Dilute suspensions near walls

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Motivations

Modeling of suspensions at small scale has various applications :

- separation techniques in analytical chemistry
- chemical engineering processes
- various micro-devices
- biological flows

At small scales :

• Reynolds number is low ; hydrodynamic interactions have a long range.

• Walls effects are more important relative to bulk.

Thus a precise account of particle-wall hydrodynamic interactions is important.

Assumptions and equations

Consider here dilute suspensions (volume fraction $\ll 1$) of solid and spherical particles.
Particles are small so that Reynolds number is low : Re$\ll 1$.
Expanding Navier-Stokes equations :

$$\mathrm{Re}\frac{\partial \mathbf{v}}{\partial t} + \mathrm{Re}\,\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = -\boldsymbol{\nabla}p + \nabla^2\mathbf{v}, \quad \boldsymbol{\nabla}\cdot\mathbf{v} = 0$$

gives

- 1st order : Stokes (creeping flow) equations, which are quasi-steady :

$$-\boldsymbol{\nabla}p + \nabla^2\mathbf{v} = 0, \quad \boldsymbol{\nabla}\cdot\mathbf{v} = 0$$

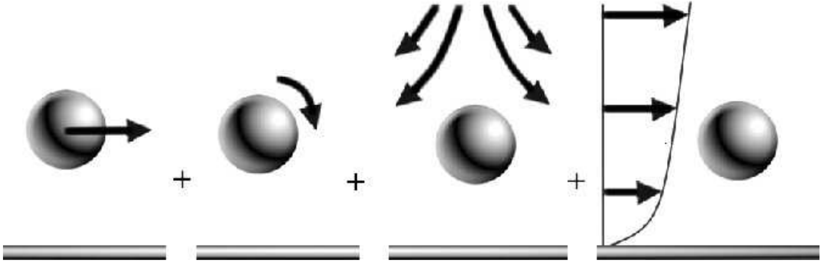
- 2nd order : small effects of inertia (both steady and unsteady problems will be considered).

Outline

Outline

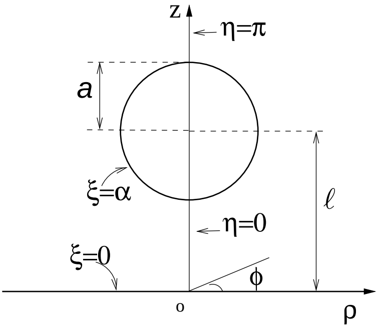
Superposition from linearity of Stokes equations

translation + rotation + unperturbed flow fields (perpendicular + parallel to wall) and sphere held fixed.



Particle fixed in an unperturbed flow field near a wall

- Perturbed flow = unperturbed flow + perturbation :
  $\mathbf{u} = \mathbf{u}_\infty + \mathbf{v}$
- Stokes for the perturbation flow : $\nabla^2 \mathbf{v} - \boldsymbol{\nabla} p = 0, \quad \boldsymbol{\nabla} \cdot \mathbf{v} = 0$
- Boundary conditions : $\mathbf{v} = -u^\infty \mathbf{e}_x (\text{or } -u^\infty \mathbf{e}_z)$ on sphere, $\mathbf{v} = 0$ on wall and at infinity.



Use bispherical coordinates :

$$\rho = c \frac{\sin \eta}{\cosh \xi - \mu}, \quad z = c \frac{\sinh \xi}{\cosh \xi - \mu}$$

$$c = \sqrt{\ell^2 - a^2}$$

$$(\eta, \xi, \phi) \in [0, \pi] \times [0, \alpha] \times [0, 2\pi]$$

Axisymmetric unperturbed flow normal to wall

L. Pasol, M. Chaoui, S. Yahiaoui, and F. Feuillebois. Phys. Fluids, 17(7) :073602, 2005.

Unperturbed flow field stream function taken as polynomial should be :

$$\psi = \sum_M \rho^2 c_M z^M$$

Consider here $M \leq 3$. $M = 0$ is a uniform flow. For $M = 1$ the no-slip condition on wall cannot be applied. Then :

degree of polynomial in z	u_ρ^∞	u_z^∞	p^∞
$M = 2$	$S_2 \rho z$	$-S_2 z^2$	$-2S_2 \mu_f z$
$M = 3$	$3S_3 \rho z^2$	$-2S_3 z^3$	$-3S_3 \mu_f (2z^2 - \rho^2)$

here in dimensional form with μ_f the fluid viscosity.

