof**

The system (85), when written explicitly, is of the form

\[
\beta \frac{d^4 \tilde{\theta}}{d\xi^4} + \mu \sigma^* \frac{d^3 \tilde{\theta}}{d\xi^3} - \frac{d^2 \tilde{\theta}}{d\xi^2} + \rho \tilde{\theta} = g \tag{86}
\]

\[
\tilde{h}'' + R'_{0,\alpha}(\sigma^*, G, G')\tilde{h}' + R'_{0,\alpha}(\sigma^*, G, G')\tilde{h} = \Phi(f, \tilde{\sigma}, \tilde{\theta}) \tag{87}
\]

where

\[
\Phi(f, \tilde{\sigma}, \tilde{\theta}) = f - \sum_{i=0}^{3} r_i(\sigma^*, G, G')\tilde{\theta}^{(i)} - \tilde{\sigma}R'_{0,\alpha}(\sigma^*, G, G') \tag{88}
\]

and \(r_i(\sigma, G, G'), i = 0, 1, 2, 3,\) are defined in (61).

The solution of Equation (86) is (see Lemma 1)

\[
\tilde{\theta}(\xi) = K(g)(\xi) \in B^4_{p-\delta, q+\delta} \tag{89}
\]

whereas the solution of (87) can be written in the form

\[
\tilde{h}(\xi) = \tilde{h}_1(\xi) + \tilde{h}_2(\xi) \tag{90}
\]

where

\[
\tilde{h}_1(\xi) = -z_1(\xi) \int_0^\xi z_2(\zeta)b(\zeta)\Phi(\zeta) \, d\zeta \tag{91}
\]

\[
\tilde{h}_2(\xi) = z_2(\xi) \int_{-\infty}^\xi z_1(\zeta)b(\zeta)\Phi(\zeta) \, d\zeta
\]
Now, using (89) as well as the assumptions imposed on the functions \( R \) and \( f \), we conclude that
\[
\Phi(f, \tilde{\sigma}, \tilde{\theta}) \in B^{1}_{p-\delta, \lambda_{0}^{-}+\delta}
\] (92)

Owing to this and (74) we obtain
\[
\tilde{h}_{1} \in B^{1.0}_{p-\delta, \lambda_{0}^{-}+\delta}
\] (93)

A similar procedure with the use of (75) yields
\[
\tilde{h}_{2} \in B^{1.0}_{p-\delta, \lambda_{0}^{-}}
\] (94)

Hence, \( \tilde{h} \in B^{2.0}_{p-\delta, \lambda_{0}^{-}+\delta} \) only if
\[
\int_{-\infty}^{\infty} z_{1}(\zeta)b(\zeta)\Phi(f, \tilde{\sigma}, \tilde{\theta})(\zeta)\,d\zeta = 0
\]

Explicitly this condition reads
\[
\tilde{\sigma} = \left[ \int_{-\infty}^{\infty} G'(\zeta)b(\zeta)\Phi(f, 0, \tilde{\theta})(\zeta)\,d\zeta \right]^{-1} \left[ \int_{-\infty}^{\infty} G'(\zeta)b(\zeta)R_{0.\delta}(\sigma^{*}, G, G')(\zeta)\,d\zeta \right]^{-1}
\] (95)

determining thus \( \tilde{\sigma} \) in a unique way. Thus the operator \( DQ(0, \sigma^{*}, 0, 0) \) is invertible on the whole \( B_{p-\delta, q+\delta}^{0} \times B_{p-\delta, \lambda_{0}^{-}+\delta}^{0} \). Due to Theorem 4.2-H, p. 180 in Reference [11] the inverse operator \((DQ(0, \sigma^{*}, 0, 0))^{-1}\) is continuous and thus bounded. The proof of the lemma is complete.

\[\Box\]

**Theorem 3**

Assume that Assumptions 1–7 hold. Then for \(|\tau|\) sufficiently small, there exist a unique (up to a translation in \( \xi \)) triple \((\sigma(\tau), \theta(\xi, \tau), n(\xi, \tau))\) satisfying (34), (35) such that:

1. the triple \((\sigma, \theta, n)\in\mathbb{R}^{1}\times B_{p-\delta, q+\delta}^{0} \times B_{0, \lambda_{0}^{-}+\delta}^{0}\) where \(\delta > 0\) is sufficiently small;
2. the functions \(\sigma(\tau), \theta(\xi, \tau)\), and \(n(\xi, \tau)\) are continuously differentiable with respect to \(\tau\) and \((\sigma(\tau), \theta(\xi, \tau), n(\xi, \tau)) \to (\sigma_{*}, 0, G(\xi))\) in the sense of \(\mathbb{R}^{1}\times B_{p-\delta, q+\delta}^{0} \times B_{0, \lambda_{0}^{-}+\delta}^{0}\);
3. the functions \((\theta(\xi, \tau), n(\xi, \tau))\) satisfy the limit conditions (49).

