SLOW VISCOUS MOTION OF A SOLID PARTICLE IN A SPHERICAL CAVITY

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Seminar at IPPT
Warsaw, 16 November 2011

Outline

1) Addressed problem and assumptions

2) Key issues and available literature

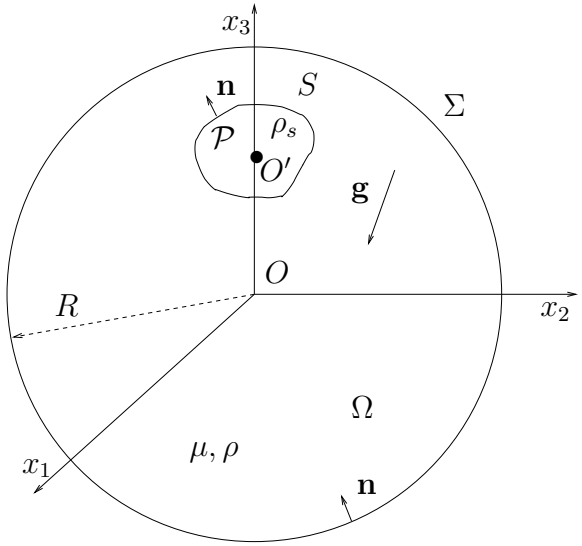
3) Boundary approach and suitable Green tensor

4) Numerical implementation and comparisons

5) Numerical results for a non-spherical particle

6) Concluding remarks

Addressed problem



- A Newtonian liquid (ρ, μ). Applied uniform gravity field $\mathbf{g}$
- The liquid is confined by a solid and motionless cavity Σ with attached Cartesian coordinates (O, x_1, x_2, x_3)
- A solid arbitrary-shaped particle $\mathcal{P}$ with center of mass O', uniform density ρ_s and smooth surface S with $\mathbf{n}$ the unit outward normal
- The particle translates at $\mathbf{U}$ (velocity of O') and rotates at $\mathbf{W}$

Basic issues

Experienced surface traction $\mathbf{f}$ on S?
Resulting hydrodynamic force $\mathbf{F}$ and torque $\mathbf{\Gamma}$ (about O') on $\mathcal{P}$?

Assumptions and governing equations

- The particle and its rigid-body motion $(\mathbf{U}, \mathbf{W})$ have length and velocity scales a and V.
- Assuming that $\mathrm{Re} = \rho V a/\mu \ll 1$ one neglects inertia effects and obtains a quasi-steady flow $(\mathbf{u}, p+\rho \mathbf{g}.\mathbf{x})$ in the liquid domain Ω

Creeping steady flow

$$\mu \nabla^2 \mathbf{u} = \nabla p \ \text{ and } \ \nabla.\mathbf{u} = 0 \text{ in } \Omega,$$

$$\mathbf{u} = \mathbf{0} \ \text{ on } \ \Sigma,$$

$$\mathbf{u} = \mathbf{U} + \mathbf{W} \wedge \mathbf{x}' \text{ on } S \text{ with } \mathbf{x}' = \mathbf{O}'\mathbf{M}$$

Introducing the stress tensor $\boldsymbol{\sigma}$ such that $\sigma_{ij} = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i})$, one looks at

$$\mathbf{f} = \boldsymbol{\sigma}.\mathbf{n} \ \text{ on } \ S,$$

$$\mathbf{F} = \int_S \mathbf{f} dS, \ \boldsymbol{\Gamma} = \int_S \mathbf{x}' \wedge \mathbf{f} dS$$

Two basic Problems

- Problem 1: $(\mathbf{U}, \mathbf{W})$ prescribed. Evaluation of $\mathbf{F}$ and $\boldsymbol{\Gamma}$?
- Problem 2: freely-suspended particle $\mathcal{P}$ with volume $\mathcal{V}$. Obtain $(\mathbf{U}, \mathbf{W})$ by enforcing

$$\mathbf{F} = (\rho - \rho_s)\mathcal{V}\mathbf{g}, \quad \boldsymbol{\Gamma} = \mathbf{0}$$