Solution for perturbation in bispherical coordinates

$$p = \frac{Q_0}{c}, \quad v_\rho = \frac{\rho Q_0}{2c} + V_1, \quad v_z = \frac{z Q_0}{2c} + W_0, \quad \text{where :}$$

$$Q_0 = c^M (\cosh\xi - \mu)^{1/2} \sum_{n=0}^{\infty} [A_n, B_n]\, P_n(\mu)$$

$$V_1 = c^M (\cosh\xi - \mu)^{1/2} \sin\eta \sum_{n=1}^{\infty} [C_n, D_n]\, P_n'(\mu)$$

$$W_0 = c^M (\cosh\xi - \mu)^{1/2} \sum_{n=0}^{\infty} [0, F_n]\, P_n(\mu)$$

with $\mu = \cos\eta$ and with shorthand notation :

$$[A_n, B_n] = A_n \cosh\left(n + \frac{1}{2}\right)\xi + B_n \sinh\left(n + \frac{1}{2}\right)\xi$$

Determination of coefficients in the series

- Boundary conditions $\Rightarrow$ Coefficients A_n, B_n, C_n, D_n are expressed in term of the F_n's.
- Continuity equation gives :

$$f_{n,n-1}(\alpha)F_{n-1} + f_{n,n}(\alpha)F_n + f_{n,n+1}(\alpha)F_{n+1} = b_n^M(\alpha), \quad n \geq 0,$$

$$(M = 0, 2, 3); \quad f_{0,-1} = 0; \quad F_0 \text{ unkown}; F_n \to 0 \text{ for } \quad n \to \infty$$

where the f's and b's are known in term of the dimensionless sphere to wall distance $\ell/a = \cosh\alpha$.

- Solve an infinite linear system for the F_n's using an iterative technique (Bhatt & O'Neill 1991, Chaoui & Feuillebois 2003, Pasol et al 2005).

Determination of coefficients F_n by iteration

The coefficients F_n are written in the form :

$$F_n = t_n + F_0 u_n \qquad (n \geq 1)$$

in which the t_n's and u_n's are defined by iteration :

$$t_1 = 0 \quad ; \quad f_{n,n-1}t_{n-1} + f_{n,n}t_n + f_{n,n+1}t_{n+1} = b_n \quad (n \geq 1)$$
$$u_1 = 1 \quad ; \quad f_{n,n-1}u_{n-1} + f_{n,n}u_n + f_{n,n+1}u_{n+1} = 0 \quad (n \geq 1)$$

$F_n \to 0$ for $n \to \infty$ gives the result : $F_0 = \lim_{n\to\infty} -\frac{t_n}{u_n}$.

Other F_n's are calculated by iteration :

$$F_1 = \frac{b_0}{f_{0,1}} - \frac{f_{0,0}}{f_{0,1}}F_0$$
$$\dots$$
$$F_n = \frac{b_{n-1}}{f_{n-1,n}} - \frac{f_{n-1,n-1}}{f_{n-1,n}}F_{n-1} - \frac{f_{n-1,n-2}}{f_{n-1,n}}F_{n-2}.$$

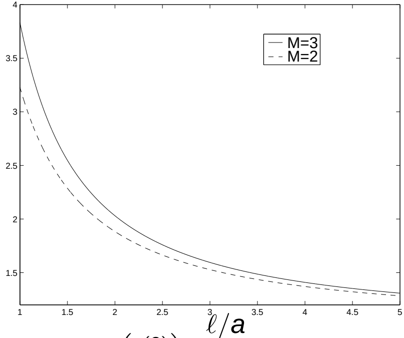
Calculation of coefficients and results

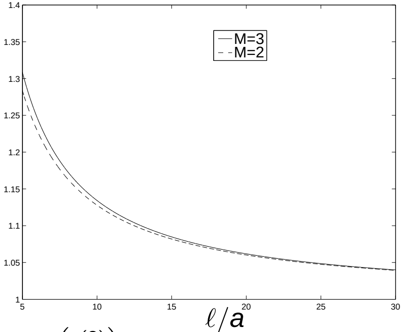
- Explicit solution, thus use Maple for “infinite” precision, allowing the number of digits to grow upon request.
- Choosing a 10^{-15} precision requires typically 35 digits.
- Results for small gaps between sphere and wall, viz in lubrication region, require to calculate typically thousands of terms in the series. This is possible here thanks to the present iteration technique.

Force acting on the particle

$\mathbf{F}^{(M)} = 6\pi\mu_f a\,(u_z^\infty)_C \left(f_{zz}^{(M)}\right)_C \mathbf{e}_z$, where $(u_z^\infty)_C$ is the unperturbed fluid velocity at the sphere center and :

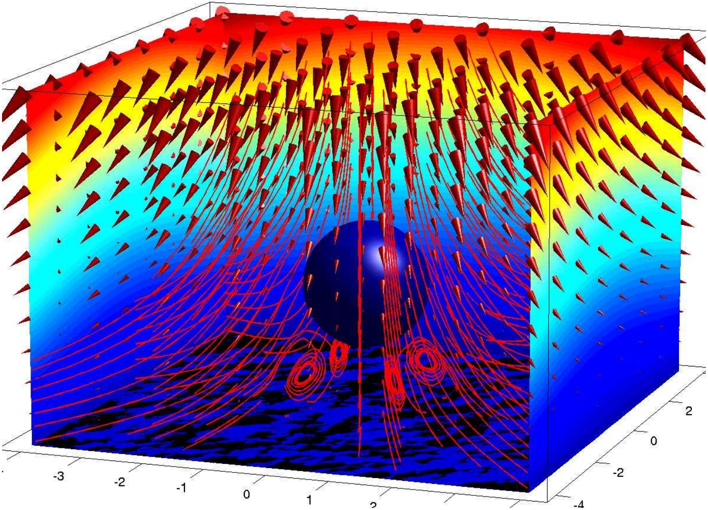
$$\left(f_{zz}^{(M)}\right)_C = -\frac{2\sqrt{2}}{3(M-1)} \tanh^M \alpha \, \sinh \alpha \sum_{n=0}^{\infty} F_n, \quad (M = 2, 3)$$