**Proof**

By Lemma 4 and the Implicit Function Theorem [10] we infer that for sufficiently small \(|\tau|\), there exist for Equations (82) a unique solution \((\sigma(\tau), \theta(\tau), h(\tau))\in\mathbb{R}^{1}\times B_{p-\delta, q+\delta}^{0} \times B_{0, \lambda_{0}^{-}+\delta}^{0}\) and thus, by (81), a unique (up to a translation) solution \((\sigma(\tau), \theta(\tau), n(\tau))\) of the system (34), (35), (49) belonging to the space \(\mathbb{R}^{1}\times B_{p-\delta, q+\delta}^{0} \times B_{0, \lambda_{0}^{-}+\delta}^{0}\). The proof is complete.

\[\Box\]

Now, we prove that the dermis cell density is monotone, which is a more difficult task than the previous one. More precisely, it is difficult to show that the dermis cell density \(n(\xi)\) is monotone and positive for large values of \(\xi\), say for \(\xi \geq \xi_{0}\). From the decomposition (81) and Theorem 2 it follows that \(n(\xi) > 0\) for \(\xi \in (-\infty, \xi_{0}]\) and sufficiently small \(0 < |\tau| \leq \tau(\xi_{0})\). That
is why we focus our attention on the case of the behaviour of the solution of our problem in
a vicinity of the critical point \(Z=(0,0,0,0,0,0)\).
To this end we will use the representation (48) of the equations (34), (35). Basing on
Assumptions 4 and 5, we rewrite this system in the form
\[
\mathbf{u}' = \mathbf{M}_Z \mathbf{u} + \mathbf{E}(\tau, \sigma, \mathbf{u})
\]
where \(\mathbf{u}=(\theta, \theta_1, \theta_2, \theta_3, n, n_1)\), \(\mathbf{M}_Z\) is the matrix given by (51) evaluated for \(c=Z\), and
\[
\mathbf{E}(\tau, \sigma, \mathbf{u}) = \begin{pmatrix}
0 \\
0 \\
0 \\
-\frac{\tau}{\beta}[P(\mathbf{u}) - P_{u_5}(Z)u_5 - P_{u_6}(Z)u_6] \\
-\frac{1}{\beta}[R(\sigma, \mathbf{u}) - R_{u_5}(\sigma, Z)u_5 - R_{u_6}(\sigma, Z)u_6]
\end{pmatrix}
\]
It follows from Assumptions 4 and 5 that the asymptotic estimate holds true
\[
|\mathbf{E}(\tau, \sigma, \mathbf{u})| = O(|\mathbf{u}|^2) \quad \text{as} \quad |\mathbf{u}| \to 0
\]
Owing to Assumptions 4, 5, the assumptions of the Hartman–Grobman Theorem [12,13]
are satisfied for our system (96). Hence, there is a homeomorphism \(H : \mathbb{R}^6 \to \mathbb{R}^6\), \(H(Z) = Z\)
defined in a neighbourhood of the point \(Z\) taking the orbits of the linear system
\[
\mathbf{v}' = \mathbf{M}_Z \mathbf{v}
\]
to those of the non-linear system (96), i.e.
\[
\mathbf{u}(\xi, \mathbf{u}_0) = H^{-1}[\exp(\mathbf{M}_Z \xi)H(\mathbf{u}_0)]
\]
for all \(\mathbf{u}_0\) from some vicinity of \(Z\). In general, \(H\) is only of \(C^0\) class, however in our case it
is of higher class. Namely, we have

**Lemma 5**
Under Assumptions 1–6, the homeomorphism \(H\) appearing in the Hartman–Grobman Theorem
(cf. (99)) is a \(C^1\) local diffeomorphism, when constrained to the stable manifold of \(Z\).