Auxiliary Stokes flows and key surface tractions

- $(\mathbf{u}_t^{(i)}, p_t^{(i)})$ and $(\mathbf{u}_r^{(i)}, p_r^{(i)})$ for $i = 1, 2, 3$. Stokes flows with

$$\mathbf{u}_t^{(i)} = \mathbf{u}_r^{(i)} = \mathbf{0} \text{ on } \Sigma, \mathbf{u}_t^{(i)} = \mathbf{e}_i \text{ and } \mathbf{u}_t^{(i)} = \mathbf{e}_i \wedge \mathbf{x}' \text{ on } S$$

- Resulting surface tractions $\mathbf{f}_t^{(i)}$ and $\mathbf{f}_r^{(i)}$ on S

Use for Problem 1

$$\mathbf{F} = -\mu\{\mathbf{A}_t.\mathbf{U} + \mathbf{B}_t.\mathbf{W}\}, \quad \mathbf{\Gamma} = -\mu\{\mathbf{A}_r.\mathbf{U} + \mathbf{B}_r.\mathbf{W}\}$$

$$-\mu A_L^{i,j} = \int_S \mathbf{f}_L^{(i)}.\mathbf{e}_j dS, \quad -\mu B_L^{i,j} = \int_S (\mathbf{x}' \wedge \mathbf{f}_L^{(i)}).\mathbf{e}_j dS$$

Use for Problem 2

The rigid-body migration $(\mathbf{U}, \mathbf{W})$ is obtained by solving

$$\mu\{\mathbf{A}_t.\mathbf{U} + \mathbf{B}_t.\mathbf{W}\} = (\rho_s - \rho)\mathcal{V}\mathbf{g}$$

$$\mu\{\mathbf{A}_r.\mathbf{U} + \mathbf{B}_r.\mathbf{W}\} = \mathbf{0}$$

- Well-posed linear system
- Unique solution $(\mathbf{U}, \mathbf{W})$

Available literature?

• Restricted to a spherical particle!

• Case of a translating sphere located at the cavity center
- Cunningham (1910), Williams (1915)
by obtaining the stream function (exact solution)

• Case of a sphere not located at the cavity center
- Use of bipolar coordinates (well adapted to the fluid domain geometry)
- Jeffery (1915), Stimson & Jeffery (1926), O'Neill & Majumdar (1970a, 1970b)
- recently: accurate calculations by Jones (2008)

• Merits
- very accurate solution (if carefully implemented)
- able to deal with small sphere-cavity gaps!
- provides very nice benchmatk tests for other methods to be developed

• Drawbacks
- cumbersome approach (tricky analytical manipulations)
- provides the net force $\mathbf{F}$ and torque $\mathbf{\Gamma}$ but still uneasy to
calculate the surface tractions $\mathbf{f}_t^{(i)}$ and $\mathbf{f}_r^{(i)}$ on S
- not possible to cope with one non-spherical particle
or with several particles!

Quite different boundary approach

Green tensors

• $\mathbf{y}$ source point or pole in the entire domain $\mathcal{D} = \Omega \cup \mathcal{P}$

• $\mathbf{x}$ observation point. For $j = 1, 2, 3$ one introduces a Stokes flows $(\mathbf{v}^{(j)}, p^{(j)})$,

$$\mu \nabla^2 \mathbf{v}^{(j)} = \nabla p^{(j)} - \delta_{3d}(\mathbf{x} - \mathbf{y})\mathbf{e}_j, \ \nabla . \mathbf{v}^{(j)} = 0 \ \text{ in } \ \mathcal{D}$$

• Resulting Green tensor $\mathbf{G}$ with Cartesian components

$$G_{kj}(\mathbf{x}, \mathbf{y}) = \mathbf{v}^{(j)}(\mathbf{x}, \mathbf{y}).\mathbf{e}_k$$

Remark, examples

• A Green tensor: not unique (no prescribed boundary conditions)