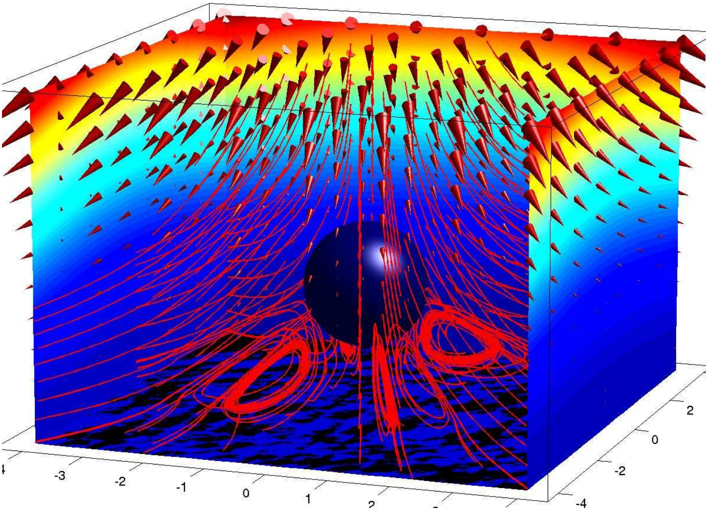


On wall : $\left(f_{zz}^{(2)}\right)_C$ 3.22933432181, $\left(f_{zz}^{(3)}\right)_C$ 3.83382526548.

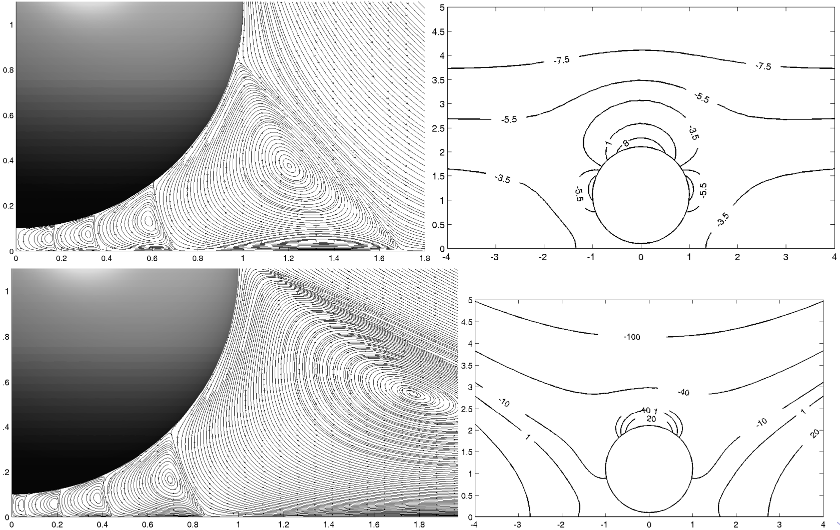
Perturbed velocity ($M = 2$ and $\ell/a = 2$)



Perturbed velocity ($M = 3$ and $\ell/a = 2$)



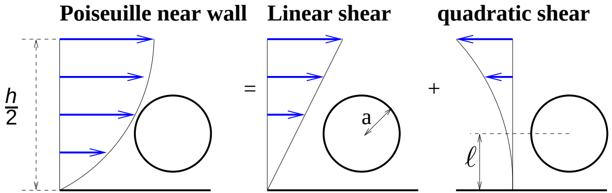
Perturbed fields (velocity and pressure)($M = 2, 3$ and $\ell/a = 1.1$)



Comments on results

- These analytical solutions for a sphere held fixed close to a wall in axisymmetric flow give directly the velocity and pressure fields.
- Alternatively the pressure was found from the solution in term of the stream function (following an idea of *Chervenivanova and Zapryanov, Int. J. Multiphase Flow 11, 721 (1985)*).
- Precision of 10^{-16} for flow field allows to discover toroïdal recirculation cells ; the number of cells increases for decreasing sphere to wall gap.

Particle held fixed in unperturbed flow along wall



$$u(z) = k_s z + k_q z^2 \quad \text{where} : k_s = \frac{h}{2\mu_f}\left(-\frac{dp}{dx}\right), \quad k_q = \frac{1}{2\mu_f}\left(\frac{dp}{dx}\right).$$

$$\frac{F_x^p}{6\pi\mu_f a < u >} = 6\frac{\ell}{h}\left[f_{xx}^s\left(\frac{a}{\ell}\right) - \frac{\ell}{h} f_{xx}^q\left(\frac{a}{\ell}\right)\right]$$

$$\frac{C_y^p}{8\pi\mu_f a^2 < u >} = 6\frac{a}{h}\left[\frac{c_{yx}^s\left(\frac{a}{\ell}\right)}{2} - \frac{\ell}{h} c_{yx}^q\left(\frac{a}{\ell}\right)\right]$$

with dimensionless force and torque (f_{xx}^s, c_{yy}^s) for linear shear flow and (f_{xx}^q, c_{yy}^q) for quadratic shear flow.