**Proof**
We treat the six dimensional space \(\mathbb{R}^6\) as a Cartesian product of \(\mathbb{R}^4\) and \(\mathbb{R}^2\), i.e. we write
\(\mathbb{R}^6 = \mathbb{R}^4 \times \mathbb{R}^2\). By choosing appropriate variables \((\mathbf{w}, \mathbf{z})\) with \(\mathbf{w} \in \mathbb{R}^4\), \(\mathbf{z} \in \mathbb{R}^2\) we can transform
the system (96) to the form
\[
\begin{pmatrix}
\mathbf{w}' \\
\mathbf{z}'
\end{pmatrix} =
\begin{pmatrix}
\mathbf{A} & 0 \\
0 & \mathbf{B}
\end{pmatrix}
\begin{pmatrix}
\mathbf{w} \\
\mathbf{z}
\end{pmatrix} + \mathbf{F}(\mathbf{w}, \mathbf{z})
\]
where \(\mathbf{A}\) is a constant \(4 \times 4\) matrix whose all eigenvalues have negative real parts, i.e. they
are \(\lambda_3^i(\sigma), \lambda_4(\sigma)\) and \(\lambda_5(\sigma)\) (see (44) and (56), respectively), \(\mathbf{B}\) is a constant \(2 \times 2\) ma-
trix whose all eigenvalues have positive real parts, i.e. they are \(\lambda_3(\sigma)\) and \(\lambda_4(\sigma)\), finally

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\( \mathbf{F}(w, z) = (\mathbf{F}_1(w, z), \mathbf{F}_2(w, z)) \) is of \( C^2 \)-class and satisfies \( \mathbf{F}(0, 0) = \mathbf{0}, \partial \mathbf{F}(0, 0) = \mathbf{0}. \) Hence the stable manifold \( P_s \) of the linear counterpart of this system coincides with the plane \( z = 0. \) According to Lemma IX. 5.1 in Reference [12] we conclude that locally \( P_s \) is at least \( C^1 \) diffeomorphic to the four-dimensional stable manifold \( U \) of the full system. To be more precise, there exists \( \psi \) at least of \( C^1 \) class, \( \psi : P_s \to U, \) such that \( \psi(w) = (w, g(w)) \in U \) for all \( w \in P_s \) with \( |w| \) sufficiently small, such that \( g : \mathbb{R}^4 \to \mathbb{R}^2, \) \( g(0) = \mathbf{0}, \partial g(0) = \mathbf{0} \) such that every point \( (w, z) \in U \) with \( |w| + |z| \) sufficiently small can be written as \( (w, z) = (w, g(w)). \) Thus replacing \( z \) with \( \psi(w) \) we can lead the considered system on \( U \) to the plane \( P_s. \) This system has the form:

\[
w' = A\mathbf{w} + \mathbf{F}_1(w, \psi(w))
\]

Now, due to the fact that \( A \) generates a contraction (for positive \( \xi \)), we conclude (see e.g. Exercise IX. 8.2 in Reference [12], as well as References [13,14]) that the homeomorphism \( H : \mathbb{R}^6 \to \mathbb{R}^6 \) from the Hartman–Großman Theorem is at least a \( C^1 \) diffeomorphism and close to identity i.e. such that \( H(0) = \mathbf{0} \) and \( \partial H(0) = \mathbf{I}. \) By means of this diffeomorphism every trajectory of the non-linear system (99) is conjugate to a trajectory of the linear system \( w' = A\mathbf{w}. \)

Now, for a given \( \tau, \) let \( S_t \) be a plane spanned by the eigenvectors \( r_0^- (\sigma(\tau)), r_1 (\sigma(\tau)), r_2 (\sigma(\tau)) \) of the matrix \( M_Z. \) Let \( S_n \) be the counter image of \( S_t \cap B(Z, r), \) where \( B(Z, r) \subset \mathbb{R}^6 \) is a ball with the centre at \( Z \) and a sufficiently small radius \( r > 0, \) i.e. \( S_n = H^{-1}(S_t \cap B(Z, r)). \) We have thus constructed a three-dimensional manifold containing the trajectories with the last two components tending to 0 with the exponents \( \lambda^-_0 (\sigma(\tau)) \) as \( \xi \to +\infty. \)

Lemma 6

\( S_n \) is a unique set of points in \( B(Z, r) \) the trajectories through which are such that their last two components tend to zero as fast as \( \exp(\lambda^-_0 (\sigma(\tau))\xi) \) as \( \xi \to +\infty. \)