• Widely-employed free-space Green tensor $\mathbf{G}^\infty$ such that

$$8\pi\mu G^\infty_{kj}(\mathbf{x}, \mathbf{y}) = \frac{\delta_{kj}}{|\mathbf{x} - \mathbf{y}|} + \frac{[(\mathbf{x} - \mathbf{y}).\mathbf{e}_j][(\mathbf{x} - \mathbf{y}).\mathbf{e}_k]}{|\mathbf{x} - \mathbf{y}|^3}$$

• Specific Green tensor $\mathbf{G}^c$ for the given cavity Σ :

$$G^c_{jk}(\mathbf{x}, \mathbf{y}) = 0 \quad \text{for } \mathbf{x} \text{ on } \Sigma$$

Relevant integral representations and boundary-integral equations

- One looks at $\mathbf{f} = f_k \mathbf{e}_k$ on S for $\mathbf{u} = \mathbf{U} + \mathbf{\Omega} \wedge \mathbf{x}'$ on S

- Due to this velocity boundary condition, one gets a single-layer integral representation

$$[\mathbf{u}.\mathbf{e}_j](\mathbf{x}) = -\int_{S \cup \Sigma} f_k(\mathbf{y}) G_{kj}(\mathbf{y}, \mathbf{x}) dS(\mathbf{y}) \quad \text{for } \mathbf{x} \text{ in } \Omega \cup S;\ j = 1, 2, 3.$$

(Here $\mathbf{x}$ is the pole)

- Associated Fredholm boundary-integral equation of the first kind

$$[\mathbf{U} + \mathbf{\Omega} \wedge \mathbf{x}'].\mathbf{e}_j = -\int_{S \cup \Sigma} f_k(\mathbf{y}) G_{kj}(\mathbf{y}, \mathbf{x}) dS(\mathbf{y}) \quad \text{for } \mathbf{x} \text{ on } S;\ j = 1, 2, 3.$$

(solution unique up to $c\mathbf{n}$ with c constant)

- Valid for any Green tensor $\mathbf{G}$!

- Because $G^c_{jk}(\mathbf{y}, \mathbf{x}) = 0$ for $\mathbf{y}$ on Σ, one replaces $S \cup \Sigma$ with S in the above integrals!

- Additional general property: $G^c_{jk}(\mathbf{x}, \mathbf{y}) = G^c_{kj}(\mathbf{y}, \mathbf{x})$ under the condition $G^c_{jk}(\mathbf{x}, \mathbf{y}) = 0$ on Σ

Green tensor G^c for the spherical cavity

Obtained (in a different form not suitable for numerics) by Oseen 1927!

- Pole $\mathbf{y}$ and obervation point $\mathbf{x}$.

$$\mathbf{y}' = \frac{R^2\mathbf{y}}{|\mathbf{y}|^2},\ \mathbf{t} = \frac{\mathbf{y}}{|\mathbf{y}|},\ \mathbf{a} = \mathbf{x} - (\mathbf{x}.\mathbf{t})\mathbf{t},\ \mathbf{h} = \frac{|\mathbf{y}|}{R}(\mathbf{x} - \mathbf{y}'),\ h = |\mathbf{h}|$$