Solutions for motions along wall using bispherical coordinates : Translation, rotation, linear shear flow

M. Chaoui and F. Feuillebois. Quart. J. Mech. Applied Math., 56 :381-410, 2003.

F_x^{t}	$= -6\pi a\mu f_{xx}^{\mathrm{t}} U_x$	Force, translation
C_y^{t}	$= 8\pi a^2\mu\, c_{yx}^{\mathrm{t}} U_x$	Torque, translation
C_y^{r}	$= -8\pi a^3\mu\, c_{yy}^{\mathrm{r}}\Omega_y$	Torque, rotation
F_x^{r}	$= 6\pi a^2\mu f_{xy}^{\mathrm{r}}\Omega_y$	Force, rotation
F_x^{s}	$= 6\pi a\mu f_{xx}^{\mathrm{s}} k_{\mathrm{s}}\ell$	Force, linear shear flow
C_y^{s}	$= 4\pi a^3\mu c_{yx}^{\mathrm{s}} k_{\mathrm{s}}$	Torque, linear shear flow

All friction factors are defined so that their limit is unity for $\ell/a \to \infty$.

Solutions for motions along wall using bispherical coordinates : Quadratic shear flow, modulated shear flow

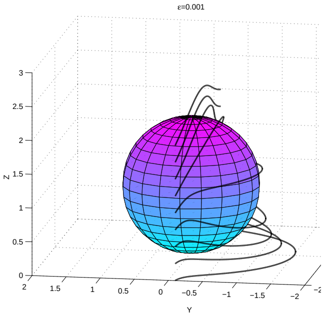
L. Pasol, A. Sellier, and F. Feuillebois. Quart. J. Mech. Applied Math., 59 : 587-614, 2006.

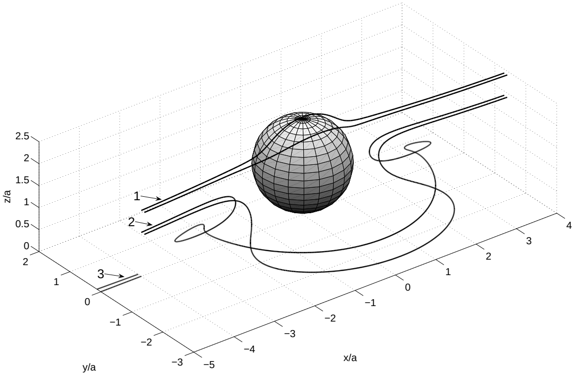
$$F_x^{\mathrm{q}} = 6\pi a \ell^2 \mu f_{xx}^{\mathrm{q}} k_{\mathrm{q}} \quad \text{Force, quadratic shear flow}$$

$$C_y^{\mathrm{q}} = 8\pi a^3 \ell \mu c_{yx}^{\mathrm{q}} k_{\mathrm{q}} \quad \text{Torque, quadratic shear flow}$$

A 'modulated shear flow' with velocity $\mathbf{u}^\infty = 2k_{\mathrm{m}} z y \mathbf{e}_x$ gives a torque on a sphere centered on the z axis, but no force by symmetry.
Excellent agreement with the results obtained by *M. Ekiel-Jeżewska and E Wajnryb, Quart. J. Mech. Applied Math., 59 : 563-585, 2006* with the method of multipoles.

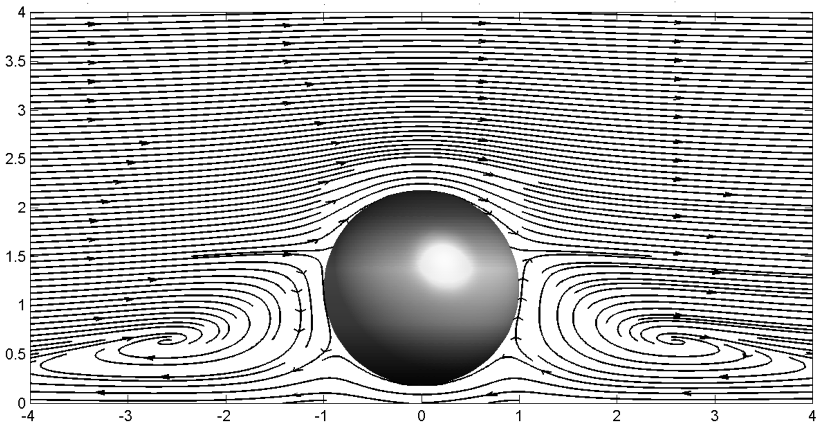
Flow trajectories around fixed sphere in linear and quadratic shear flows





Flow trajectories around sphere held fixed in a quadratic shear flow

In symmetry plane :



Superposition of unperturbed flows normal and parallel to the wall

Example of application : trajectories of a particle in ambient axisymmetric flow

$$\mathbf{u}_\infty = (S_2 \rho z + 3S_3 \rho z^2)\mathbf{e}_\rho - (S_2 z^2 + 2S_3 z^3)\mathbf{e}_z$$

The particle at (X, Y, Z) off the z axis also 'sees' locally these axisymmetric flows plus ambient linear and quadratic shear flows. In fixed frame centered at $(X, Y, 0)$ at considered time :

$$\begin{aligned} \mathbf{u}'_\infty &= (S_2 z' + 3S_3 z'^2)(x'\mathbf{e}_x + y'\mathbf{e}_y) - (S_2 z'^2 + 2S_3 z'^3)\mathbf{e}_z \\ &\quad + XS_2 z'\mathbf{e}_x + YS_2 z'\mathbf{e}_y + 3XS_3 z'^2\mathbf{e}_x + 3YS_3 z'^2\mathbf{e}_y \end{aligned}$$

Take $\tau_e = (S_2 a)^{-1}$ as characteristic time for flow ;
Let $\lambda_3 = aS_3/S_2$.

