Anizotropowa Turbulencja w Nadciekłym Helu i Dynamika Wirów Dyskretnych

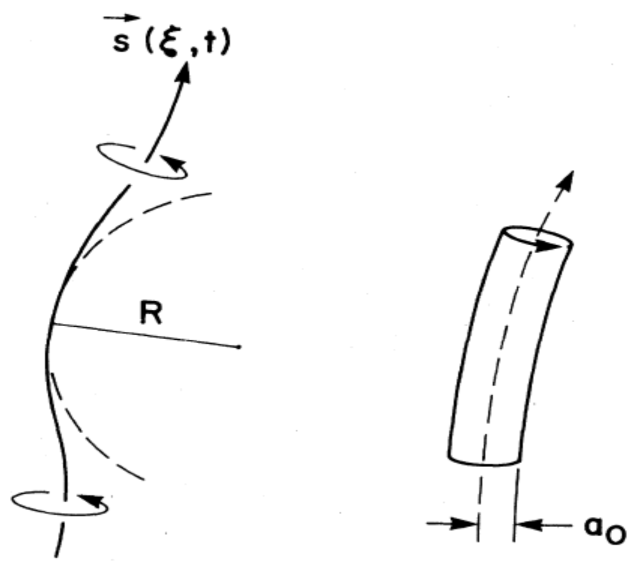
Tomasz Lipniacki

He II:

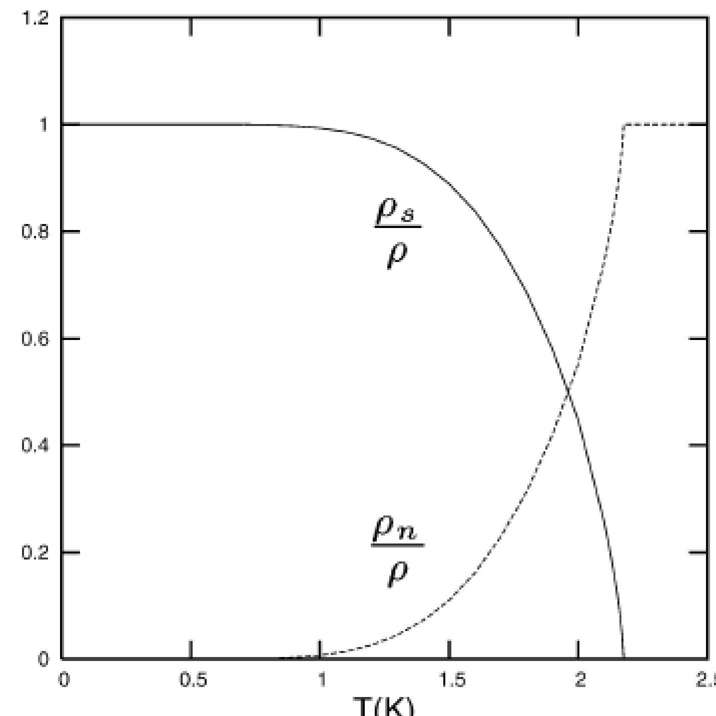
-- normal component,

-- superfluid component,

-- superfluid vortices.



vortex line



Relative proportion of normal fluid and superfluid as a function of temperature.

Localized Induction Approximation (LIA)

$$\boldsymbol{V}_i(\boldsymbol{s}(\xi, t)) = \frac{\kappa}{4\pi} \int \frac{(\boldsymbol{s}(\xi, t) - \boldsymbol{s}(\bar{\xi}, t)) \times \boldsymbol{s}'(\bar{\xi}, t)}{|\boldsymbol{s}(\xi, t) - \boldsymbol{s}(\bar{\xi}, t)|^3} \mathrm{d}\bar{\xi}$$

$$\dot{\mathbf{s}}(\xi, t) = \beta \mathbf{s}' \times \mathbf{s}'' = \beta c \mathbf{b}$$

c - curvature

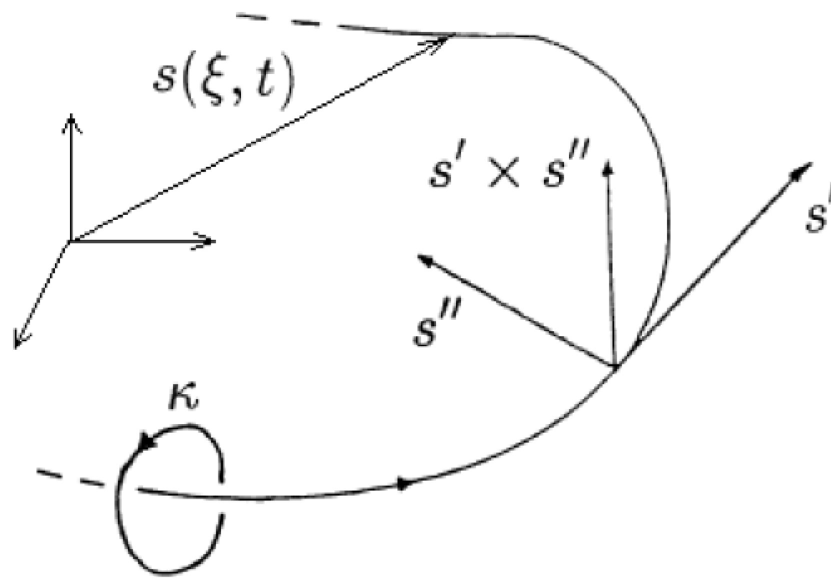
. τ torsion

where $\dot{\mathbf{s}} = \frac{d\mathbf{s}}{dt}$ and $\mathbf{s}' = \frac{d\mathbf{s}}{d\xi}$,