$$\begin{aligned}
G^c_{jk}(\mathbf{x},\mathbf{y}) &= G^\infty_{jk}(\mathbf{x},\mathbf{y}) - \frac{\delta_{jk}}{h} - \frac{(\mathbf{x}.\mathbf{e}_j)(\mathbf{x}.\mathbf{e}_k)}{h^3} + \frac{(\mathbf{t}.\mathbf{e}_j)(\mathbf{t}.\mathbf{e}_k)}{h}[\frac{|\mathbf{x}|^2}{h^2} - 1] \\
&-[\frac{2|\mathbf{y}|\mathbf{t}.\mathbf{x}}{h^3}](\mathbf{t}.\mathbf{e}_j)(\mathbf{t}.\mathbf{e}_k) + |\mathbf{y}|[\frac{(\mathbf{t}.\mathbf{e}_j)(\mathbf{x}.\mathbf{e}_k) + (\mathbf{t}.\mathbf{e}_k)(\mathbf{x}.\mathbf{e}_j)}{h^3}] \\
&-\frac{[|\mathbf{x}|^2 - R^2][|\mathbf{y}|^2 - R^2]}{2}\{\frac{\delta_{jk}}{R^3h^3} - \frac{3}{R^2}[\frac{(\mathbf{h}.\mathbf{e}_j)(\mathbf{h}.\mathbf{e}_k)}{h^5}] \\
&-2\frac{\mathbf{t}.\mathbf{e}_k}{R^2}[\frac{\mathbf{t}.\mathbf{e}_j}{h^3} - \frac{3(\mathbf{h}.\mathbf{e}_j)(\mathbf{h}.\mathbf{t})}{h^5}] + \frac{3E}{R^4h}[\delta_{jk} - (\mathbf{t}.\mathbf{e}_k)(\mathbf{t}.\mathbf{e}_j)] \\
&+\frac{3\mathbf{a}.\mathbf{e}_k}{R}[-\frac{E}{R^3h}\{\frac{|\mathbf{y}|\mathbf{h}.\mathbf{e}_j}{Rh^2} + \frac{2\mathbf{a}.\mathbf{e}_j}{|\mathbf{a}|^2}\} + \frac{\mathbf{E}.\mathbf{e}_j}{R^4h^2[|\mathbf{x}|^+_-(\mathbf{x}.\mathbf{t}))]} + \mathbf{a}.\mathbf{e}_j[\frac{(2R^2)^+_-|\mathbf{y}||\mathbf{x}|}{R^4h^2|\mathbf{a}|^2}]]\}
\end{aligned}$$

$$E = \{|\mathbf{x}|^+_-\frac{2R^2\mathbf{x}.\mathbf{t}}{R^2 + Rh^-_+|\mathbf{x}||\mathbf{y}|}\}/\{|\mathbf{x}|^+_-\mathbf{x}.\mathbf{t}\},\ \mathbf{E} =^-_+ |\mathbf{y}|\mathbf{x} + [|\mathbf{y}||\mathbf{x}|^+_-(1^-_+2)R^2]\mathbf{t}^+_-2[\frac{2R^2|\mathbf{y}|\mathbf{x} + [R^3h^-_+R^2|\mathbf{y}||\mathbf{x}|]\mathbf{t}}{R^2 + Rh^-_+|\mathbf{y}||\mathbf{x}|}]$$

with upperscripts or subscripts for $\mathbf{x}.\mathbf{t} \geq 0$ or $\mathbf{x}.\mathbf{t} < 0$, respectively

Numerical strategy

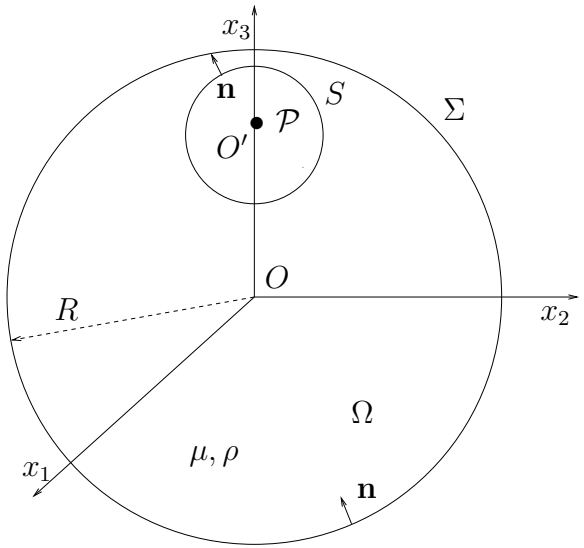
- Isoparametric triangular curvilinear Boundary Elements on S and, if needed, on the cavity Σ
- Discretize each boundary-integral equation. This requires to accurately deal with the case of a source $\mathbf{x}$ on a boundary element (a refined treatment is needed with the use of local polar coordinates)
- Solve each resulting linear systems $AX = Y$ by Gaussian elimination
- The use of $\mathbf{G}^c$ permits one to solely mesh the particle's surface (worth for a large cavity)

Benchmarks are needed!