Stokes number

Take $\tau_e = (S_2 a)^{-1}$ as characteristic time for flow ;
Let $\tau_p = m_p/(6\pi a \mu_f)$ be the characteristic time for accelerating a particle, where m_p = mass of particle.
Define the *Stokes number* :

$$\mathcal{S} = \frac{\tau_p}{\tau_e} = \frac{2}{9}\frac{S_2 a^3}{\nu_f}\frac{\rho_p}{\rho_f}$$

Note

$$\frac{S_2 a^3}{\nu_f} = \text{Re} \ll 1$$

is a small Reynolds number relative to the particle.
$\mathcal{S}$ may be non small for

$$\frac{\rho_p}{\rho_f} \gg 1$$

that is.for a particle in a gas.

Solving the system

Introducing all forces and torques due to the preceding ambient flows in the equations of motion of the particle, in dimensionless form :

$$\mathcal{S}\frac{dV_x}{dt} = -f^{t}_{xx}\,V_x + f^{r}_{xy}\,\Omega_y - f^{s}_{xx}\,XZ - 3f^{q}_{xx}\,\lambda_3 XZ^2$$

$$\mathcal{S}\frac{dV_y}{dt} = -f^{t}_{xx}\,V_y - f^{r}_{xy}\,\Omega_x - f^{s}_{xx}\,YZ - 3f^{q}_{xx}\,YZ^2$$

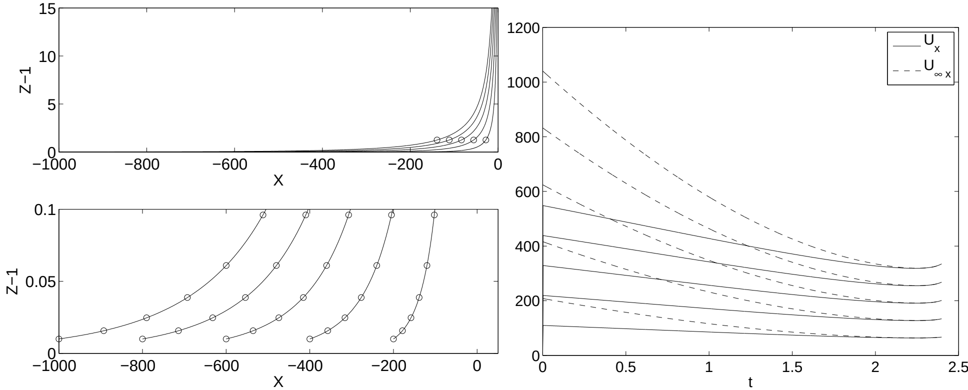
$$\mathcal{S}\frac{dV_z}{dt} = -f^{t}_{zz}\,V_z + f^{(2)}_{zz} + f^{(3)}_{zz}\lambda_3 - V_S$$

$$\mathcal{S}\frac{d\Omega_y}{dt} = \frac{10}{3}c^{t}_{yx}\,V_x - \frac{10}{3}c^{r}_{yy}\,\Omega_y - \frac{5}{3}c^{s}_{yx}X - 10\,c^{q}_{yx}\lambda_3 XZ$$

$$\mathcal{S}\frac{d\Omega_x}{dt} = -\frac{10}{3}c^{t}_{yx}V_y - \frac{10}{3}c^{r}_{yy}\Omega_x + \frac{5}{3}c^{s}_{yx}Y + 10\,c^{q}_{yx}\lambda_3 YZ$$

Difficulty : f^{t}_{zz} normalized force for motion normal to wall varies like $1/\xi$, the inverse normalized gap. Thus the system is stiff.

Trajectories of particles entrained from wall, low Stokes number $\mathcal{S} \sim \text{Re} \ll 1$



(i) the particle spends a long time near wall (lubrication region) ;
(ii) for low $\mathcal{S}$ the motion is reversible : motion towards wall gives no collision on wall.

Possible collision on wall due to particle inertia, Stokes number $\mathcal{S} = O(1)$?

For particle on axis, when gap becomes small $\xi \ll 1$:

$$\mathcal{S}\xi'' = -\frac{\xi'}{\xi} - 1$$

Unity on rhs comes from the first order term of the force due to axisymmetric flow field, after normalizing.
Integrating from some $\xi_0 \ll 1$ with velocity ξ_0' :

$$\mathcal{S}(\xi' - \xi_0') + \log\frac{\xi}{\xi_0} + \tau = 0$$

1st term is bounded ; $\xi \to 0$ in 2nd term only if normalized time $\tau \to \infty$ in 3rd term.
Thus no collision in finite time, for any value of the Stokes number

Then how can small particles in a liquid or a gas collide with a wall ?

