METRO
MEtallurgical TRaining On-line

Spatial discretization techniques

Arkadiusz Nagórka
CzUT

Education and Culture

Introduction

- Many physical phenomena are described by partial differential equations (PDEs).
- Analytical solutions are impossible to obtain except for linear equations on simple geometries.
- Since the computer memory is limited, discretization of the problem is necessary. Numerical methods can give *approximate* solutions of PDEs.
- Spatial discretization - obtain the solution in a set of points rather that in the entire domain.

Discretization of a simple problem

Simple one-dimensional Poisson equation

$$\frac{d^2 u}{dx^2} = 2, \quad 0 < x < 1$$

with Dirichlet boundary conditions

$$u(a) = u(0) = 0, \quad u(b) = u(1) = 0$$

will be solved numerically using

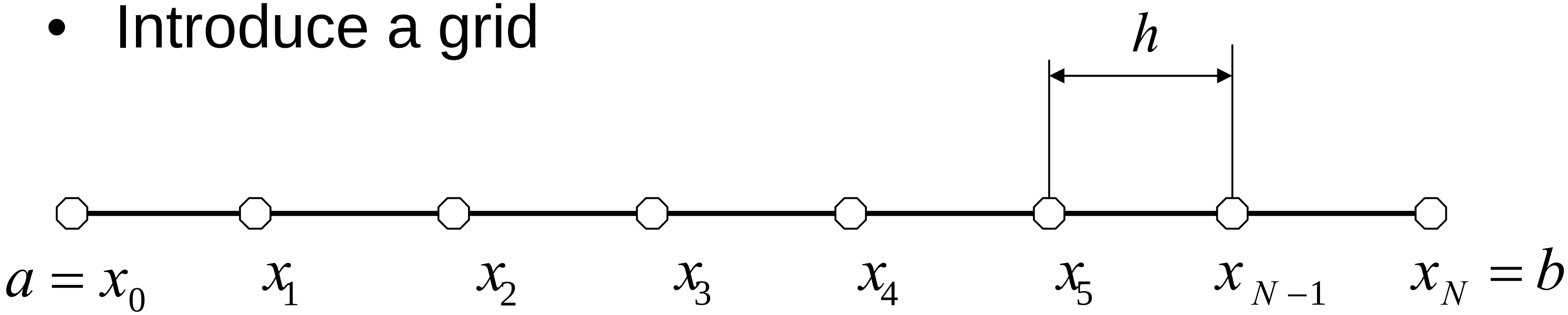
– the finite difference method (FDM),

– the finite element method (FEM),

– the finite volume method (FVM).

Solution of a simple problem using the finite difference method

- Introduce a grid



$$h_i = x_{i+1} - x_i = h, \qquad i = 0,1,\ldots N-1$$

- Approximate the differential equation at each grid point
- Solve the resulting system of algebraic equations

Solution of a simple problem using FDM, cont.

- By definition the derivative $\dfrac{du}{dx} = \lim_{h \to 0} \dfrac{u(x+h) - u(x)}{h}$

- Finite difference approximation at point x_i

$$\left(\frac{du}{dx}\right)_i \approx \frac{u(x_i + h) - u(x_i)}{h} = \frac{u_{i+1} - u_i}{h}$$ forward difference

$$\left(\frac{du}{dx}\right)_i \approx \frac{u_i - u_{i-1}}{h}$$

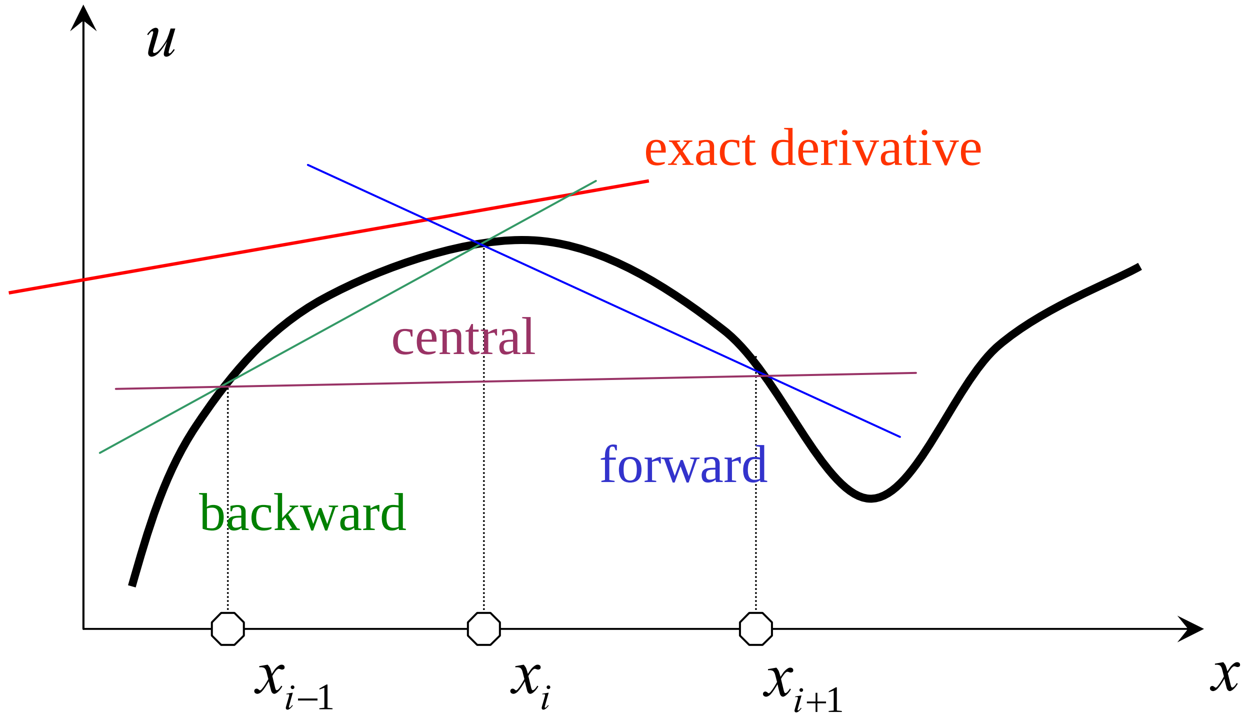
$$\left(\frac{du}{dx}\right)_i \approx \frac{u_{i+1} - u_{i-1}}{2h}$$

backward difference

central difference

Solution of a simple problem using FDM, cont.