- As seen before, $\mathbf{G}^c$ is available for a spherical cavity
- Comparisons with both analytical and numerical results for a spherical particle (previously-mentioned literature)
- Sphere located or not located at the cavity center

Case of a spherical particle

Adopted notations



- A spherical cavity with center O and radius R
- A spherical particle with radius a and center O'

$$\mathbf{OO'} = d\mathbf{e}_3 \quad \text{and } 0 \leq d < R - a$$

- $R - (d + a)$ is the sphere-cavity gap
- Normalized sphere-cavity gap $\eta = (R - d - a)/a$

Numerical comparisons for a sphere located at the cavity center

- Here $O = O'$ and $d = 0$. Sphere with radius $a < R$ translating at the velocity $\mathbf{e}_i$.

$$\mathbf{F} = -6\pi\mu a c(a/R)\mathbf{e}_i, \qquad \mathbf{\Gamma} = \mathbf{0}$$

  - Analytical formula for the occurring dimensionless resistance coefficient c

$$c(\beta) = \frac{1-\beta^5}{1-\frac{9\beta}{4}+\frac{5\beta^3}{2}-\frac{9\beta^5}{4}+\beta^6}, \qquad \beta = a/R < 1.$$

- A $N-node$ mesh on the sphere and, if needed, 1058 nodal points on the cavity Σ

Two computed values of the above coefficient c

- c_s : using the Green $\mathbf{G}^\infty$ and putting Stokeslets on both S and Σ
- c_c : using the Green tensor $\mathbf{G}^c$ and Stokeslets on S
- Notation: $\Delta c_l = |c_l/c - 1|$

A translating sphere
located at the cavity center

N	R/a	c_s	Δc_s	c_c	Δc_c
74	1.1	3258.137	1.00613	2097.155	0.29128
242	1.1	2124.983	0.30842	1949.547	0.01030
1058	1.1	1777.331	0.09436	1676.260	0.00353
exact	1.1	1624.089	0	1624.089	0
74	2.	7.223525	0.00968	7.218993	0.01030
242	2.	7.289179	0.00068	7.284937	0.00126
1058	2.	7.297493	0.00046	7.293273	0.00012
exact	2.	7.294118	0	7.294118	0
74	5.	1.749799	0.00344	1.749640	0.00353
242	5	1.755232	0.00035	1.755073	0.00044
1058	5.	1.755937	0.00005	1.755777	0.00004
exact	5.	1.755845	0	1.755845	0

Computed quantities $c_s, \Delta c_s, c_c$ and Δc_c
versus the number N of collocation points on S

Arbitrarily-located sphere

• Here $\mathbf{OO'} = d\mathbf{e}_3$ with $0 \leq d < R - a$.

For symmetry reasons one confines the attention to four cases.

• (i) A sphere translating at the velocity $\mathbf{e_1}$: $\mathbf{F} = -6\pi\mu a c_1 \mathbf{e}_1$ and $\mathbf{\Gamma} = 8\pi\mu a^2 s \mathbf{e}_2$

• (ii) A sphere translating at the velocity $\mathbf{e_3}$: $\mathbf{F} = -6\pi\mu a c_3 \mathbf{e}_3$ and $\mathbf{\Gamma} = \mathbf{0}$

• (iii) A sphere rotating at the velocity $\mathbf{e_1}$: $\mathbf{F} = -8\pi\mu a^2 s \mathbf{e}_2$ and $\mathbf{\Gamma} = -8\pi\mu a^3 t_1 \mathbf{e}_1$

• (iv) A sphere rotating at the velocity $\mathbf{e_3}$: $\mathbf{F} = \mathbf{0}$ and $\mathbf{\Gamma} = -8\pi\mu a^3 t_3 \mathbf{e}_3$

Comparisons for the computed coefficients c_1, c_3, t_1, t_3 and s

• Accurate computations obtained elsewhere by using the bipolar coordinates (Jones 2008, here labelled Jones in each reported table)

• $R = 4a$ and two values of the normalized gap $\eta = (R - d - a)/a$ are selected: $\eta = 0.5$ and $\eta = 0.1$ (small sphere-cavity gap).