Particle comes so close to wall that other physical effects become effective :

- Roughness of particle or wall
- Short range forces (van der Waals, etc.)

Thus, calculate the time for ξ to reach some small value, e.g. ξ^* due to roughness.
As we will see, this time gets smaller for larger $\mathcal{S}$.

Matched asymptotic expansions for the collision problem

Consider again the model equation for the gap variation :

$$\mathcal{S}\xi\xi'' + \xi' + \xi = 0$$

with initial conditions $\xi = \xi_0 \ll 1, \xi' = \xi_0'$.
At 1st order, 1st term is negligible ; cannot apply both initial conditions, thus singular perturbation problem.
Let $\xi = \xi_0\,\tilde{\xi}$ with $\tilde{\xi} = 0(1)$.

Outer equation $\tilde{\xi}' + \tilde{\xi} = 0$ has the solution $\tilde{\xi} = Ce^{-\tau}$.

Inner equation. Let the inner variable $\hat{\tau} = \tau/\xi_0$. Rename the unknown $\tilde{\xi}$ as $\hat{\xi}$.
Inner equation : $\mathcal{S}\hat{\xi}\hat{\xi}'' + \hat{\xi}' = 0$

Matched asymptotic expansions for the collision problem (cont'd)

Inner equation : $\mathcal{S}\hat{\xi}\hat{\xi}'' + \hat{\xi}' = 0$. Solution using initial conditions :

$$\hat{\tau} = \mathcal{S}e^{\mathcal{S}\xi_0'} \int_{-\mathcal{S}\xi_0' + \log \hat{\xi}}^{-\mathcal{S}\xi_0'} \frac{e^{\theta}}{\theta} \, d\theta$$

Matching : $\lim_{\hat{\tau}\to\infty} \hat{\xi} = \lim_{\tau\to 0} \tilde{\xi}$

• The limit $\hat{\tau} \to \infty$ is obtained as $\mathcal{S}\xi_0' + \log \hat{\xi} \to 0$ because then the integral diverges in $\theta = 0$.

• $\lim_{\tau\to 0} \tilde{\xi} = C$. Thus the matching gives the constant C.

Result for outer solution, in original variable :

$$\xi = \xi_0 e^{\mathcal{S}\xi_0' - \tau}$$

(with $\xi_0' < 0$) says that for larger Stokes number $\mathcal{S}$ a value ξ^* is attained more rapidly. But in any case, the value $\xi = 0$ would be attained only after an infinite time.

Outline

Order O(Re) steady terms in motion along wall.

Steady Navier-Stokes equations in frame of sphere center, for $\mathbf{u}_\infty$ and $\mathbf{U}_p$ along wall :

$$\boldsymbol{\nabla}.\mathbf{v} = 0, \qquad \Delta \mathbf{v} - \nabla p = Re\left(\mathbf{v}.\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}.\boldsymbol{\nabla}\mathbf{u}_\infty + (\mathbf{u}_\infty - \mathbf{U}_p).\boldsymbol{\nabla}\mathbf{v}\right)$$

with boundary conditions :

$$\begin{aligned} \mathbf{v} &= \mathbf{U}_p + \boldsymbol{\Omega}_p \wedge \mathbf{r} - \mathbf{u}_\infty \qquad \text{on sphere surface} \\ \mathbf{v} &= 0 \qquad \text{on plane,} \qquad \mathbf{v} \to 0 \qquad \text{at infinity} \end{aligned}$$

Consider here the ambient flow field along the wall :
$\mathbf{u}_\infty = \mathbf{u}^{\mathrm{s}}_\infty + \mathbf{u}^{\mathrm{q}}_\infty$ where $\mathbf{u}^{\mathrm{s}}_\infty = k_{\mathrm{s}}\, z\, \mathbf{e}_x$ and $\mathbf{u}^{\mathrm{q}}_\infty = k_{\mathrm{q}}\, z^2\, \mathbf{e}_x$
(Example : Poiseuille flow near wall).
Consider the $O(Re)$ terms in the expansions :

$$\mathbf{v} = \mathbf{v}^0 + Re\,\mathbf{v}^1 + O(Re^2), \qquad p = p^0 + Re\,p^1 + O(Re^2)$$

$$\mathbf{U}_p = \mathbf{U}^0_p + Re\,\mathbf{U}^1_p + O(Re^2), \qquad \boldsymbol{\Omega}_p = \boldsymbol{\Omega}^0_p + Re\,\boldsymbol{\Omega}^1_p + O(Re^2)$$

Reciprocity theorem for the regular problem

Problem is regular (unlike Oseen's) because the wall is in the inner region of expansion (distance to wall $\ll$ 1/Re). Because of their image in the wall, perturbations decay fast enough (Cox & Brenner 1968).