$\mathbf{s}' = \mathbf{t}$ - tangent,

$\mathbf{s}'' = c\mathbf{n}$ - normal,

$\mathbf{s}' \times \mathbf{s}'' = c\mathbf{b}$ - binormal



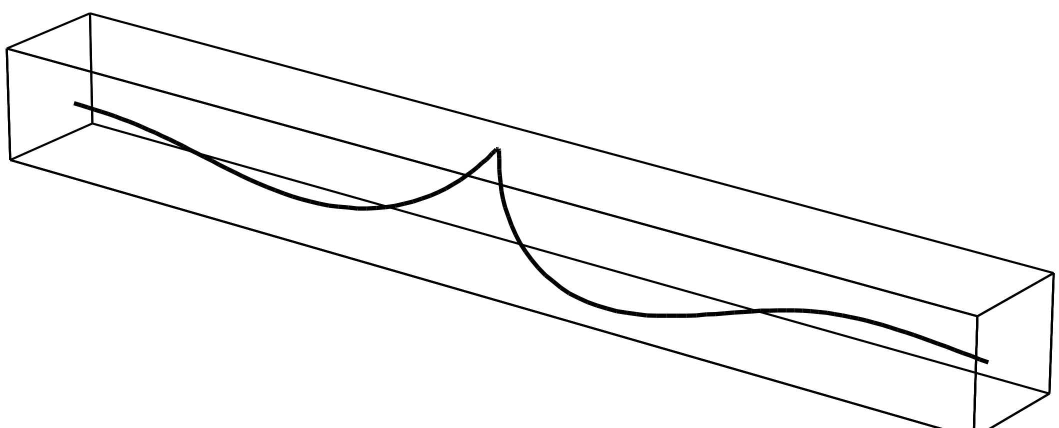
Hasimoto soliton on ideal vortex (1972)

$$\dot{\mathbf{s}}(\xi, t) = \beta \mathbf{s}' \times \mathbf{s}''$$

Totally integrable,
equivalent to Non-linear Schrödinger equation

$$c = c_0 \operatorname{sech}\left(c_0(\xi - Vt)/2\right), \quad \tau = \tau_0$$

$$V = 2\beta\tau_0, \quad \omega = \beta\left(c_0^2/4 - \tau_0^2\right)$$



Similar solutions for quantum vortices ?

$$\dot{\mathbf{s}}(\xi,t)=\beta\mathbf{s}'\times\mathbf{s}''+\alpha\mathbf{s}''=c(\beta\mathbf{b}+\alpha\mathbf{n})$$

Quantum vortex shrinks:

$$\dot{\xi}=-\alpha\int_0^{\xi}c^2d\bar{\xi}$$

$$\frac{\partial c}{\partial t}=-\beta\left(2c'+c\tau'\right)+\alpha\left(c''-c\tau^2+c^3\right)+\alpha c'\int_0^{\xi}c^2d\bar{\xi}$$

$$\frac{\partial \tau}{\partial t}=-\beta\left(\frac{c''-c\tau^2}{c}+\frac{c^2}{2}\right)+\alpha\left[\left(\frac{2c'\tau+c\tau'}{c}\right)'+2\tau c^2\right]+\alpha\tau'\int_0^{\xi}c^2d\bar{\xi}$$

Frenet Seret equations $\mathbf{t}'=c\mathbf{n},\quad \mathbf{n}'=-c\mathbf{t}+\tau\mathbf{b},\quad \mathbf{b}'=-\tau\mathbf{n}$

Shape-preserving solutions

$$t \to \lambda^2 t, \qquad \xi \to \lambda \xi,$$

$$c = \frac{1}{\sqrt{t}} K\left(\frac{\xi}{\sqrt{t}}\right), \qquad \tau = \frac{1}{\sqrt{t}} T\left(\frac{\xi}{\sqrt{t}}\right), \qquad l = \frac{\xi}{\sqrt{t}}.$$

$$\mathbf{s}(\xi, t) = \sqrt{t}\,\Omega(t)\,\mathbf{S}\left(\frac{\xi}{\sqrt{t}}\right)$$

+ solutions with decreasing scale

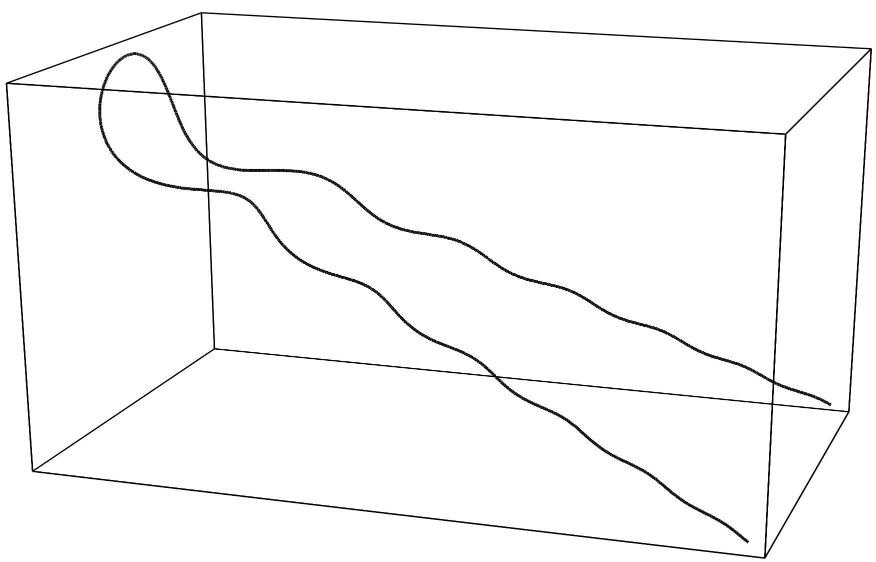
$$-\beta\left(2K'T + KT'\right) + \alpha\left(K'' - KT^2 + K^3\right) + \alpha K' \int_0^l K^2 d\bar{l} = \frac{K + lK'}{2}$$

$$\beta\left(\frac{K'' - KT^2}{K} + \frac{K^2}{2}\right) + \alpha\left[\left(\frac{2K'T + KT'}{K}\right) + 2TK^2\right] + \alpha T' \int_0^l K^2 d\bar{l} = \frac{T + lT'}{2}$$

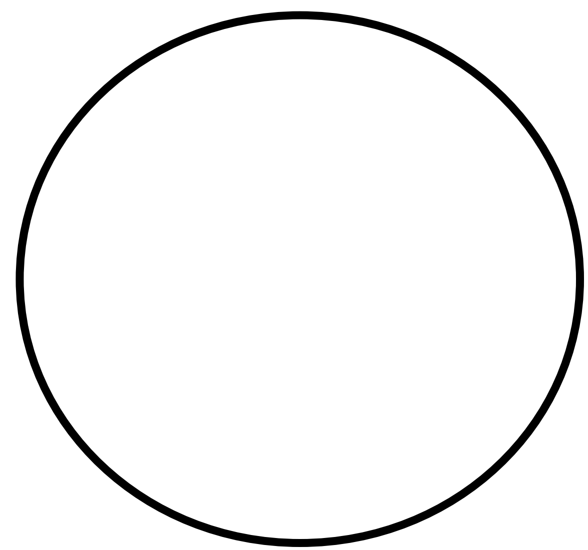
Analytic result: shape-preserving solution