Proof

Let us note that every other trajectory (if it exists) satisfying the above condition must be asymptotically tangent to \( S_n. \) \( S_n \) is in turn tangent to \( S_t \) at \( Z. \) As \( S_n = H^{-1}(S_t \cap B(Z, r)), \) it follows that every trajectory tangent asymptotically to \( S_n, \) but not contained in it must be locally mapped by the diffeomorphism \( H \) to a trajectory not lying in \( S_t. \) (Note that \( S_t \) is invariant with respect to the linear counterpart of the considered system.) But, as \( H \) is of \( C^1 \) class, the tangent vector of this trajectory must be asymptotically tangent to \( S_t. \) However, analysing the flow generated by the linear system (98) we conclude that this is impossible. Indeed, there is no trajectory of the form

\[
c_0^- \exp(\lambda^-_0 \xi) r_0^- + c_0^+ \exp(\lambda^+_0 \xi) r_0^+ + c_1 \exp(\lambda_1 \xi) r_1 + c_2 \exp(\lambda_2 \xi) r_2
\]

with \( c_0^+ \neq 0 \) tending to a trajectory which can be described by

\[
d_0^- \exp(\lambda^-_0 \xi) r_0^- + d_1 \exp(\lambda_1 \xi) r_1 + d_2 \exp(\lambda_2 \xi) r_2
\]

as \( \xi \to +\infty \) due to the fact that \( \lambda^-_0 (\sigma(\tau)) < \lambda^+_0 (\sigma(\tau)). \) The proof is complete. \( \Box \)
Lemma 7
Let the Assumptions 1–7 be satisfied. Then the plane \( M^0 = \{ n = 0, n’ = 0 \} \) is invariant with respect to the system (34), (35). Moreover, if there exists a submanifold of \( M^0 \) invariant (with respect to this system), then it is spanned by appropriate subset of the eigenvectors \( r_1^0(\sigma(\tau)), r_2^0(\sigma(\tau)), r_3(\sigma(\tau)) \) of the matrix \( M_z \) corresponding to the eigenvalues of this matrix which have negative real part.

Proof
The proof follows from Assumption 5(b) and the fact that on \( M^0 \) the system (34), (35) degenerates to a linear equation.

Theorem 4
Under the assumptions of Theorem 3, for all \( |\tau| \) sufficiently small the dermis cell density \( n(\xi) \) is a positive and decreasing function of \( \xi \in \mathbb{R}^1 \).

Proof
Let us fix \( \tau \) sufficiently small and denote the solution obtained in Theorem 3 by \( u_\tau = (\theta, \theta_1, \theta_2, \theta_3, n, n_\tau, n_1, n_2, \ldots) \). Note that \( u_\tau \) is contained in the common part of the stable manifold of \( Z \) and the set of solutions for which \( n_\tau(\xi) \) and \( n_1(\xi) \) tend zero faster than \( \exp((\lambda_0^+(\sigma(\tau)) - \delta)\zeta) \), \( \delta > 0 \). This manifold, which will be denoted as \( M(\tau) \), is three dimensional, according to Lemmas 5 and 6. For \( \tau = 0 \) it is spanned by the vectors \( r_1(\sigma^+), r_2(\sigma^+) \) and \( r_0^- (0, \sigma^+) \). The manifold \( M_{12} \) spanned by \( r_1(\sigma^+) \) and \( r_2(\sigma^+) \) is also invariant with respect to the full system (96) and is contained in \( M^0 \). When \( |\tau| > 0, |\sigma - \sigma^+| > 0 \), then the vectors \( r_1(\sigma), r_2(\sigma) \) and \( r_0^- (\tau, \sigma) \) with \( r_0^- (\tau, \sigma) \) perpendicular to \( r_1(\sigma) \) and \( r_2(\sigma) \) up to \( O(\tau) \) terms (see (57), (58)) span the plane \( M_{12} \), defined as the plane tangent to \( M(\tau) \) at the point \( Z \). According to the Hartman-Grobman Theorem [12,13] in the vicinity of \( Z \) the flow \( u_\tau(\xi, u_0) \) generated by the non-linear system (96) is conjugated to that generated by the linearized system (98). This conjugacy is established by (99), where, according to Lemma 5, \( H \) is at least \( C^1 \) diffeomorphism satisfying the condition of non-vanishing Jacobian at \( Z \). In consequence, due to Lemma 6, \( M(\tau) \) is of \( C^1 \) class and at \( Z \) its tangent three-dimensional plane is the union of \( M_{12} \) and two half-spaces with \( \{ n > 0 \} \) and \( \{ n < 0 \} \). Similarly, sufficiently close to \( Z \), \( M(\tau) \) is the sum of \( M(\tau)_+ \) and \( M(\tau)_- \) having the same properties. To see this let us remind that \( M_{12} \) is invariant with respect to the full system (95), thus for \( |\tau| \) small enough every vector \( u \in S_n \) can be represented as a sum of a vector \( \tilde{r}(u, \tau) \in M_{12} \) and \( c(u, \tau) \tilde{r}_0(u, \tau) \), where \( \tilde{r}_0(u, \tau) = r_0^- (\tau, \sigma^+) + o(1) \) as \( |u| + |\tau| \to 0 \). Hence, if \( u \in S_n \) is sufficiently close to \( Z \), then either \( u \in M_{12} \) or \( u_5 \neq 0 \). In consequence, if \( u_5 \) fell into \( M(\tau)_- \) then it would never be able to reach \( M(\tau)_+ \), because the solution is unique and \( M_{12} \) is an invariant subspace. The proof is complete.