- Forward, backward and central differences



- Central difference is the most accurate

Solution of a simple problem using FDM, cont.

- Approximation of second derivative

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx}\right) \approx \frac{(du/dx)(x+h) - (du/dx)(x)}{h}$$

- Use central differences for du/dx

$$\frac{d^2u}{dx^2} \approx \frac{1}{h}\left[\frac{u_{i+1} - u_i}{h} - \frac{u_i - u_{i-1}}{h}\right] = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}$$

Solution of a simple problem using FDM, cont.

- In our problem assume three grid points

$$x_0 = 0, \quad x_1 = 0.5, \quad x_2 = 1, \quad \mathrm{h} = 0.5$$

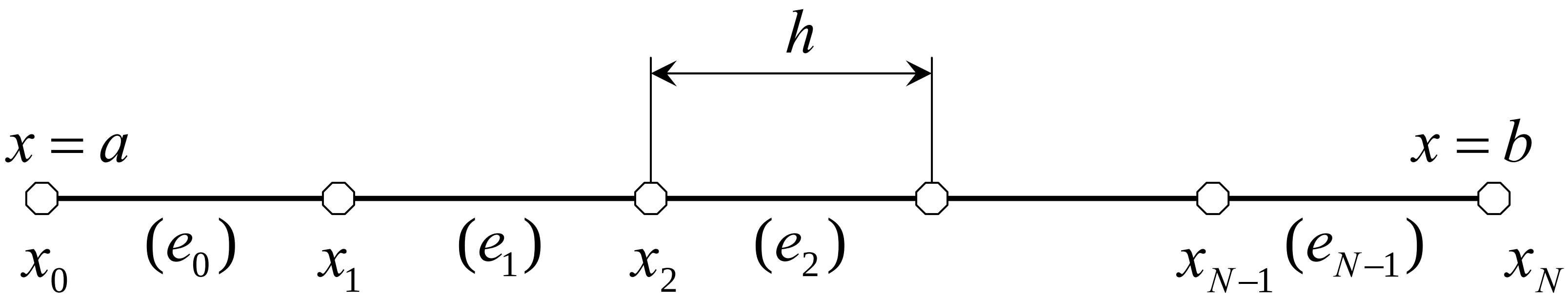
- Difference equation for the point x_1

$$\frac{u_0 - 2u_1 + u_2}{h^2} = 2 \qquad \text{or} \qquad \frac{0 - 2u_1 + 0}{0.5^2} = 2$$

- One equation is needed for one unknown
- The solution is $u_1 = -0.25$

Solution of a simple problem using the finite element method

Subdivide the domain into finite elements
(e.g. line segments in 1D,
triangles or quadrilaterals in 2D, tetrahedra in 3D)



Solution of a simple problem using FEM, cont.

- Assume approximating function in each element, e.g. a linear polynomial

$$u_h^{(0)}(x) = \alpha_1 + \alpha_2 x \qquad \text{for the element 0}$$

- Parameters α_j can be determined easily

$$u_h^{(0)}(0) = \alpha_1 + \alpha_2 0 = u_0$$

$$u_h^{(0)}(h) = \alpha_1 + \alpha_2 h = u_1 \quad \text{, hence} \quad \alpha_1 = u_0, \ \alpha_2 = \frac{u_1 - u_0}{h}$$

- Substituting yields

$$u_h^{(0)}(x) = \left(1 - \frac{x}{h}\right)u_0 + \frac{x}{h}u_1 = N_0(x)u_0 + N_1(x)u_1$$

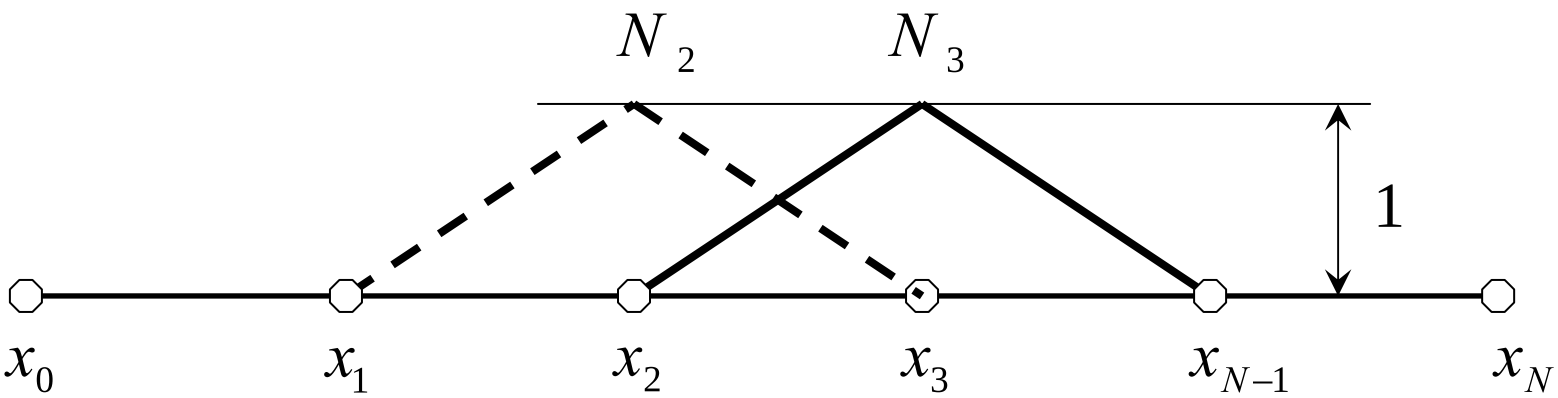
Solution of a simple problem using FEM, cont.