In the case when transformation is a pure homothety we get analytic solution in implicit form:

$$l(K) = \mp\sqrt{\frac{\alpha^2+\beta^2}{\alpha}} \int_{K_0}^{K} \frac{d\overline{K}}{\overline{K}\sqrt{\ln(K_0/\overline{K}) + p\alpha(K_0^2 - \overline{K}^2)}}, \qquad \mathrm{T} = \frac{\beta K'}{\alpha K}$$

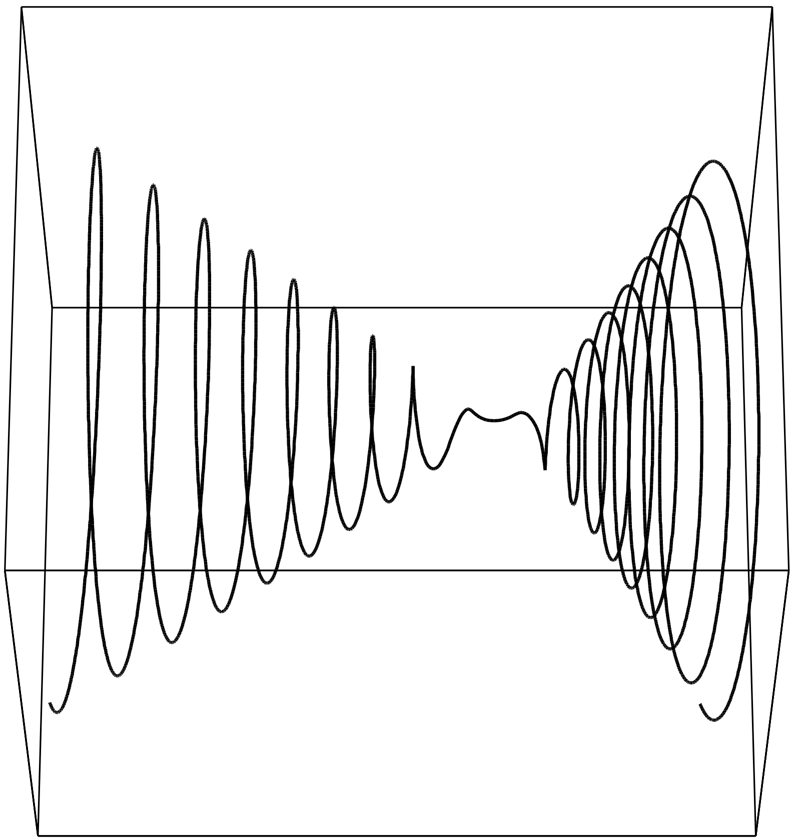


$p = 1$ growing scale

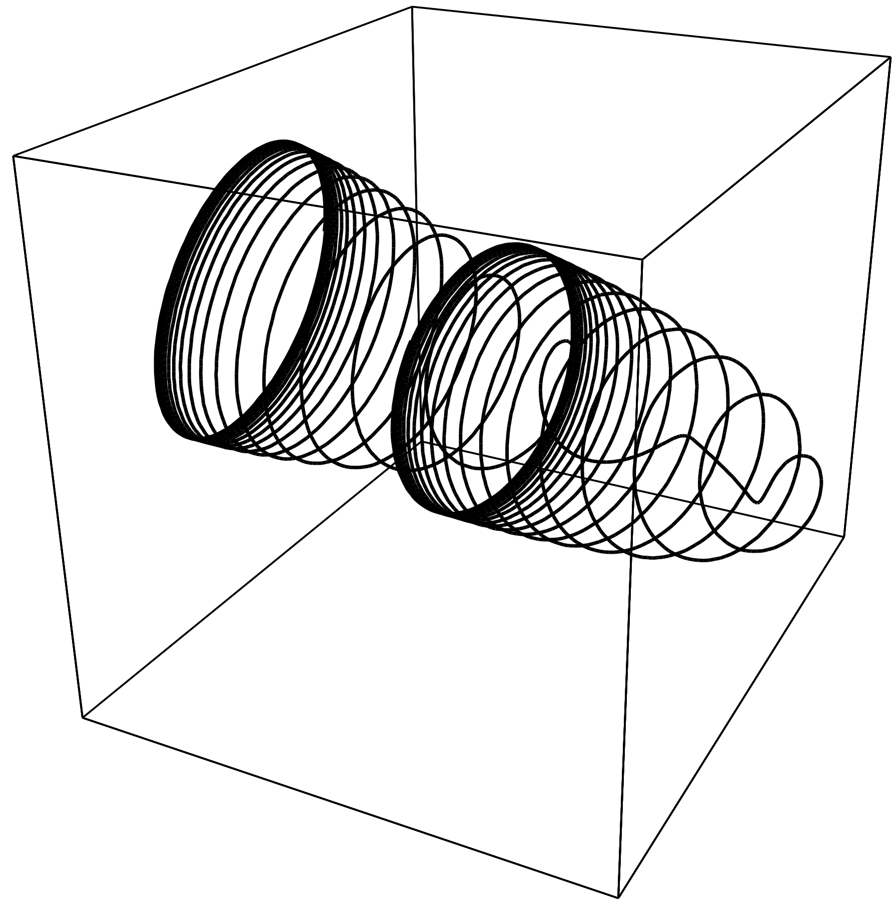


$p = -1$ decreasing scale, $\mathrm{T} = 0$

Shape preserving solution: general case



$p = 1$ growing scale



$p = -1$ decreasing scale

Logarithmic spirals on cones

Results in general case

I. 4-parametric class of solutions, determined by initial condition

$$K(0), \quad \mathrm{K}'(0), \quad T(0), \quad T'(0)$$

II. Each solution corresponds to a specific similarity transformation

III. The asymptotics for $l \to \infty$

is given the initial condition of the original problem (with time).