• 4098 nodal points are put on the cavity Σ when using the Green tensor $\mathbf{G}^{\infty}$

Comparisons for a sphere not located at the cavity center with $\eta = (R - d - a)/a = 0.5$

N	Method	c_1	c_3	t_1	t_3	s
74	$\mathbf{G}^{\infty}$	2.6330	4.6730	1.1640	1.0789	0.11870
74	$\mathbf{G}^{c}$	2.6327	4.6714	1.1639	1.0789	0.11861
242	$\mathbf{G}^{\infty}$	2.6473	4.7107	1.1639	1.0755	0.11927
242	$\mathbf{G}^{c}$	2.6471	4.7090	1.1639	1.0755	0.11920
1058	$\mathbf{G}^{\infty}$	2.6488	4.7144	1.1639	1.0755	0.11938
1058	$\mathbf{G}^{c}$	2.6486	4.7127	1.1639	1.0755	0.11932
Jones	Bipolar	2.6487	4.7131	1.1639	1.0755	0.11933

Comparisons for a sphere not located at the cavity center with $\eta = (R - d - a)/a = 0.1$

N	Method	c_1	c_3	t_1	t_3	s
74	$\mathbf{G}^{\infty}$	3.9016	15.552	1.6065	1.1960	0.20206
74	$\mathbf{G}^{c}$	3.9009	15.413	1.6052	1.1960	0.20138
242	$\mathbf{G}^{\infty}$	3.9273	18.886	1.6145	1.1939	0.19108
242	$\mathbf{G}^{c}$	3.9237	18.636	1.6134	1.1938	0.19001
1058	$\mathbf{G}^{\infty}$	3.9159	18.832	1.6171	1.1945	0.18494
1058	$\mathbf{G}^{c}$	3.9121	18.711	1.6160	1.1945	0.18353
Jones	Bipolar	3.9121	18.674	1.6163	1.1945	0.18344

Numerical results for a non-spherical particle

- Ellipsoid with semi-axis (a_1, a_2, a_3) and surface admitting the equation

$$(x_1/a_1)^2 + (x_2/a_2)^2 + ([x_3 - d]/a_3)^2 = 1$$

- Ellipsoid-cavity normalized separation parameter λ with

$$0 < \lambda = d/a_3 < (R - a_3)/a_3$$

- 8 friction coefficients c_i, t_i, s_1 and s_2 such that

$$\mathbf{A}_T^{(i)} = 6\pi\mu a_3 c_i \mathbf{e}_i, \ \mathbf{B}_R^{(i)} = 8\pi\mu a_3^3 t_i \mathbf{e}_i,$$
$$\mathbf{B}_T^{(1)} = -8\pi\mu a_3^2 s_1 \mathbf{e}_2, \ \mathbf{B}_T^{(2)} = 8\pi\mu a_3^2 s_2 \mathbf{e}_1, \ \mathbf{B}_T^{(3)} = \mathbf{0},$$
$$\mathbf{A}_R^{(1)} = 8\pi\mu a_3^2 s_2 \mathbf{e}_2, \ \mathbf{A}_R^{(2)} = -8\pi\mu a_3^2 s_1 \mathbf{e}_1, \ \mathbf{A}_R^{(3)} = \mathbf{0}$$