Reciprocity theorem (Ho & Leal 1974) gives the lift force :

$$
\begin{aligned}
F_z &= -Re \int_{V_f} \left(\mathbf{v}^0 \cdot \nabla \mathbf{v}^0 + \mathbf{v}^0 \cdot \nabla \mathbf{u}_\infty + (\mathbf{u}_\infty - \mathbf{U}_p) \cdot \nabla \mathbf{v}^0 \right) . \mathbf{w} \, dV \\
&= -Re \, L
\end{aligned}
$$

where $\mathbf{v}^0 = T_t \mathbf{v}^t + T_r \mathbf{v}^r + T_s \mathbf{v}^s + T_q \mathbf{v}^q$, the T's are constants and $\mathbf{w}$ is the velocity field of Stokes flow for a sphere moving normal to wall.

Integral L calculated using our precise results for Stokes flow in bispherical coordinates.

Expression of lift is composed of various coupling terms

$$
\begin{aligned}
L = {} & T_s^2\, L_s \;+\; T_q^2\, L_q \;+\; T_r^2\, L_r \;+\; T_t^2\, L_t \\
& + T_s\, T_q\, L_{sq} \;+\; T_s\, T_r\, L_{sr} \;+\; T_s\, T_t\, L_{st} \\
& + T_q\, T_r\, L_{qr} \;+\; T_q\, T_t\, L_{qt} \;+\; T_r\, T_t\, L_{rt}\,.
\end{aligned}
$$

with :

$$L_s = \int_{V_f} \left(\mathbf{v}_s.\nabla\mathbf{v}_s^\infty + \left(\mathbf{v}_s + \mathbf{v}_s^\infty\right).\nabla\mathbf{v}_s\right).\mathbf{w}\, dV$$

$$L_t = \int_{V_f} \left(\left(\mathbf{v}_t - e_x\right).\nabla\mathbf{v}_t\right).\mathbf{w}\, dV$$

$$L_{sr} = \int_{V_f} \left(\mathbf{v}_r.\nabla\left(\mathbf{v}_s + \mathbf{v}_s^\infty\right) + \left(\mathbf{v}_s + \mathbf{v}_s^\infty\right).\nabla\mathbf{v}_r\right).\mathbf{w}\, dV$$

$$L_r = \int_{V_f} \left(\mathbf{v}_r.\nabla\mathbf{v}_r\right).\mathbf{w}\, dV$$

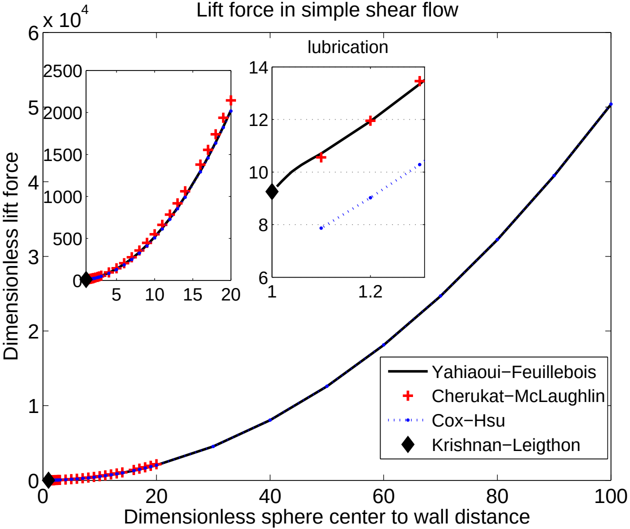
$$L_{st} = \int_{V_f} \left(\mathbf{v}_t.\nabla\left(\mathbf{v}_s + \mathbf{v}_s^\infty\right) + \left(\mathbf{v}_s + \mathbf{v}_s^\infty\right).\nabla\mathbf{v}_t - e_x.\nabla\mathbf{v}_s\right).\mathbf{w}\, dV$$

$$L_{rt} = \int_{V_f} \left(\mathbf{v}_r.\nabla\mathbf{v}_t + \left(\mathbf{v}_t - e_x\right).\nabla\mathbf{v}_r\right).\mathbf{w}\, dV$$

$$L_{sq} = \int_{V_f} \left(\left(\mathbf{v}_s + \mathbf{v}_s^\infty\right).\nabla\mathbf{v}_q + \left(\mathbf{v}_q + \mathbf{v}_q^\infty\right).\nabla\mathbf{v}_s + \mathbf{v}_q.\nabla\mathbf{v}_s^\infty + \mathbf{v}_s.\nabla\mathbf{v}_q^\infty\right).\mathbf{w}\, dV$$

L_q, L_{qr}, L_{qt} are formally like L_s, L_{sr}, L_{st}.

Example : result for the lift in shear flow on a fixed sphere



Comparison with :
Cox & Hsu (1977)
Cherukhat & Mc Laughlin (1994)
Krishnan & Leighton (1995)

Outline

Order $O(Re)$ unsteady terms in motion normal to wall

Expand Navier-Stokes equations :

$$\mathrm{Re}\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\right) = -\boldsymbol{\nabla} p + \nabla^2\mathbf{v}, \quad \boldsymbol{\nabla}\cdot\mathbf{v} = 0$$

with boundary conditions :

$\mathbf{v} = \mathbf{U}_p = U_z\mathbf{e}_z$ on sphere surface , $\mathbf{v} = 0$ on plane and at infinity.