Quasi-static solutions

$$\mathbf{s}(\xi,t) = \mathbf{W}(t) + \Omega(t)\,\mathbf{s}(\xi,0)$$

$$c(\xi,t) = c(\xi), \qquad \tau(\xi,t) = \tau(\xi)$$

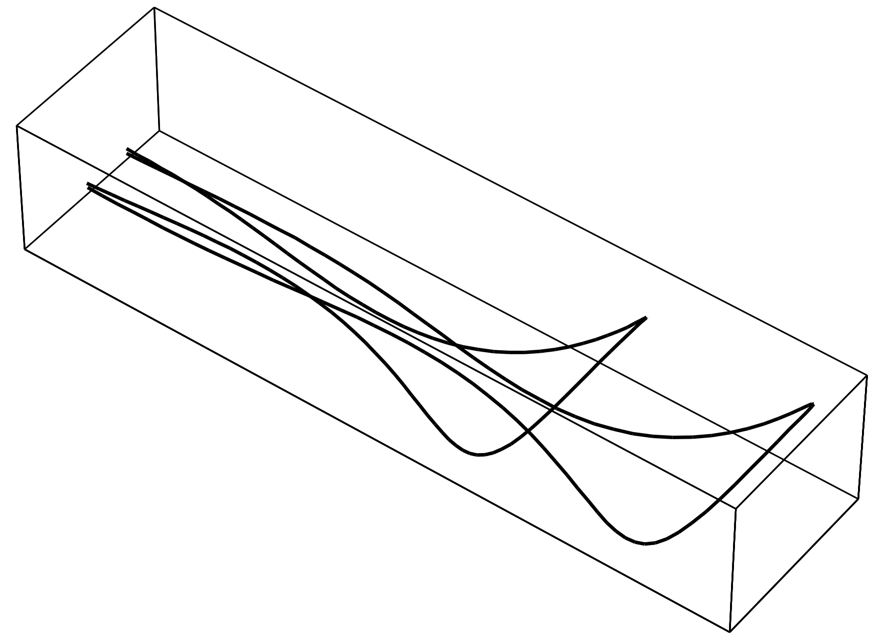
$$-\beta\left(2c' + c\tau'\right) + \alpha\left(c'' - c\tau^2 + c^3\right) + \alpha c' \int_0^{\xi} c^2 \, d\bar{\xi} = 0$$

$$-\beta\left(\frac{c'' - c\tau^2}{c} + \frac{c^2}{2}\right) + \alpha\left[\left(\frac{2c'\tau + c\tau'}{c}\right)' + 2\tau c^2\right] + \alpha\tau' \int_0^{\xi} c^2 \, d\bar{\xi} = 0$$

Analytic result: quasi-static solution

In the case when transformation is a pure translation we get analytic solution:

$$c = c_0 \operatorname{sech}(A\xi), \qquad \tau = B\tanh(A\xi)$$

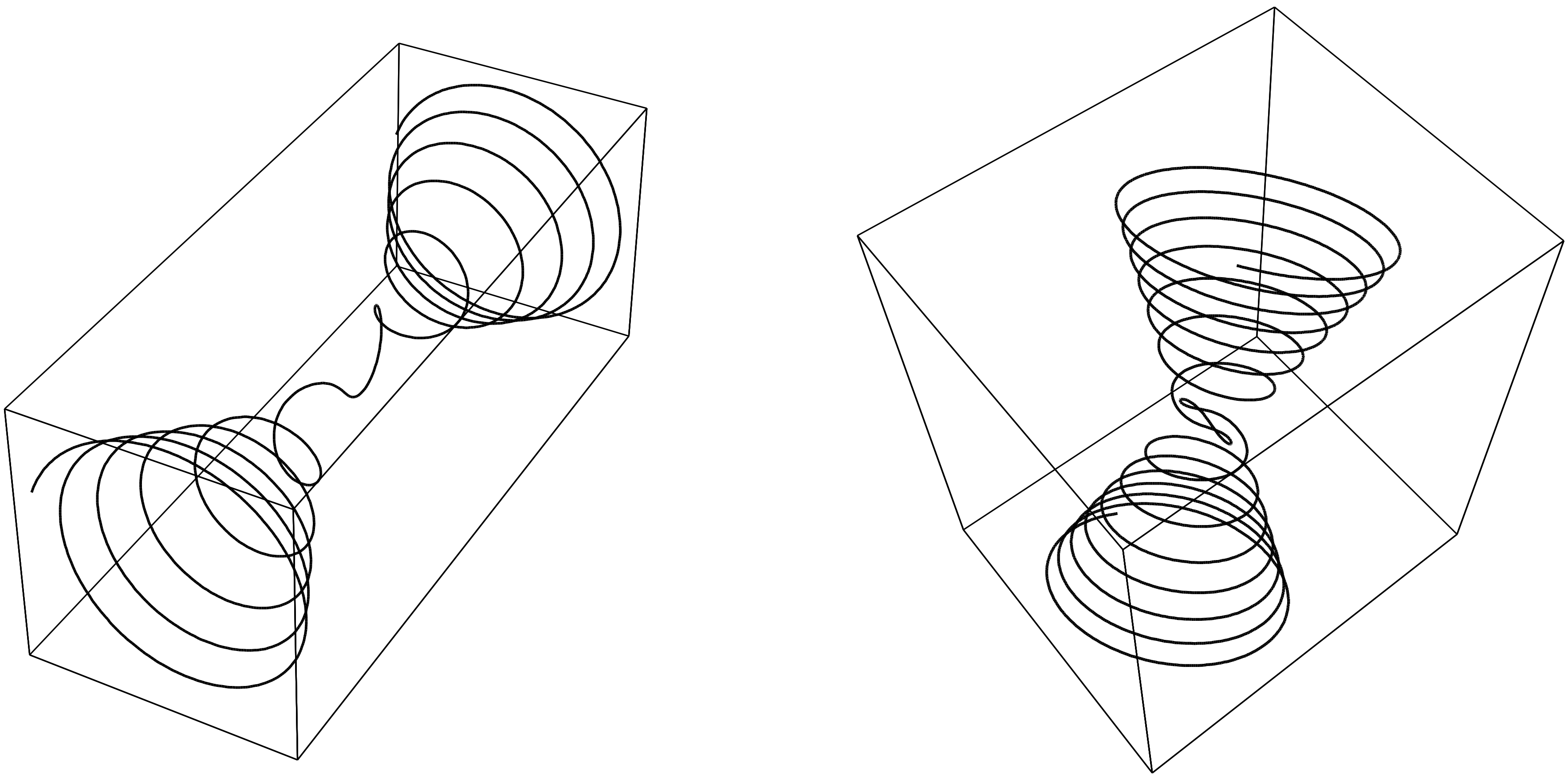


$$A = \frac{\alpha c_0}{\sqrt{\alpha^2+\beta^2}}, \qquad B = \frac{\beta c_0}{\sqrt{\alpha^2+\beta^2}}$$

$$\mathbf{s}(\xi,t) = \left(\int_{-\infty}^{\xi} R\cos(\mathrm{q}\bar{\xi})\, d\bar{\xi}, \int_{-\infty}^{\xi} R\sin(\mathrm{q}\bar{\xi})\, d\bar{\xi}, tc_0\sqrt{\alpha^2+\beta^2} + \ln(\cosh(A\xi))/A \right)$$