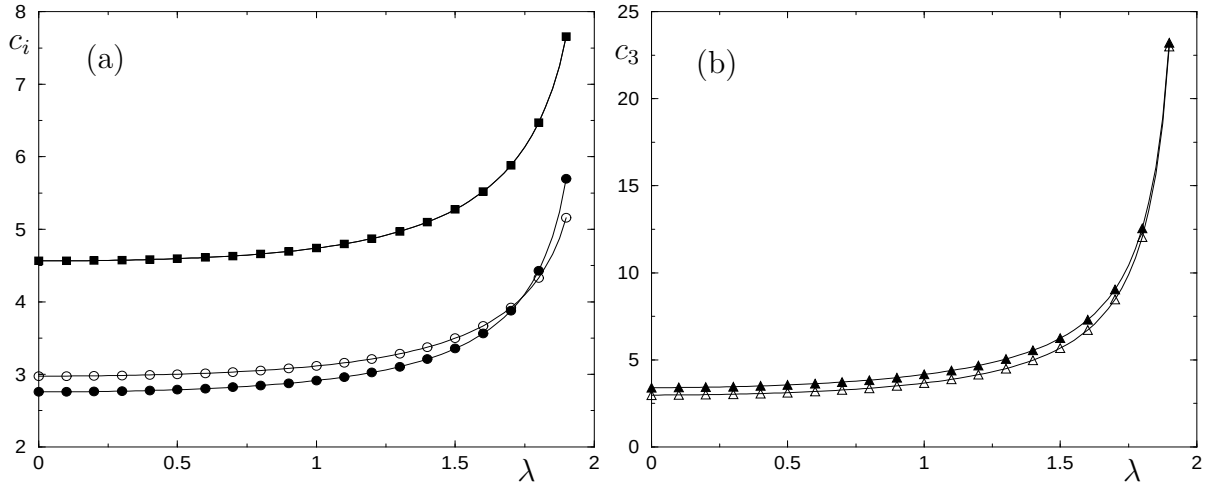
Comparisons for two selected ellipsoids

- A sphere with radius a_3 (clear symbols)
- The ellipsoid $a_1 = 5a_3/3, a_2 = 0.6a_3$

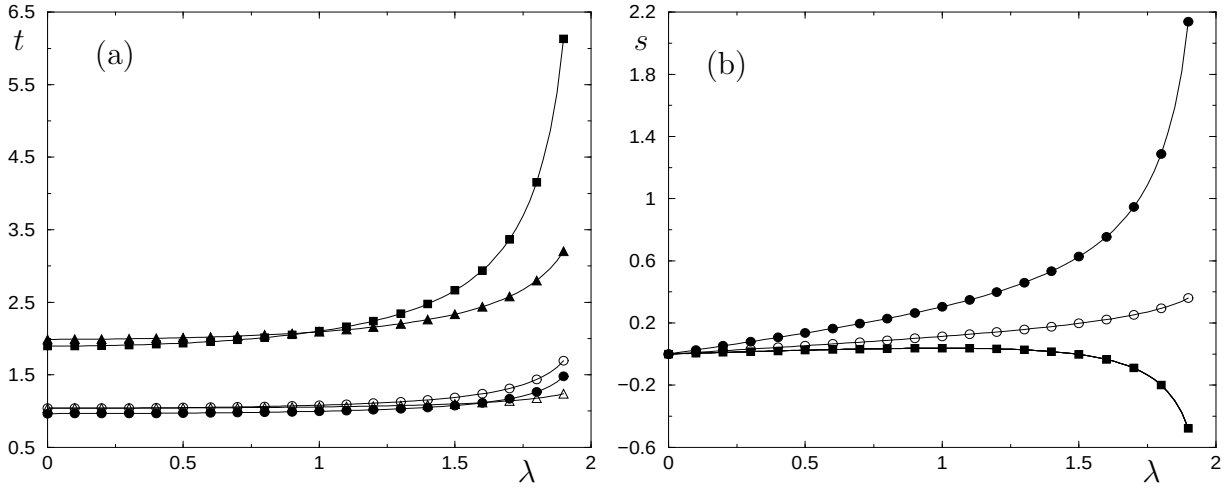
having the same volume as the sphere (filled symbols)

Friction coefficients

Normalized coefficients c_i for the sphere (clear symbols) and the ellipsoid (filled symbols).