The function $N_j(x)$ is a *trial* function. It has the property

$$N_j(x) = \begin{cases} 1 & \text{at } x_j \\ 0 & \text{elsewhere} \end{cases}$$

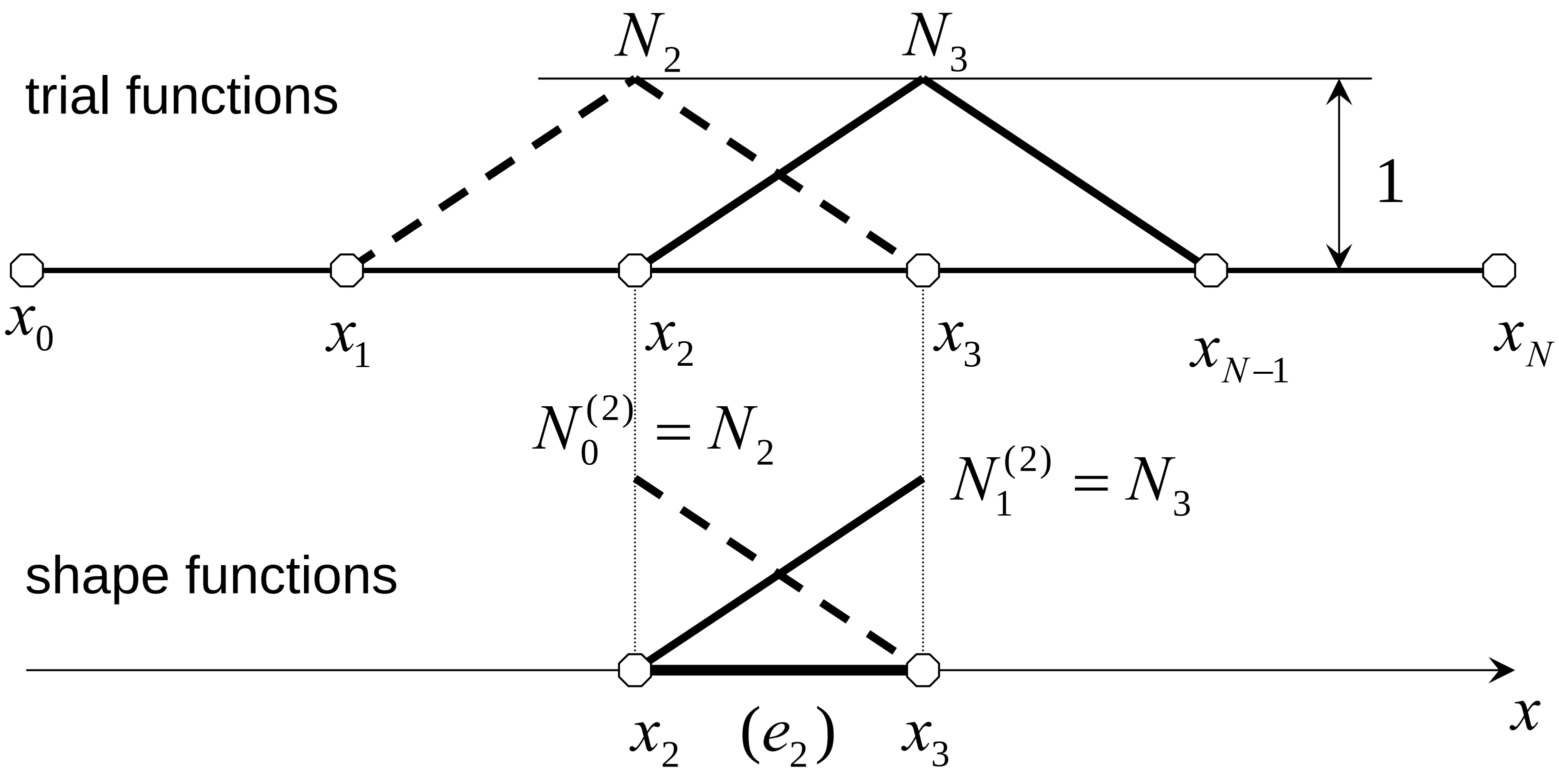
Then

$$u_h(x) = \sum_{j=0}^{N} N_j(x) u_j$$



Solution of a simple problem using FEM, cont.

Trial function is called *shape* function when restricted to a finite element



Solution of a simple problem using FEM, cont.

Obtain an integral form of the problem,
for example using the *weighted residual method*

Make the residual $R(x) = \frac{d^2 u_h}{dx^2} - 2$ "vanish", e.g.

$\int_\Omega W(x) R(x) d\Omega = 0,$ $W(x)$ is a *test* or *weighting* function

$W(x) = N(x)$ is a possible choice. Then

$$\boxed{\int_\Omega N_i(x) R(x) d\Omega = 0, \quad i = 0,1,\ldots,N}$$

Galerkin finite element method

Solution of a simple problem using FEM, cont.

Assumed solution is piecewise linear and does not have second-order derivative. Integrate by parts

$$\int_a^b N_i \left(\frac{d^2 u_h}{dx^2} - 2 \right) dx = \int_a^b \left(-\frac{dN_i}{dx}\frac{du_h}{dx} - 2N_i \right) dx + N_i \frac{du_h}{dx} \Big|_a^b$$

Now only the first derivative is necessary

$\frac{du_h(b)}{dx} \cdot 1$ and $\frac{du_h(a)}{dx} \cdot (-1)$ are *normal fluxes* q_n

imposed in *Neumann* boudary conditions

Solution of a simple problem using FEM, cont.

- The weak form of the problem reads

$$\int_a^b \frac{dN_i}{dx}\frac{du_h}{dx}dx = \int_a^b (-2)N_i dx + N_i q_n(a) + N_i q_n(b)$$

$$i = 0,1,\ldots,N$$

- Derivative of the solution is also approximated

$$u_h = \sum_{j=0}^{N} N_j u_j \qquad \frac{du_h}{dx} = \sum_{j=0}^{N} \frac{dN_j}{dx} u_j$$

Solution of a simple problem using FEM, cont.