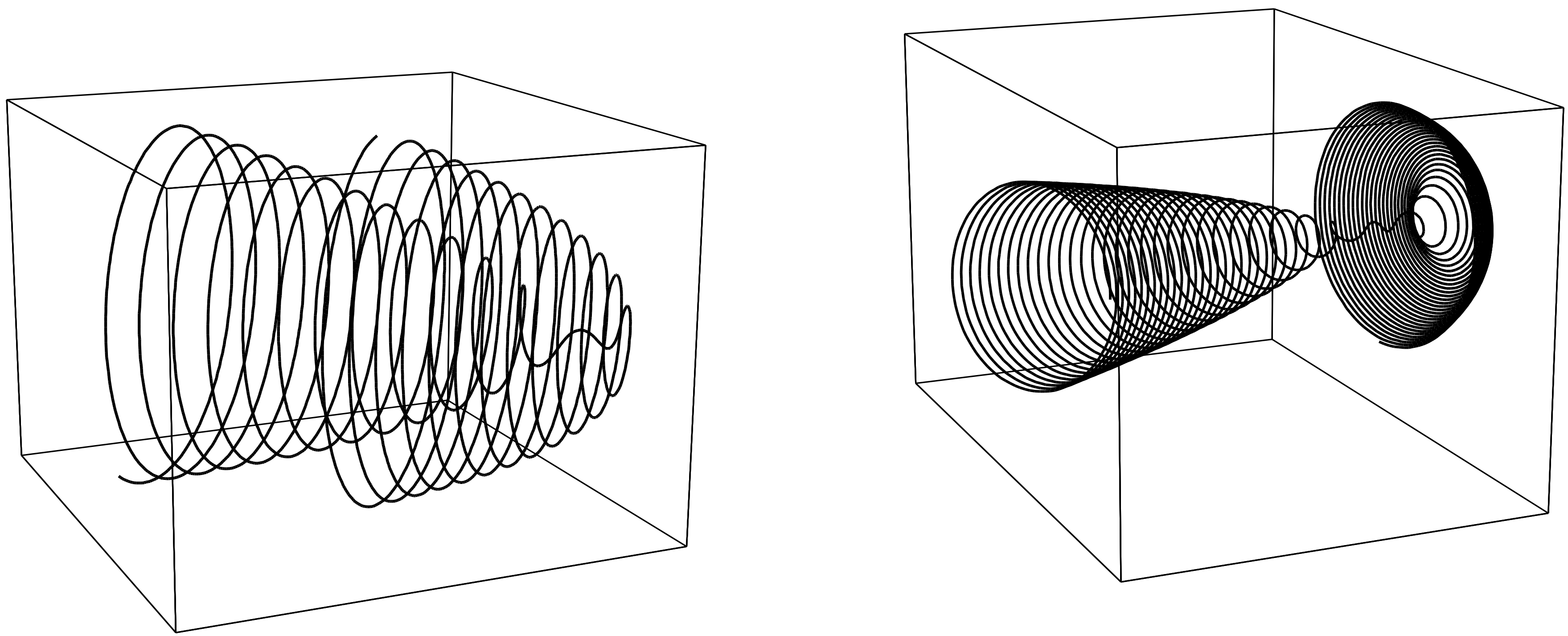
where $\quad R = \operatorname{sech}(A\xi), \qquad \mathrm{q} = \sqrt{c_0^2 - A^2}$

Quasi-static solution: pure rotation



Asymptotically vortex wraps over cones

Quasi-static solution: general case



Asymptotically vortex wraps over paraboloids

Results in general case

I. 4-parametric class of solutions, determined by initial condition

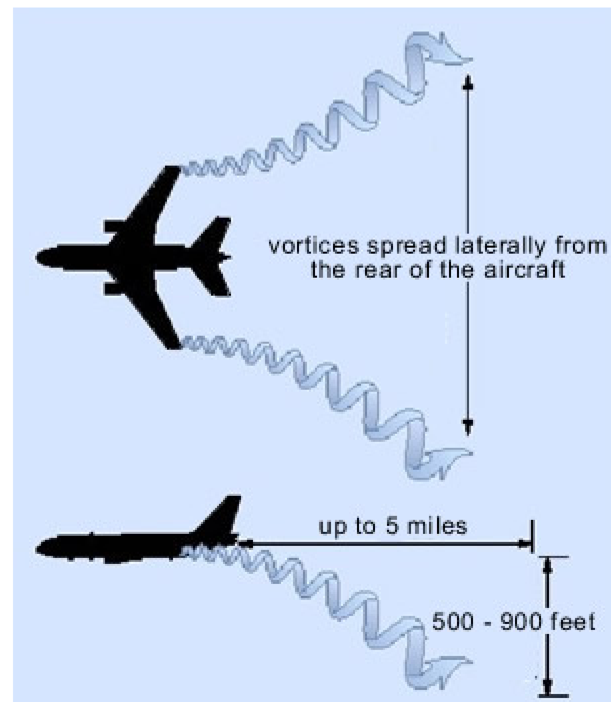
$$c(0), \quad c'(0), \quad \tau(0), \quad \tau'(0)$$

II. Each solution corresponds to a specific isometric transformation Γ (t) related analytically to initial condition.

III. The asymptotic for $\xi \to \infty$

is related analytically via transformation Γ(t) to initial condition.

Wing tip vortices



Macroscopic description

-- Euler equation for superfluid component

-- Navier-Stokes equation for normal component

$$\operatorname{div} \mathbf{V}_n = \operatorname{div} \mathbf{V}_s = 0,$$

$$\rho_s \left(\frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}_s \cdot \nabla \mathbf{V}_s \right) = -\nabla p_s - \mathbf{F}_{ns},$$

$$\rho_n \left(\frac{\partial \mathbf{V}_n}{\partial t} + \mathbf{V}_n \cdot \nabla \mathbf{V}_n \right) = -\nabla p_n + \mathbf{F}_{ns} + \eta \Delta^2 \mathbf{V}_n,$$

Coupled by mutual friction force $F_{ns} \sim V_{ns} L$

where $V_{ns} = V_n - V_s$

L is superfluid vortex line density.