- At order O(1), Stokes equations apply with these conditions.
- At O(Re) :

$$\nabla.\mathbf{v}^{(1)} = 0 \quad , \quad \Delta\mathbf{v}^{(1)} - \nabla p^{(1)} = \frac{\partial \mathbf{v}^{(0)}}{\partial t} + \mathbf{v}^{(0)}\cdot\nabla\mathbf{v}^{(0)} \qquad (1)$$

with boundary conditions :

$\mathbf{v}^{(1)} = \mathbf{U}_p^{(1)} = 0$ on sphere surface , $\mathbf{v}^{(1)} = 0$ on plane and at infinity.

The problem is regular because of the wall. Note the unsteady terms remain in any reference frame (wall or sphere).

Expanding the reciprocity theorem to the unsteady case

Expanding the reciprocity theorem gives the $O(Re)$ term in the expansion for the drag force :

$$F^{(1)} = -\left[\int_{V_f} \frac{\partial \mathbf{v}^{(0)}}{\partial \delta} \frac{\partial \delta}{\partial t} \cdot \mathbf{u} \, dV + \int_{V_f} \left(\mathbf{v}^{(0)} \cdot \nabla \mathbf{v}^{(0)} \right) \cdot \mathbf{u} \, dV \right]$$

where :

$\delta = \ell / a$: dimensionless sphere center to wall distance ;

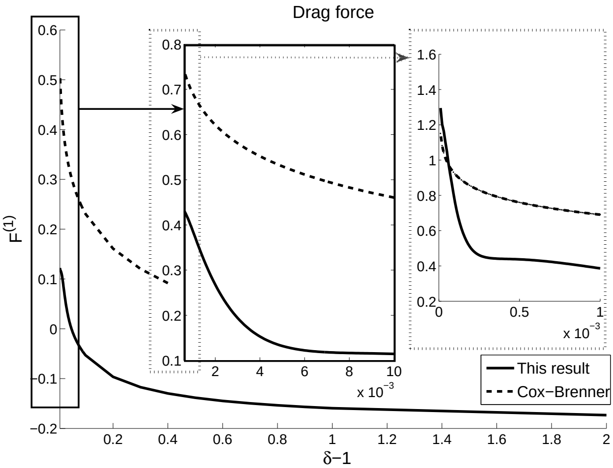
$\partial \delta / \partial t = \text{sign}(U_z)$ thus drag force depends on direction of motion ;

$\mathbf{u}$: creeping flow for sphere moving towards wall with unit velocity.

Last integral vanishes by symmetry.

O(Re) term in the drag force, for $U_z = -1$

and Cox& Brenner (1967) result for lubrication.



O(Re) term in the normalized drag force

- Results for small gap are close to those of Cox & Brenner (1967), although not exactly the same.
- For a given distance from wall, the drag is different for a sphere moving away from it (for then the wake interacts with wall) and towards it.
- Close to wall ($\xi < 0.05$), drag is increased for motion towards the wall, decreased for motion away from wall.
- For distances $\xi > 0.05$, this is the reverse.
- Far from wall, $\xi \gg 1$, dimensionless drag force at O(Re) converges to $-\frac{3}{16}$ for motion towards wall and $\frac{3}{16}$ for motion away from wall.

Drag force including inertial and unsteady terms

- Far from wall, total dimensional drag force for constant particle velocity : $\mathbf{F}_t = -6\pi a \mu_f \, \mathbf{U}_p \, f_t$

$$f_t = \left\{ \underbrace{1}_{\texttt{Stokes}} + \underbrace{\frac{9}{8}\frac{1}{\delta}}_{\texttt{Lorentz}} - \operatorname{sign}(U_z)\frac{3}{16}|Re| + o\left(Re, \frac{1}{\delta}\right) \right\}$$

- Future work : consider non-steady particle velocity.
- Close to wall : no possibility of collision is brought by the O(Re) term since it is small compared with quasi-steady lubrication term.

Outline

Conclusion

- Comprehensive results have been given for a sphere in creeping flow : translating, rotating, held fixed in unperturbed flows that are axisymmetric (polynomial up to degree 3) and along the wall (polynomial up to degree 2).
- All these creeping flows may be added by linearity of Stokes equations.
- Solutions for all these flow fields were calculated in bispherical coordinates using an iteration technique which provides very precise results ; it makes it possible to obtain details of the slow flow field and obtain results for the drag force for small gaps, in the lubrication region.

Conclusion (cont'd)

- An example of trajectories of a sphere moving in an axisymmetric flow field was given. A discussion shows that there is in principle no collision in creeping flow, even for O(1) Stokes number ; "collision" occurs because of various physical effects.
- O(Re) terms for a sphere moving along a wall in a parabolic unperturbed flow along the wall give a lift force. This force involves all couplings between translation, rotation, linear shear flow, quadratic shear flow. Classical results for the lift force on a fixed sphere in linear shear flow were improved.
- O(Re) unsteady drag force on a sphere moving normal to the wall was calculated. Drag for motions towards wall and away from wall are different. This term does not create any possibility of collision with wall. Thus a non-small Re is essential for a possibility of collision due to hydrodynamics.