Substituting and shifting sum gives for $i = 0,1,\ldots,N$

$$\sum_{j=0}^{N}\left(\int_a^b \frac{dN_i}{dx}\frac{dN_j}{dx}dx\right)u_j = \int_a^b (-2)N_i dx + N_i q_n(a) + N_i q_n(b)$$

which is a system of linear algebraic equations

$$\sum_{j=0}^{N} k_{ij}u_j = f_i + g_i, \quad i = 0,1,\ldots,N \qquad \text{or} \qquad \boxed{\mathbf{Ku} = \mathbf{f} + \mathbf{g}}$$

$\mathbf{K}$ - stiffness matrix

$\mathbf{u}$ - unknown nodal values

$\mathbf{f}$ - source vector

$\mathbf{g}$ - flux vector

Solution of a simple problem using FEM, cont.

- Integrals can be evaluated by summing contributions from individual elements

$$\int_{\Omega} (\cdot)\, d\Omega = \sum_{elements} \int_{\Omega_e} (\cdot)\, d\Omega$$

- Integrals involving N_i are nonzero only in elements containing the node i, e.g.

$$\int_{\Omega} (-2) N_i \, d\Omega = \int_{\Omega_i} (-2) N_1^{(i)} d\Omega + \int_{\Omega_{i+1}} (-2) N_0^{(i+1)} d\Omega$$

Solution of a simple problem using FEM, cont.

This leads to a local system of equations for each element *e*

$$\boxed{\mathbf{K}^{(e)}\mathbf{u}^{(e)} = \mathbf{f}^{(e)} + \mathbf{g}^{(e)}}$$

$$k_{ij}^{(e)} = \int_{\Omega_e} \frac{dN_i^{(e)}}{dx}\frac{dN_j^{(e)}}{dx}dx, \qquad i, j = 0,1$$

$$f_i^{(e)} = \int_{\Omega_e} (-2) N_i^{(e)} dx$$

$$g_i^{(e)} = N_i^{(e)} q_n$$

(only in elements with Neumann boundary condition)

Solution of a simple problem using FEM, cont.

- The element integrals can be evaluated easily

$$\mathbf{K}^{(e)} = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{f}^{(e)} = -h\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Assuming the mesh with 3 nodes and 2 elements, the local systems are

$$\frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_0^{(0)} \\ u_1^{(0)} \end{Bmatrix} = -h\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_0^{(1)} \\ u_1^{(1)} \end{Bmatrix} = -h\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution of a simple problem using FEM, cont.

Using global indices and augmenting gives

$$\frac{1}{h}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_0 \\ u_1 \\ u_2 \end{Bmatrix} = -h\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

for element (0)

$$\frac{1}{h}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_0 \\ u_1 \\ u_2 \end{Bmatrix} = -h\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

for element (1)

Solution of a simple problem using FEM, cont.

- *Assemble* the global system by summing up local systems

$$\frac{1}{h}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_0 \\ u_1 \\ u_2 \end{Bmatrix} = -h\begin{bmatrix} 1 \\ 1+1 \\ 1 \end{bmatrix}$$

- Local contributions are overlapped (superimposed)

Solution of a simple problem using FEM, cont.

- In our example $u_0 = u_2 = 0$

- Equation for the node 1

$$\frac{-u_0 + 2u_1 - u_2}{0.5} = -2 \cdot 0.5 \qquad \text{hence} \qquad u_1 = -0.25$$

- The solution is identical to that of FDM (for this simple problem and geometry)

Solution of a simple problem using the finite volume method

- Partial differential equations that we solve express conservation of a quantity (energy, mass,etc.)

- Conservation equations can be written in integral form, e.g. for the Poisson equation

$$\int_0^1 \frac{d^2u}{dx^2}\,dx = \int_0^1 2\,dx$$

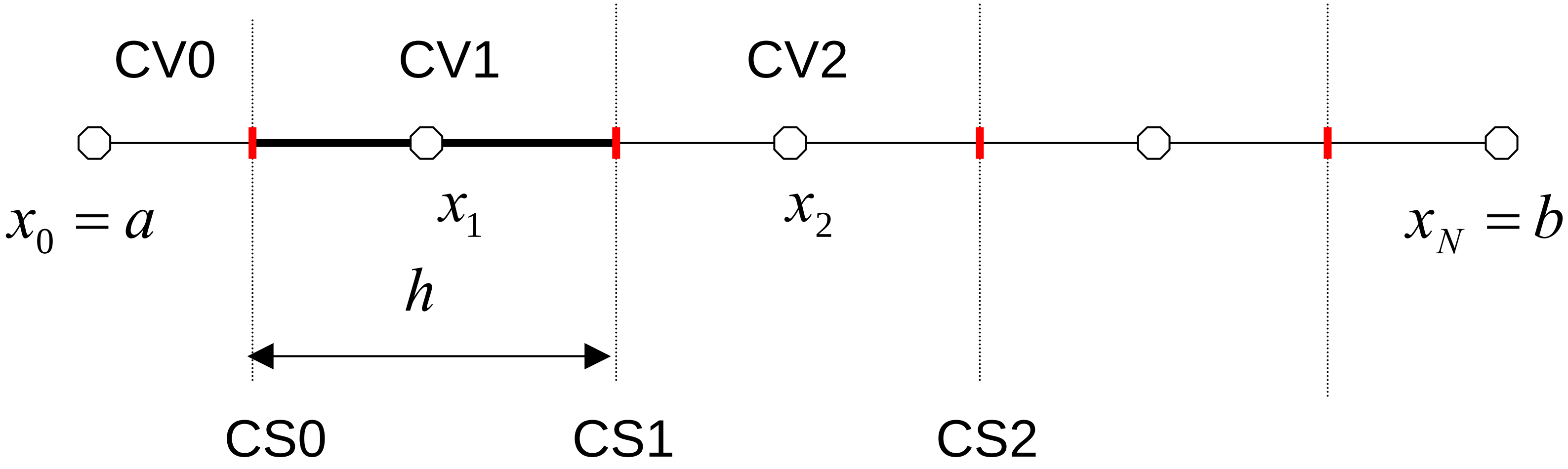
- Change the volume integral to a surface integral

$$\left.\frac{du}{dx}\right|_0^1 = \int_0^1 2\,dx$$

Balance: flux of a quantity entering and leaving the domain equals the amount produced internally by the source

Solution of a simple problem using FVM, cont.