Microscopic description of tangle superfluid vortices

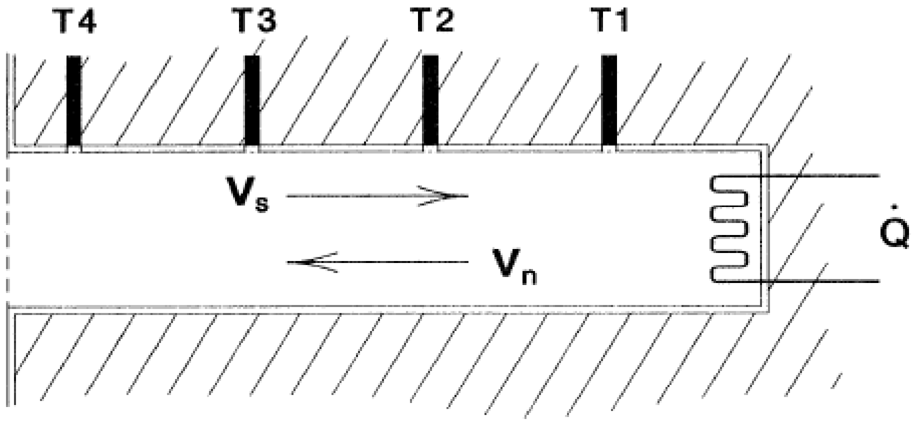
Aim

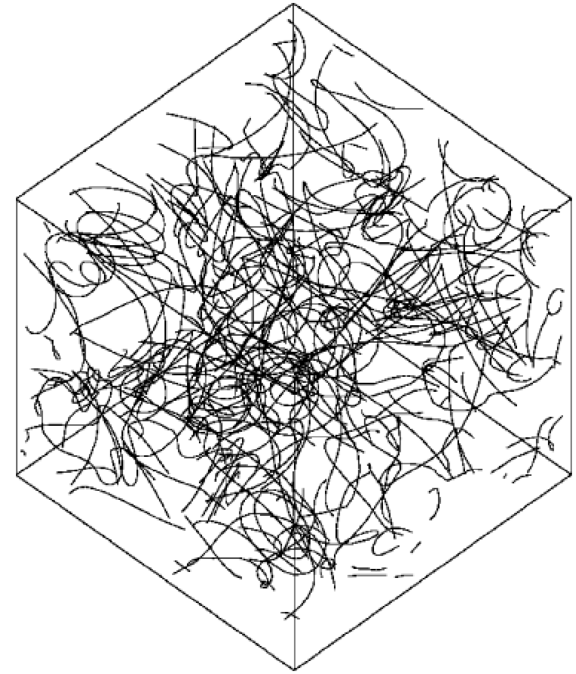
Assuming that quantum tangle is “close” to statistical equilibrium derive equations for quantum line length density L and anisotropy parameter of the tangle.

Two cases

I – rapidly changing counterflow, but uniform in space

II – slowly changing counterflow, not uniform in space





$$V_n = \frac{dQ}{dt} / A\rho ST$$

Model of quantum tangle

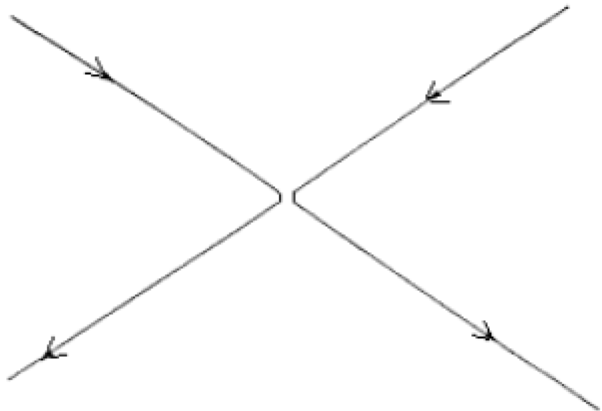
“Particles” = segments of vortex line of length equal to characteristic radius of curvature.

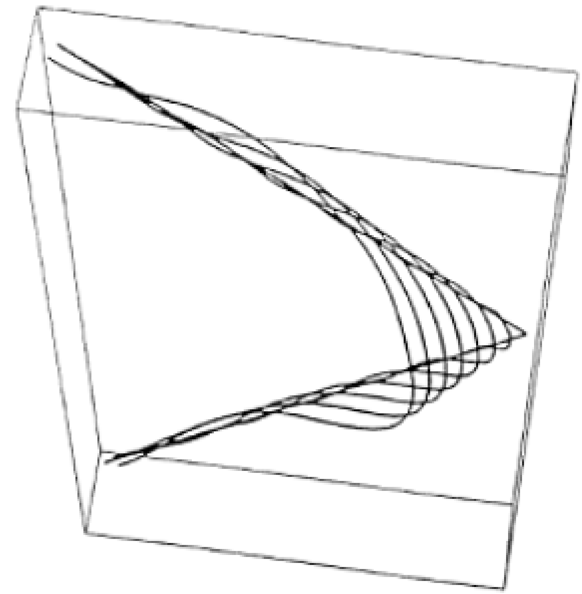
Particles are characterized by their tangent **t**, normal **n**, and binormal **b** vectors.

Velocity of each particle is proportional to its binormal (+collective motion).

Interactions of particles = reconnections.

Reconnections





1. Lines lost their identity:
two line segments are replaced by two new line segments

2. Introduce new curvature to the system

Motion of vortex line in the presence of counterflow

$$V_{ns} = V_n - V_s$$

$$\dot{\mathbf{s}} = \beta(\mathbf{s}' \times \mathbf{s}'' + \alpha \mathbf{s}'') + \alpha(\mathbf{s}' \times \mathbf{V}_{ns}).$$

Evolution of its line-length $l = \int d\xi$ is

$$\frac{\partial l}{\partial t} = \int (\alpha \mathbf{V}_{ns} \cdot (\mathbf{s}' \times \mathbf{s}'') - \alpha\beta |\mathbf{s}''|^2) d\xi.$$

Evolution of line length density $L = \frac{1}{\Omega}\int d\xi$

$$\frac{dL}{dt} = \alpha \; L^{3/2} c_1 \mathbf{I} \cdot \mathbf{V}_{ns} - \beta \alpha c_2^2 \; L^2 .$$

where $c_1 = \frac{1}{\Omega L^{3/2}} \int |\mathbf{s}''| d\xi$ average curvature

$c_2^2 = \frac{1}{\Omega L^2} \int |\mathbf{s}''|^2 d\xi$ average curvature squared

$\mathbf{I} = \frac{\langle \mathbf{s}' \times \mathbf{s}'' \rangle}{\langle |\mathbf{s}''| \rangle}$ Average binormal vector (normalized)