5.5. Perturbation of non-monotonic solutions
We complete Assumption 3 with the following point

Assumption 3(d)
Let \( \sigma’ > \sigma > \sigma_c \), then there exists a unique (except for translation) non-monotone heteroclinic solution \( n = G(\xi) \) of (36) tending for \( \xi \to -\infty \) to 1 and for \( \xi \to \infty \) to 0 and entering the node \((0,0)\) in the direction \( dn_1/dn = \lambda_0^+(\sigma) \).
Motivation: The cited Theorem 1 does not cover the case \( \sigma^* > \sigma > \sigma_c \). However by means of the Implicit Function Theorem it can be proved that at least for \( \sigma \) close to \( \sigma^* \) the travelling wave solution to Equation (28) exists and enters the node \((0,0)\) in the direction \( \frac{dn_1}{dn} = \lambda^+_0(\sigma) \).

According to the first part of this theorem, this solution must be non-monotone. But one can prove

Lemma 8
Let \( G(\xi) \) be a non-monotone solution of Equation (36) corresponding to \( \sigma \in (\sigma_c, \sigma^*) \) and tending for \( \xi \to -\infty \) to 1 and for \( \xi \to \infty \) to 0, corresponding to \( \sigma \in (\sigma_c, \sigma^*) \). Then \( G(\xi) \) is negative for some values of its argument.

Proof
First, let us notice that from Assumption 1 it follows that the function \( R_0(\sigma, n, 0) \) takes positive values only in the interval \( 0 < n < 1 \), it vanishes exactly at two points \( n = 0 \) and \( n = 1 \), and is negative for all other values of \( n \). Secondly, if at \( \xi = \xi_M \) the function \( G(\xi) \) attains a local maximum, then

\[
0 < G(\xi_M) < 1
\]  

(101)

Indeed, let \( \xi = \xi_M \) be the point of a local maximum, then \( G'(\xi_M) = 0 \), \( G''(\xi_M) < 0 \). But \( G(\xi) \) satisfies Equation (36), hence it must be \( R_0(\sigma, G(\xi_M), 0) > 0 \), what is possible only if (101) holds. Thirdly, if at \( \xi = \xi_M \) the function \( G(\xi) \) attains a local minimum, then \( G(\xi_M) < 0 \). In this case we have \( G'(\xi_m) = 0 \), \( G''(\xi_m) > 0 \), what implies \( R_0(\sigma, G(\xi_M), 0) < 0 \). But this is possible only if \( G(\xi_0) < 0 \) or if \( G(\xi_0) > 1 \). In the first case, the lemma is proved. We show that the second case is impossible. Indeed, let a second case hold. Then, such a point of minimum has to be accompanied by a point of maximum with the value of the considered function exceeding one. According to (101) this is a contradiction. The proof is complete.