The method utilises control volumes (finite volumes) and control surfaces



CV1 around node 1 has control surfaces CS0 and CS1.
Control surfaces are located at line midpoints.

Solution of a simple problem using FVM, cont.

Balance equation is valid for each control volume

$$\left.\frac{du}{dx}\right|_{a}^{CS\,0} = \int_{CV\,0} 2dx$$ for control volume 0

$$\left.\frac{du}{dx}\right|_{CS\,0}^{CS\,1} = \int_{CV\,1} 2dx$$ for control volume 1

$$\left.\frac{du}{dx}\right|_{CS\,1}^{CS\,2} = \int_{CV\,2} 2dx$$ for control volume 2

Solution of a simple problem using FVM, cont.

- Flux is evaluated at control surfaces using e.g. finite differences
- By summing up equations for control volumes we obtain the global equation

$$\left.\frac{du}{dx}\right|_{a}^{CS_0} + \left.\frac{du}{dx}\right|_{CS_0}^{CS_1} + \left.\frac{du}{dx}\right|_{CS_1}^{CS_2} + \ldots \left.\frac{du}{dx}\right|_{CS_{N-1}}^{b} = \left.\frac{du}{dx}\right|_{a}^{b}$$

$$\int_{VC_0} (\cdot)dx + \int_{VC_1} (\cdot)dx + \ldots + \int_{VC_N} (\cdot)dx = \int_{\Omega} (\cdot)dx$$

(since fluxes through control surfaces cancel out)

Solution of a simple problem using FVM, cont.

- Flux is conserved between CV0 and CV1 through CS0 and between CV1 and CV2 through CS1 etc. provided approximation of du/dx is the same on both sides

Automatic conservation of a physical quantity is a distinct feature of FVM

- Flux at the control surface can be approximated in many ways, e.g.

$$\left.\frac{du}{dx}\right|_{x=CS_i} \approx \frac{u_{i+1}-u_i}{h}$$

Solution of a simple problem using FVM, cont.

- Assume three nodes and three control volumes for our problem. Balance equation for the volume 1 is

$$\left.\frac{du}{dx}\right|_{CS0}^{CS1} = \int_{CV1} 2dx \Rightarrow \left.\frac{du}{dx}\right|_{x=CS1} - \left.\frac{du}{dx}\right|_{x=CS0} = \int_{CV1} 2dx$$

- Approximate derivatives with finite differences

$$\frac{u_2 - u_1}{h} - \frac{u_1 - u_0}{h} = 2h \Rightarrow \frac{u_0 - 2u_1 + u_2}{h} = 2h$$

- The solution is the same as in FDM and FEM (for this simple problem and geometry)

Imposing the Neumann condition

Consider again the Poisson equation

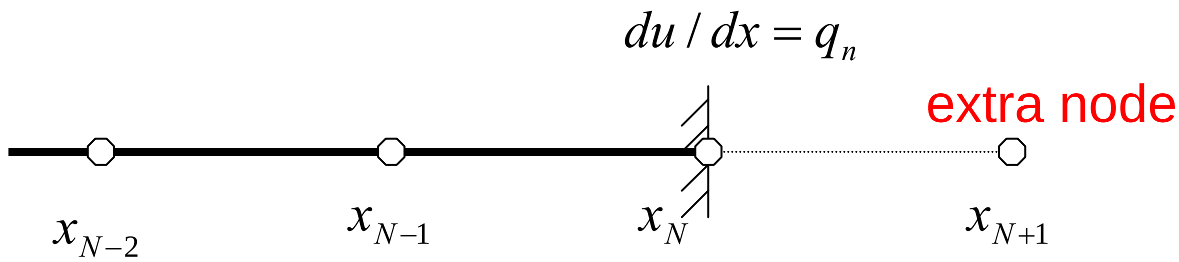
$$\frac{d^2u}{dx^2} = f(x) = 2, \quad 0 < x < 1$$

$$u(0) = u_0 = 0$$ Dirichlet boundary condition

$$\frac{du}{dx}(1) = q_n = 1$$ Neumann boundary condition

Imposing the Neumann condition in FDM

- One approach is to add a fictitious node on the right



$$\frac{u_{N-1} - 2u_N + u_{N+1}}{h^2} = f(x_N), \qquad \frac{u_{N+1} - u_{N-1}}{2h} = q_n$$

- By eliminating u_{N+1} we obtain the equation for the node *N*

$$\frac{u_{N-1} - u_N}{h^2} = \frac{1}{2} f(x_N) - \frac{q_n}{h}$$

Imposing the Neumann condition in FDM, cont.

- Us usual, assume 3 nodes in our problem
- Difference equation for the internal node 1

$$\frac{u_0 - 2u_1 + u_2}{h^2} = 2 \quad \text{or} \quad \frac{0 - 2u_1 + u_2}{0.5^2} = 2$$

- Equation for the node on the Neumann boundary

$$\frac{u_1 - u_2}{0.5^2} = \frac{2}{2} - \frac{1}{0.5}$$

- The solution is $u_1 = -0.25, \quad u_2 = 0$

Imposing the Neumann condition in FEM

- Neumann condition apears “naturally” in FEM equations as the result of integration by parts
- Since u_0 is known (Dirichlet b.c.) it can be eliminated from the system of equations
- In our two-element mesh only the last element *(e_1)* contributes to the flux vector ***g***

$$\frac{1}{h}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = -h\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$N_2(b)q_n = 1 \cdot q_n = 1$$

- The solution is the same like in FDM (for this problem)

Imposing the Neumann condition in FVM

- Neumann data q_n appears explicitly in the balance equation for the boundary volume

$$\left.\frac{du}{dx}\right|_{CS_1}^{b} = \int_{CV2} 2dx \Rightarrow q_n - \left.\frac{du}{dx}\right|_{x=CS1} = \int_{CV2} 2dx$$