Rapidly changing counterflow

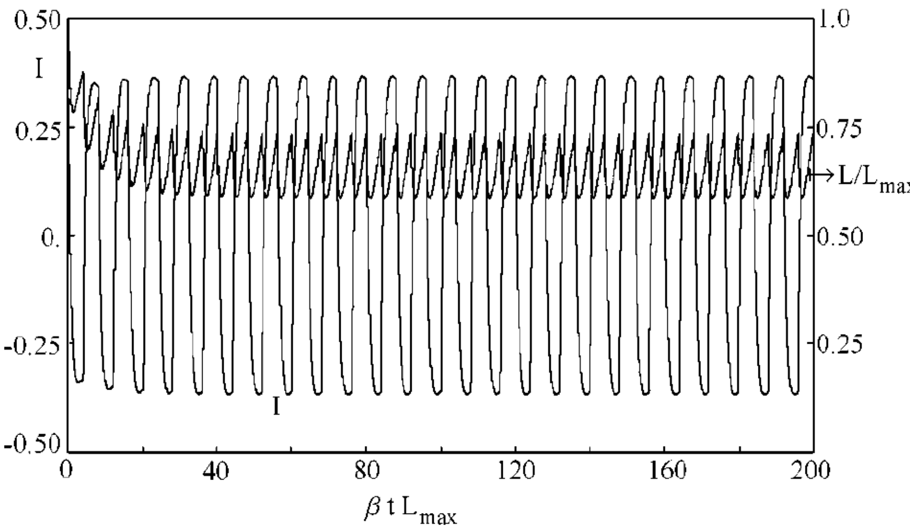
$$\frac{d\mathbf{I}}{dt} = \left[\frac{d\mathbf{I}}{dt}\right]_{gen} - \left[\frac{d\mathbf{I}}{dt}\right]_{dec}$$

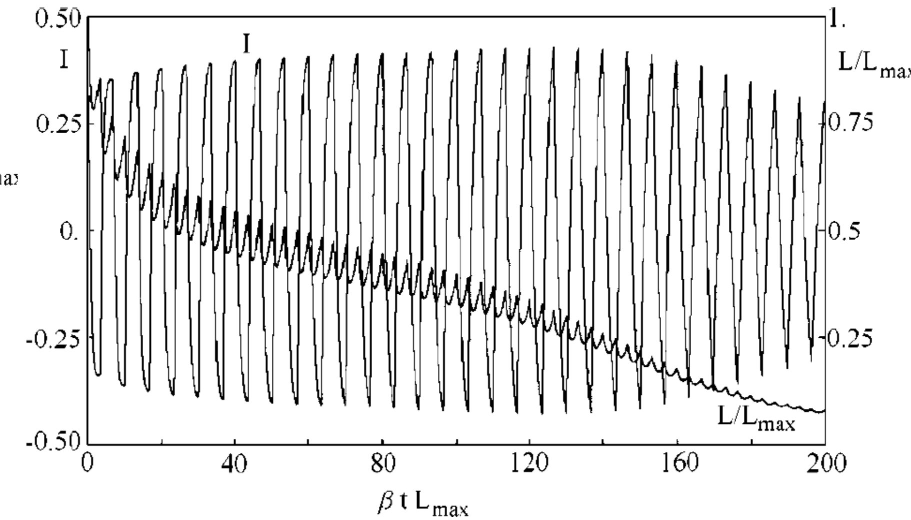
Generation term: polarization of a tangle by counterflow

Decaying term: relaxation due to reconnections

$$\frac{d\mathbf{I}}{dt} = c_1 \quad \alpha \quad L^{1/2}\left(\frac{2}{3}\mathbf{V}_{ns}(1 - \mathbf{I}\cdot\mathbf{I}) + \mathbf{I}\times(\mathbf{V}_{ns}\times\mathbf{I})\right) - d\beta L\mathbf{I}$$

Model prediction: there exists critical frequency of counterflow above which tangle may not be sustained





$$\omega = 0.78\,\beta L_{\max}$$

$$\omega = 0.94\,\beta L_{\max}$$

Ladik Skrbek experiment

Quantum turbulence with high net macroscopic vorticity

We assume $I \cdot V_{ns} = I_0 |V_{ns}|$

Let $\mathbf{q} = \dfrac{\boldsymbol{\omega}_s}{\kappa L} = \dfrac{\int \mathbf{s}' \, d\xi}{\int d\xi}$ Anisotropy of the tangent s'

Vinen type equation for anisotropic turbulence

$$\frac{\partial L}{\partial t} = \alpha I_0 c_1 (1 - q^2) |\mathbf{V}_{ns}| L^{3/2} - \beta \alpha c_2{}^2 (1 - q^2)^2 L^2 - \operatorname{div}(L \mathbf{V}_L)$$

Helium II dynamical equations

$$\operatorname{div} \mathbf{V}_n = \operatorname{div} \mathbf{V}_s = 0,$$

$$\rho_s \left(\frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}_s \cdot \nabla \mathbf{V}_s \right) = -\nabla p_s - \mathbf{F}_{ns},$$

$$\rho_n \left(\frac{\partial \mathbf{V}_n}{\partial t} + \mathbf{V}_n \cdot \nabla \mathbf{V}_n \right) = -\nabla p_n + \mathbf{F}_{ns} + \eta \Delta^2 \mathbf{V}_n,$$

with

$$\mathbf{F}_{ns} = \alpha \kappa \rho_s L \left(\mathbf{q} \times (\mathbf{q} \times \mathbf{V}_{ns}) - \frac{2}{3} \mathbf{V}_{ns} (1 - q^2) + \beta I_0 c_1 (1 - q^2) L^{1/2} \widehat{\mathbf{V}}_{ns} \right),$$

$$\frac{\partial L}{\partial t} = \alpha I_0 c_1 (1 - q^2) |\mathbf{V}_{ns}| L^{3/2} - \beta \alpha c_2{}^2 \, (1 - q^2)^2 L^2 - \operatorname{div}(L \mathbf{V}_L),$$

where

$$\mathbf{V}_L = \mathbf{V}_s + \alpha \mathbf{q} \times \mathbf{V}_{ns} + \beta \alpha I_0 c_1 (1 - q^2) L^{1/2} \widehat{\mathbf{V}}_{ns},$$

$$\mathbf{V}_{ns} = \mathbf{V}_n - \mathbf{V}_s,$$

$$\mathbf{q} = \frac{\nabla \times \mathbf{V}_s}{\kappa L}, \qquad q = |\mathbf{q}|,$$