Theorem 5
Let Assumptions 1–7 and 3(d) hold and let \( \sigma \) be an arbitrary but fixed real number such that \( \sigma_c < \sigma < \sigma^* \). Then for every \( \sigma \in (\sigma, \sigma^*) \) there is \( t_3(\sigma) > 0 \) such that for all \( |t| < t_3(\sigma) \):

1. the limit value problem (34)−(35), (49) has a unique solution \((\theta, n) \in \mathbb{R}^1 \times B^{4}_{\min(\Re \lambda_2, \lambda_1^+)} - \delta, \max(\Re \lambda_2, \lambda_1^+) \times B^{2,0}_0, \lambda_0 \) where \( \lambda_0^+ \) is the larger of the two solutions of Equation (42), \( \lambda_1^+ \) is the positive root of Equation (40), and \( \delta \) is positive sufficiently small real number;

2. these functions are continuously differentiable with respect to \( \tau \) and such that \((\theta(\xi, \tau), n(\xi, \tau)) \to (0, G(\xi)) \) in the norm of \( B^{4}_{\min(\Re \lambda_2, \lambda_1^+)} - \delta, \max(\Re \lambda_2, \lambda_1^+) \times B^{2,0}_0, \lambda_0 \), where \( G(\xi) \) is the solution of Equation (36);

3. there is \( 0 < t_3(\sigma) < t_3(\sigma) \) such that for every \( \sigma \in (\sigma, \sigma^*) \) and \( \tau \in [0, t_4(\sigma)] \) the dermis cell density function \( n(\xi) \) is negative for some \( \xi \in \mathbb{R}^1 \).

Proof
To prove the first two points of the thesis we could follow the lines of the proof of Theorem 2, however, we have to modify the form (72) of the second solution of Equation (70), since now the integrand in (72) becomes divergent at the points of extrema of the function \( G(\xi) \). It is
proved in Appendix that there exists a solution \( z_2(\xi) \) of Equation (70) such that \( z_2(\xi) \in B_{\lambda_1,-\lambda_0}^2 \) for \( \sigma < \sigma < \sigma^* \), and the solutions \( z_1(\xi), z_2(\xi) \) are linearly independent. This is sufficient to prove points (1) and (2) of the theorem. The truth of point (3) can be proved as follows. We use the decomposition (60). Since at \( \xi = \xi_m \) the function \( G(\xi) \) is negative, as this is the point of a local minimum, it is negative on some non-zero interval \( [\xi_1, \xi_2] \ni \xi_m \). But from (2) it follows that the function \( h(\xi, \tau) \) can be represented as \( h(\xi, \tau) = \tilde{h}(0, \tau) \), where \( \tilde{h}(\xi, \tau) \) is a bounded and continuous function of both variables. Therefore, for \( |\tau| \) sufficiently small the sum \( n(\xi) = G(\xi) + h(\xi) \) is also negative on some non-zero interval \( [\xi_3, \xi_4] \subset [\xi_1, \xi_2] \). The proof is complete.

6. FINAL REMARKS

The considerations conducted in Sections 5.3–5.5 do not solve all problems concerning the positiveness of the dermis cell density. For instance, it cannot be excluded at the moment that the function \( \tau_2(\sigma) \), introduced in Theorem 2, vanishes for \( \sigma = \sigma^* \). Also the function \( \tau_4(\sigma) \) of Theorem 4 may vanish at the same point.

Let us fix \( |\tau| \) sufficiently small. We will prove that the heteroclinic solutions corresponding to \( \sigma \in [\sigma(\tau), \infty) \), satisfy the condition \( n(\xi) > 0 \) for all \( \xi \), under a simplifying assumption concerning the eigenvalues at point \( Z \).

Assumption 8

The eigenvalues at point \( Z \) admit the following ordering

\[
\lambda_0^- < \lambda_0^+ < \Re \lambda_1 < \Re \lambda_2 < 0 < \Re \lambda_3 < \Re \lambda_4
\]