(a) Coefficients c_1 (circles) and c_2 (squares). (b) Coefficients c_3 (triangles)



(a) Coefficients t_1 (circles), t_2 (squares) and t_3 (triangles). (b) Coefficients s_1 (circles) and s_2 (squares)

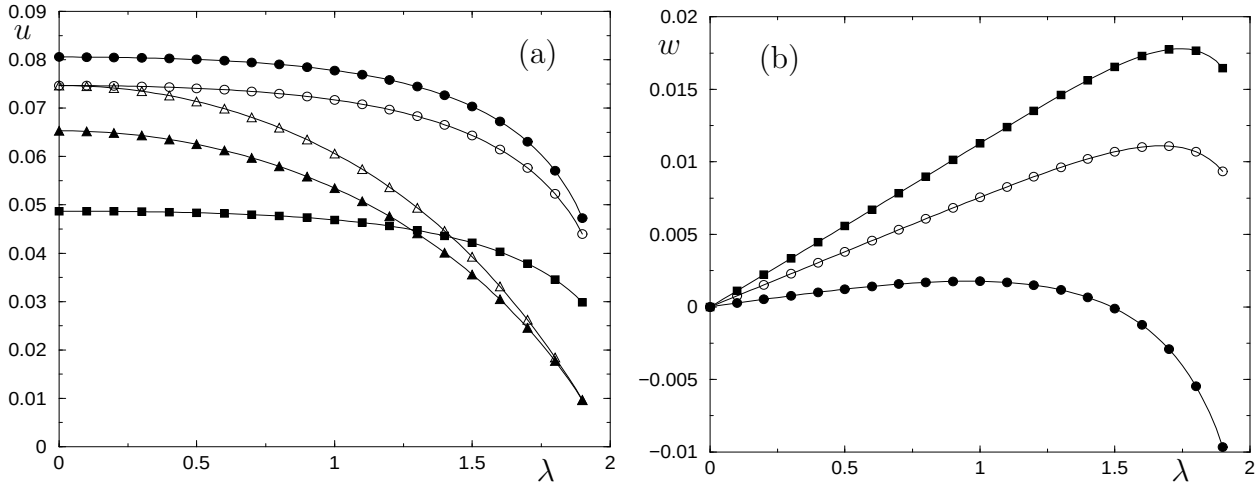
Settling normalized translational and angular velocities

Setting $U_s' = (\rho_s - \rho)a^2 g/\mu$ one gets

(i) If $\mathbf{g} = g\mathbf{e}_1 : \mathbf{U} = U_s' u_1 \mathbf{e}_1,\ \mathbf{W} = aU_s' w_2 \mathbf{e}_2$

(ii) If $\mathbf{g} = g\mathbf{e}_2 : \mathbf{U} = U_s' u_2 \mathbf{e}_2,\ \mathbf{W} = -aU_s' w_1 \mathbf{e}_1$

(iii) If $\mathbf{g} = g\mathbf{e}_3 : \mathbf{U} = U_s' u_3 \mathbf{e}_3,\ \mathbf{W} = \mathbf{0}$



Normalized velocities for the sphere (clear symbols) and the ellipsoid (filled symbols).
(a) Translational velocities u_1 (circles), u_2 (squares) and u_3 (triangles).
(b) Angular velocities w_1 (circles) and w_2 (squares)

Concluding remarks

- A new approach based on a boundary-integral formulation

- Valid for arbitrarily-shaped particles!

- Easy implementation and nicely retrieves for a spherical particle
results obtained elsewhere using a quite different (bipolar coordinates) approach

- Two tested approaches resorting to the free-space Green tensor and
the Green tensor complying with the no-slip condition on the motionless spherical cavity

- The second one makes it possible to solely mesh the particle surface
and offers more accurate results

- Numerical results reveal that a particle behaviour is slightly sensitive to its shape

- In future: cope with the challenging case of a collection of particles!