Specific cases: stationary rotating turbulence

Pure heat driven turbulence

$$L_H = V_{ns}^2 \left(\frac{c_1 \; I_0}{\beta \; c_2^2} \right)^2$$

Pure rotation

$$L_\omega = \frac{\omega_s}{\kappa} = \frac{2\Omega}{\kappa}$$

„Sum”

$$L = L_H + \frac{2L_\omega}{L} - \frac{L_\omega^4}{L^3}$$

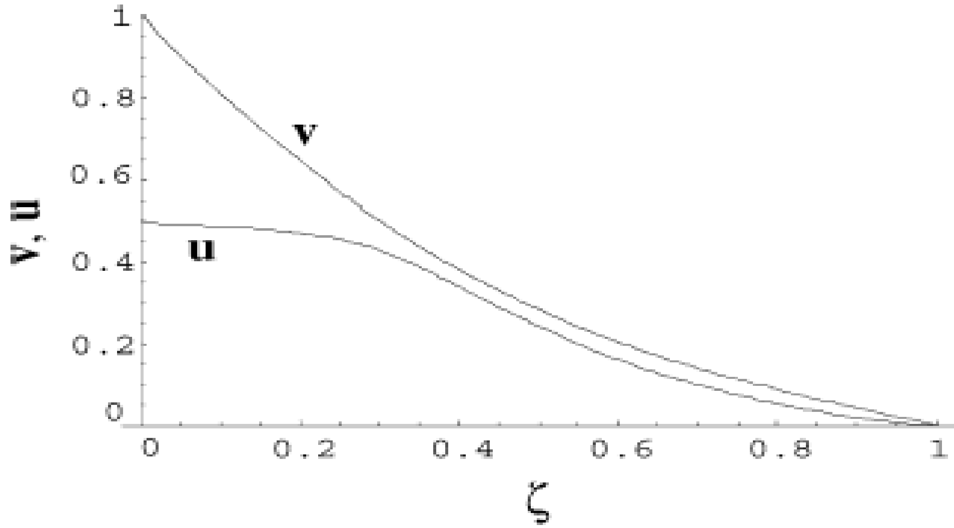
Slow rotation

$$L = L_H \left(1 + 2 \left(\frac{L_\omega}{L_H} \right)^2 \right)$$

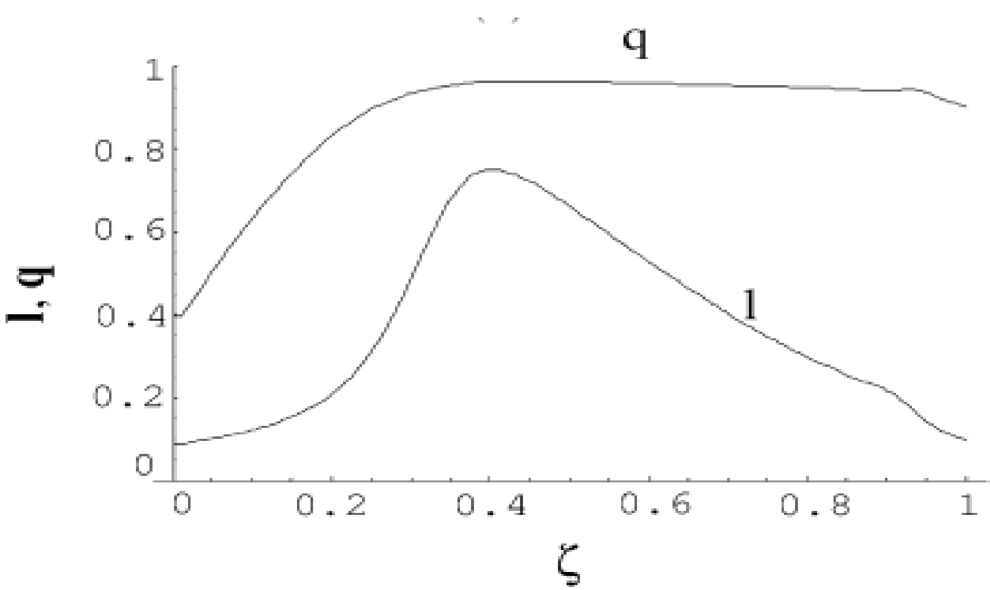
Fast rotation

$$L = L_\omega \left(1 + \frac{1}{2} \left(\frac{L_H}{L_\omega} \right)^{1/2} \right)$$

Plane Couette flow



V- normal velocity
U-superfluid velocity



q- anisotropy
l-line length density

Summary

III. Novel (some analytical) solutions of quantum vortex motion

II. System of equations describing vortex tangle Evolution in the case of rapidly changing counterflow

III. Helium II dynamical equations in the case of anisotropic quantum turbulence.

Dissertation

[H1] T. Lipniacki, Quasi-static solution for quantum vortex motion under the localized induction approximation *J. Fluid Mech.* **477**: 321–337 (2003).

[H2] T. Lipniacki, Shape-preserving solutions for quantum vortex motion under localized induction approximation *Phys. of Fluids* **15** (6): 1381–1395 (2003).

[H3] T. Lipniacki, Evolution of line-length density and anisotropy of quantum tangle, *Phys. Rev. B* **64**: 214516–1–9 (2001).

[H4] T. Lipniacki, Dynamics of superfluid He: Two-scale approach, *European Journal of Mechanics, B/Fluids* **25**: 435–458 (2006).

Supporting material

[T1] T. Lipniacki, Evolution of quantum vortices following reconnection, *European Journal of Mechanics, B/Fluids* **19**: 361–378 (2000).

[T2] T. Lipniacki, On quantum turbulence in superfluid Helium, *Arch. Mech.* **53** (1): 23–43 (2001).

Non-linear Schrodinger equation (Gross, Pitaevskii)

For $\Psi = A\exp(i\Theta)$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\left(\frac{\hbar^2}{2m}\right)\nabla^2\Psi + \left(V_0\Psi\Psi^* - E_b\right)\Psi$$

Madelung Transformation

$V_s = (\hbar / m)\nabla\Theta$ superfluid velocity

$\rho_s = mA^2 = m\Psi\Psi^*$ superfluid density

Modified Euler equation

$$\rho_s \left(\frac{\partial V_s}{\partial t} + V_s \nabla V_s \right)_j = \frac{\partial P}{\partial x_j} + \frac{\partial \Sigma_{jk}}{\partial x_k}$$

With

pressure $$P = \frac{V_0 \rho_s^2}{2m^2}$$

Quantum stress $$\Sigma_{jk} = \frac{\hbar^2}{4m^2} \rho_s \frac{\partial^2 \ln(\rho_s)}{\partial x_j \partial x_k}$$

Mixed turbulence

Kolgomorov spectrum for each fluid

$$E(k) = C\frac{\varepsilon^{2/3}}{k^{5/3}}$$

ε - energy flux per unit mass