- The system of equations is then

$$\frac{u_2 - u_1}{h} - \frac{u_1 - u_0}{h} = 2h, \qquad q_n - \frac{u_2 - u_1}{h} = 2\frac{h}{2}$$

- The solution is the same as in FDM and FEM (for this problem)

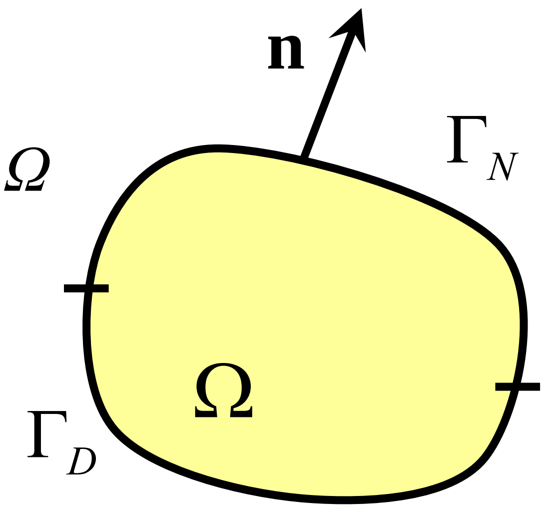
FDM, FEM and FVM for a 1D equation - conclusions

- FDM is the easiest to understand.
- Derivation of FEM equations is tedious.
- Only FVM has inherent conservative property.
- Neumann conditions are approximated in FDM, but appear directly in the formulation in FEM and FVM.
- All the methods gave identical solution. However, this happens only for simple problems and geometries.

Two-dimensional problems

Consider the Poisson equation

$$-\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(k\frac{\partial u}{\partial y}\right)=f(x,y)\quad \text{in } \Omega$$



with boundary conditions

$$u=u_D \quad \text{on } \Omega_D$$ (Dirichlet)

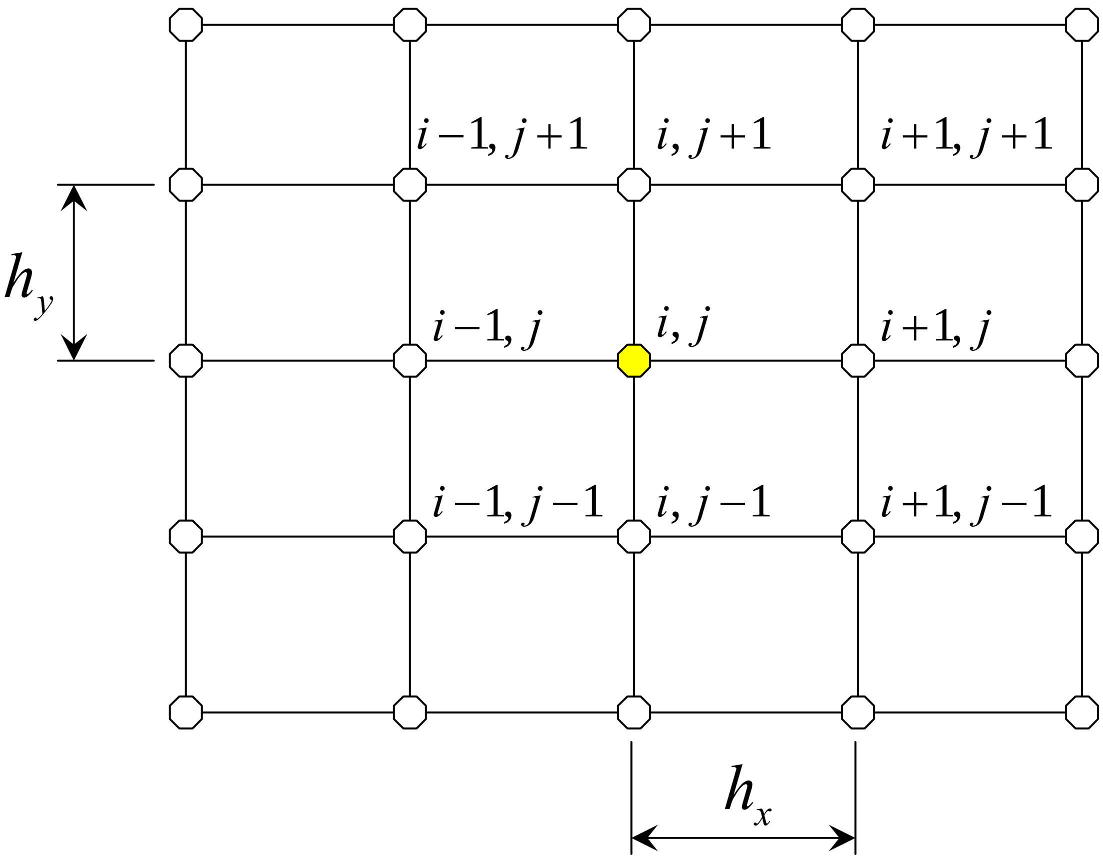
$$-k\frac{\partial u}{\partial n}=-k\left(\frac{\partial u}{\partial x}n_x+\frac{\partial u}{\partial y}n_y\right)=q_n \quad \text{on } \Omega_N$$ (Neumann)

For constant *k*

$$-k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f(x,y)\quad \text{in } \Omega$$

The finite difference method in 2D

Each grid point has two indices



$$x_i = x_0 + i\,h_x$$

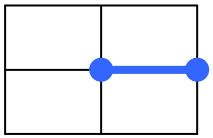
$$y_j = y_0 + j\,h_y$$

$$u_{i,j} = u\left(x_i, y_j\right)$$

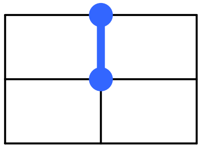
The finite difference method in 2D, cont.

Finite difference formulas used for 1D problems can be used to approximate the partial derivatives at *i,j,* e.g.

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(x+h, y) - u(x, y)}{h} \approx \frac{u_{i+1,j} - u_{i,j}}{h_x}$$



$$\frac{\partial u}{\partial y} = \lim_{h \to 0} \frac{u(x, y+h) - u(x, y)}{h} \approx \frac{u_{i,j+1} - u_{i,j}}{h_y}$$



(forward differences)

The finite difference method in 2D, cont.