As we proved in Theorem 2, for \( |\tau| \) sufficiently small and \( \sigma \in (\sigma^*, \tilde{\sigma}) \) the heteroclinic solution of our problem satisfies the condition \( n(\xi) > 0 \) for all \( \xi \), even without Assumption 8. The same is true for \( \sigma = \sigma(\tau) \) (cf. Theorem 4). To prove that \( n(\xi) > 0 \) for all \( \xi \sigma \in (\sigma(\tau), \tilde{\sigma}) \) we proceed as follows. We take an arbitrary but fixed \( \tilde{\sigma} \geq \sigma^* \) and such that \( n(\tilde{\sigma}, \xi, \xi) > 0 \) and prove that \( n(\xi) > 0 \) for \( \sigma \in (\sigma(\tau), \tilde{\sigma}) \) making use of the fact the solution depends continuously on \( \sigma \) in the above defined spaces of exponentially vanishing functions \( (B_2^\infty) \). Moreover, the cell number density tends to zero faster than \( \exp(\Re \lambda_1 \xi) \) as \( \xi \to \infty \). Thus they are contained in the invariant manifold \( I_\sigma \) corresponding to the eigenvalues corresponding to \( \lambda_0^- (\sigma), \lambda_0^+ (\sigma) \). This invariant manifold, which is \( C^1 \) \( O(\tau) \)-close to the plane \( \{n, n'\} \) of the phase space. This fact will be crucial below. First, let us note that the trajectories near point \( Z \) resemble the trajectories of the linearized system. Let \( c_\sigma \) denote the positive part of the invariant manifold generated by \( \lambda_0^- (\sigma) \). Let us note that the solution contained in \( I_\sigma \) must lie (locally near \( Z \)) above \( c_\sigma \). Thus its trajectory is a curve lying in the set \( \{n > 0, n' > 0\} \). Note that, if the trajectory crossed the line \( I_\sigma \cap \{n' = 0\} \) for \( n < 1 \), then this trajectory could not reach the point \( Z \). This is due to the fact that the solutions are unique, hence do not intersect and due to the structure of trajectories near \( Z \). Thus existence and lying above \( c_\sigma \) implies monotonicity outside the point \( Z \). Continuing the above reasoning we conclude that either \( n \) is (locally) monotone (and lies above \( c_\sigma \) for \( \sigma \in (\sigma(\tau), \tilde{\sigma}) \)) or there exists \( \sigma_0 \in (\sigma(\tau), \tilde{\sigma}) \) such that the orbit corresponding to \( \sigma_0 \) coincides with \( c_n \). However, according to the implicit function theorem for \( |\tau| \) and \( |\sigma - \sigma^*| \) sufficiently small, there exists only one heteroclinic coinciding with the invariant manifold corresponding to the eigenvalue \( \lambda_0^- \). Now we consider the second case. Let
us note that for all $\tau > 0$ sufficiently small there exists $\tilde{\sigma} \in (\sigma, \sigma(\tau))$ such that $n(\tilde{\sigma}, \tau, \xi)$ is not monotone. One can prove that $n(\sigma, \tau, \xi)$ is not monotone also for all $\sigma \in (\tilde{\sigma}, \sigma(\tau))$. As above, the proof comes from the fact that for $\sigma = \tilde{\sigma}$ the considered trajectory lies below $c_\sigma = c_{\tilde{\sigma}}(\theta)$ near the point $Z$. By the continuity the same condition is satisfied for all $\sigma > \tilde{\sigma}$ sufficiently close to $\tilde{\sigma}$. Thus either the solution is non-monotone in $n$ and lies below $c_\sigma$ or it coincides with $c_\sigma$. But this would mean that $\sigma = \sigma(\tau)$.

Due to the above considerations the wave-speed $\sigma = \sigma(\tau)$ can be called the minimal speed of the system (34), (35).

**APPENDIX**

To avoid complicated calculations we take a simplifying assumption that the function $G(\xi)$ has only one point of extremum, which as it was said in the proof of Lemma 8, has a point of minimum. So, let $\xi = \xi_m$ be the only extremal point of the function $G(\xi)$. Instead of the function $z_2(\xi)$ defined by the second formula in (72) we introduce

$$z_2(\xi) = \begin{cases} z_+(\xi) & \text{for } \xi > \xi_m \\ A & \text{for } \xi = \xi_m \\ z_-(\xi) & \text{for } \xi < \xi_m \end{cases} \quad (A1)$$

where

$$z_+(\xi) = G'(\xi) \int_{\xi_m}^{\xi} \frac{1}{b(\xi)G''(\xi)} \, d\xi', \quad \xi > \xi_m$$

$$z_-(\xi) = -G'(\xi) \int_{\xi_m}^{\xi} \frac{1}{b(\xi)G''(\xi)} \, d\xi' + BG'(\xi), \quad \xi < \xi_m \quad (A2)$$

where $z_0$ is an arbitrary constant less than $\xi_m$, and $A$, $B$ are constants, which will be found from the conditions of continuity of $z_2(\xi)$ and $z'_2(\xi)$ at the point $\xi_m$. From the l’Hôpital rule we obtain

$$\lim_{\xi \uparrow \xi_m} z_-(\xi) = \lim_{\xi \downarrow \xi_m} z_+(\xi) = \frac{1}{G''(\xi_m)b(\xi_m)}$$

So, we take in (A1)

$$A = \frac{1}{G''(\xi_m)b(\xi_m)} \quad (A3)$$

This choice of the constant $A$ guarantees the continuity of the function $z_2(\xi)$ for any $\xi \in \mathbb{R}^1$.