Approximation of partial derivatives

$$\left[\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)\right]_{i,j} \approx \frac{\left(k\frac{\partial u}{\partial x}\right)_{i+1/2,j} - \left(k\frac{\partial u}{\partial x}\right)_{i-1/2,j}}{h_x}$$

(central difference)

$$\left[\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)\right]_{i,j} \approx \frac{k_{i+1/2,j}\dfrac{u_{i+1,j}-u_{i,j}}{h_x} - k_{i-1/2,j}\dfrac{u_{i,j}-u_{i-1,j}}{h_x}}{h_x}$$

(central differences)

$$k_{i+1/2,j} = k\left(x_i + \frac{1}{2}h_x, y_j\right)$$

evaluated at the midpoint

The finite difference method in 2D, cont.

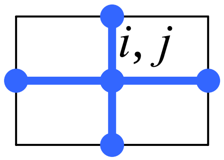
- Approximation of the Poisson equation at *i,j*

$$
-\frac{k_{i+1/2,j}\dfrac{u_{i+1,j}-u_{i,j}}{h_x} - k_{i-1/2,j}\dfrac{u_{i,j}-u_{i-1,j}}{h_x}}{h_x}\ldots
$$

$$
-\frac{k_{i,j+1/2}\dfrac{u_{i,j+1}-u_{i,j}}{h_y} - k_{i,j-1/2}\dfrac{u_{i,j}-u_{i,j-1}}{h_y}}{h_y} = f_{i,j}
$$

for each grid point except on the boundary

- Five unknowns per an equation

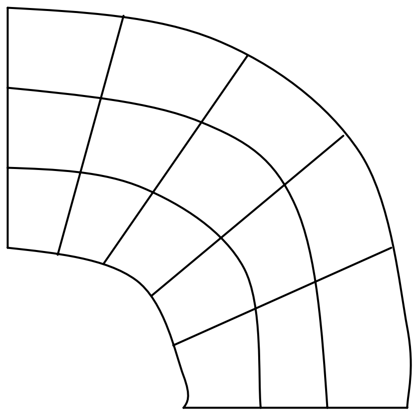


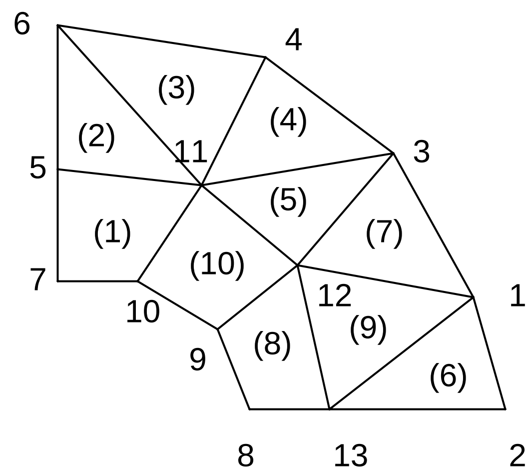
The finite difference method in 2D, cont.

- FDM is the best suited for *structured* grids, where it is possible for the grid point *i,j* to
  - compute its coordinates,
  - determine its neighbours.
- In a structured grid all internal the nodes
  - have the same number of neighbours,
  - have the same number of cells around them.
- A grid point is located at intersection of two lines, each belonging to one of the two families of lines.
- The lines are not limited to horizontal/vertical.

Structured and unstructured grids

Unstructured grids are more flexible but *connectivity* must be given explicitly





<u>connectivity</u>

elem (1): 5 7 10 11

elem (2): 11 6 5

elem (3): 4 6 11

etc.

structured grid

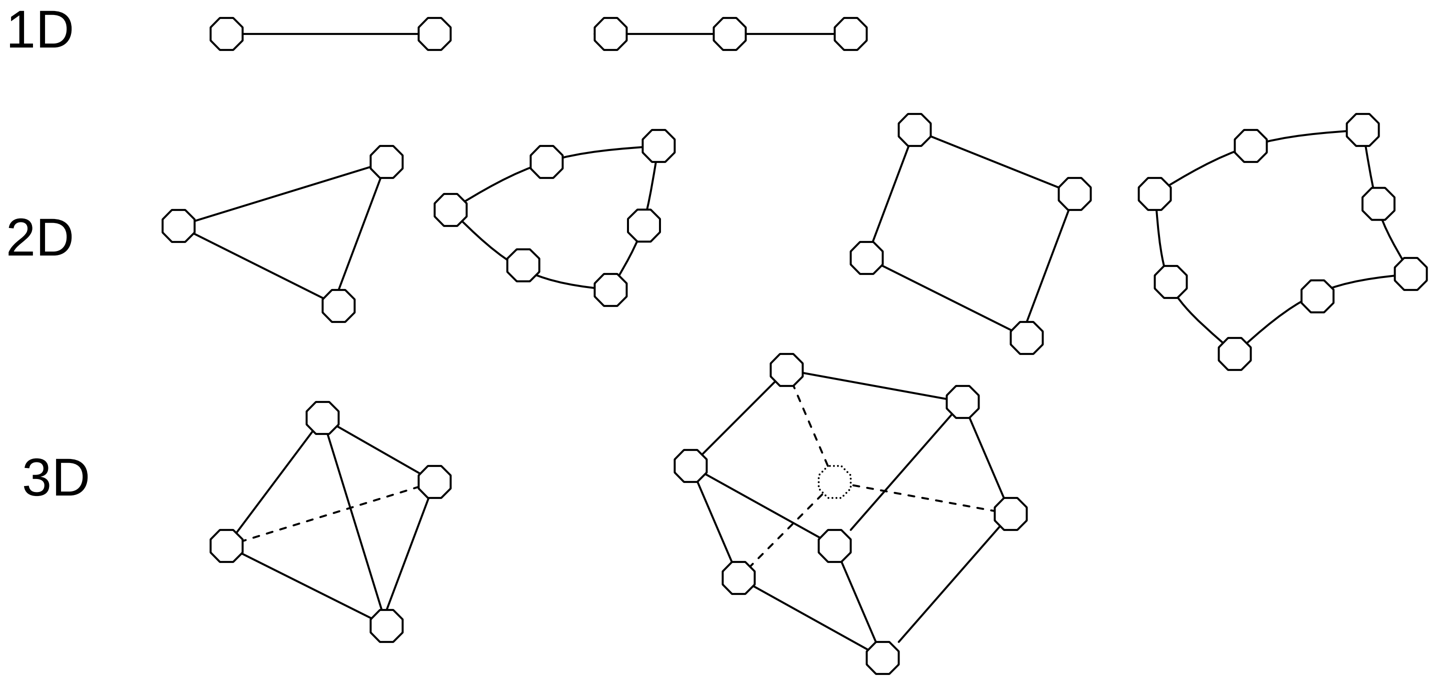
(can compute indices of element nodes)

unstructured grid

(no pattern in numbering)

The finite element method in 2D, cont.