Next we have, obviously

$$z'_+(\xi) = G''(\xi) \int_{\xi_m}^{\xi} \frac{1}{b(\xi)G''(\xi)} \, d\xi' - \frac{1}{b(\xi)G'(\xi)}, \quad \xi > \xi_m$$

$$z'_-(\xi) = -G'(\xi) \int_{\xi_m}^{\xi} \frac{1}{b(\xi)G''(\xi)} \, d\xi' - \frac{1}{b(\xi)G'(\xi)} + BG'(\xi), \quad \xi < \xi_m$$
First, we show that the limit \( \lim_{\xi \downarrow \xi_m} z'_+(\xi) \) exists. We take a constant \( \beta \) such that \( \beta > \xi_m \) and that \( G''(\xi) \neq 0 \) for \( \xi \in [\xi_m, \beta] \), and write

\[
z'_+(\xi) = G''(\xi) \int_\beta^\infty \frac{1}{b(\xi)G'^2(\xi)} \, d\xi + G''(\xi) \int_\xi^\beta \frac{1}{b(\xi)G'^2(\xi)} \, d\xi - \frac{1}{b(\xi)G'(\xi)}
\]

Integrating by parts we have

\[
\int_\xi^\beta \frac{1}{b(\xi)G'^2(\xi)} \, d\xi = - \left[ \frac{1}{G'(\xi)G''(\xi)b(\xi)} \right]_\xi^\beta + \int_\xi^\beta \left( \frac{1}{G''(\xi)b(\xi)} \right) \, G'(\xi) \, d\xi
\]

We check easily that

\[
\left( \frac{1}{G''(\xi)b(\xi)} \right)' = \frac{R'_{0n}(G(\xi), G'(\xi))}{b(\xi)G'^2(\xi)} G'(\xi)
\]

Hence, we obtain

\[
\int_\xi^\beta \frac{1}{b(\xi)G'^2(\xi)} \, d\xi = - \left[ \frac{1}{G'(\xi)G''(\xi)b(\xi)} \right]_\xi^\beta + \int_\xi^\beta \frac{R'_{0n}(G(\xi), G'(\xi))'}{b(\xi)G'^2(\xi)} \, d\xi
\]

Using this formula in (A4) we get

\[
z'_+(\xi) = G''(\xi) \int_\beta^\infty \frac{1}{b(\xi)G'^2(\xi)} \, d\xi + G''(\xi) \int_\xi^\beta \frac{R'_{0n}(\sigma, G(\xi), G'(\xi))}{b(\xi)G'^2(\xi)} \, d\xi - \frac{G''(\xi)}{G'(\beta)G''(\beta)b(\beta)}, \quad \xi > \xi_m.
\]

(A5)

In the similar way we arrive at

\[
z'_-(\xi) = -G''(\xi) \int_{x_0}^x \frac{1}{b(\xi)G'^2(\xi)} \, d\xi - G''(\xi) \int_x^\xi \frac{R'_{0n}(\sigma, G(x), G'(\xi))}{b(\xi)G'^2(\xi)} \, d\xi - \frac{G''(\xi)}{G'(x)G''(\xi)b(x)} + BG''(\xi), \quad \xi < \xi_m
\]

(A6)

where \( \kappa \) is a real number such that \( \kappa < \xi_m \) and that \( G''(\xi) \neq 0 \) for \( \xi \in [\xi_m, \kappa] \). Now, from the condition of the continuity of the derivatives at the point \( \xi = \xi_m \) we obtain using (A5) and (A6)

\[
B = \int_{x_0}^x \frac{d\xi}{b(\xi)G'^2(\xi)} + \int_\beta^\infty \frac{d\xi}{b(\xi)G'^2(\xi)} + \int_x^\beta \frac{R'_{0n}(\sigma, G(\xi), G'(\xi))}{b(\xi)G'^2(\xi)} \, d\xi - \frac{1}{G'(\tau)G''(\tau)b(\tau)} \bigg|_{\tau = \beta}^{\tau = \xi}
\]

(A7)
Formulae (A1)–(A3) and (A7) define correctly the second linearly independent solution of the linear equation. Similarly as previously we show that \( z_2(\xi) \in B_{\lambda_{-1} - \lambda_0}^2 \) for \( \sigma_c < \sigma < \sigma^* \).

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