- Finite element method shows its advantages on unstructured grids and complex geometries
- A great variety of shapes can be used



The finite element method in 2D, cont.

- Derivation of the weak problem and finite element equations is similiar to that for 1D

- The weak form

$$\int_{\Omega} k\left(\frac{\partial N_i}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial u}{\partial y}\right)d\Omega = \int_{\Omega} N_i f d\Omega - \int_{\Gamma} N_i q_n ds, \quad i = 0,1,\ldots,N$$

- Approximate solution in a triangular element *(e)* in terms of shape functions and nodal values

$$u_h^{(e)} = \sum_{j=0}^{2} N_j^{(e)}(x,y)\, u_j^{(e)} \qquad \frac{\partial u_h^{(e)}}{\partial x} = \sum_{j=0}^{2} \frac{\partial N_j^{(e)}(x,y)}{\partial x} u_j^{(e)}$$

The finite element method in 2D, cont.

- Finite element equations for the element *(e)*

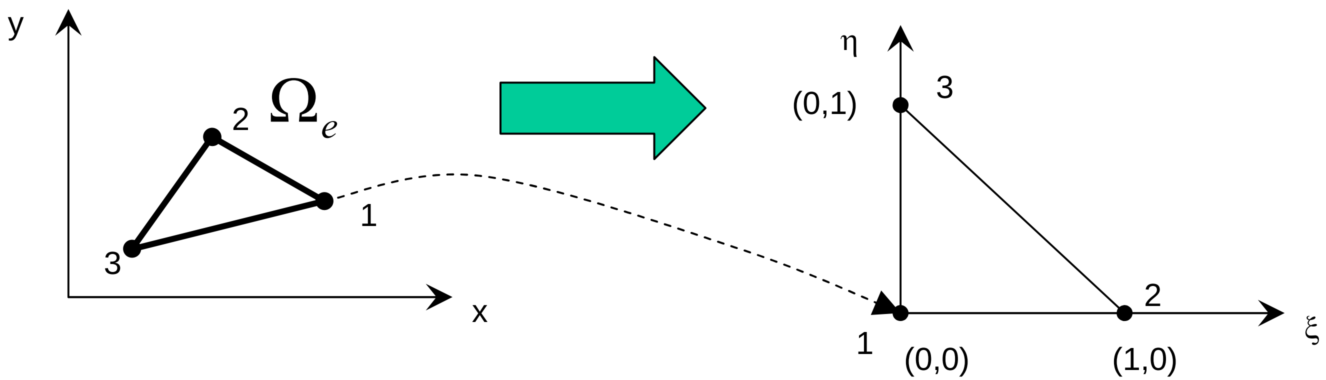
$$\int_{\Omega_e} k\left(\frac{\partial N_i^{(e)}}{\partial x}\frac{\partial N_j^{(e)}}{\partial x}+\frac{\partial N_i^{(e)}}{\partial y}\frac{\partial N_j^{(e)}}{\partial y}\right)d\Omega=\int_{\Omega_e} N_i^{(e)} f d\Omega-\int_{\Gamma_N\cap\partial\Omega_e} N_i^{(e)} q_n ds$$

$$i, j = 0,1,2$$

- It is a local system of equation. The size equals the number of nodes in the element (3 - triangle, 4 -quadrilateral, 6 - a second-order triangle)
- As usual, the global system of equations is assembled from such local systems

The finite element method in 2D, cont.

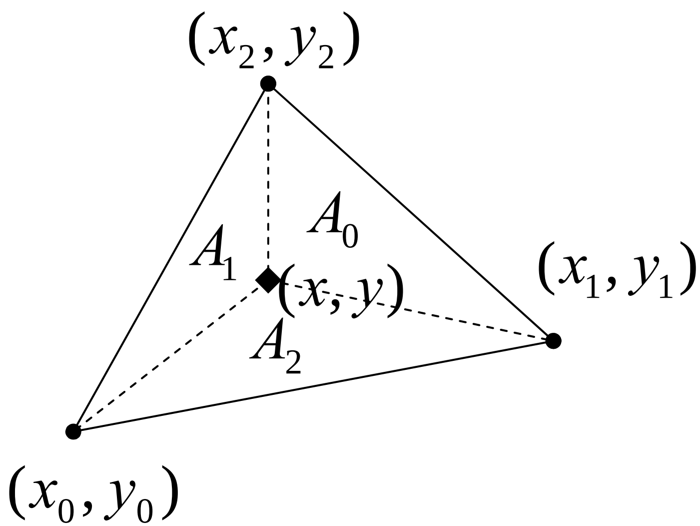
- Numerical integration is used for evaluation of volume and surface integrals
- It is much easier to integrate over regular triangles or squares. This involves *mapping*



$$\int_{\Omega_e} f(x,y)\,d\Omega = \int_0^1 \int_0^{1-\xi} f\left[x(\xi,\eta),\, y(\xi,\eta)\right] \left|\mathbf{J}(\xi,\eta)\right| d\eta\, d\xi$$

Shape functions of the linear triangular element

- Area coordinates



$$L_0(x,y) = A_0 / A$$
$$L_1(x,y) = A_1 / A$$
$$L_2(x,y) = A_2 / A$$

A - element area

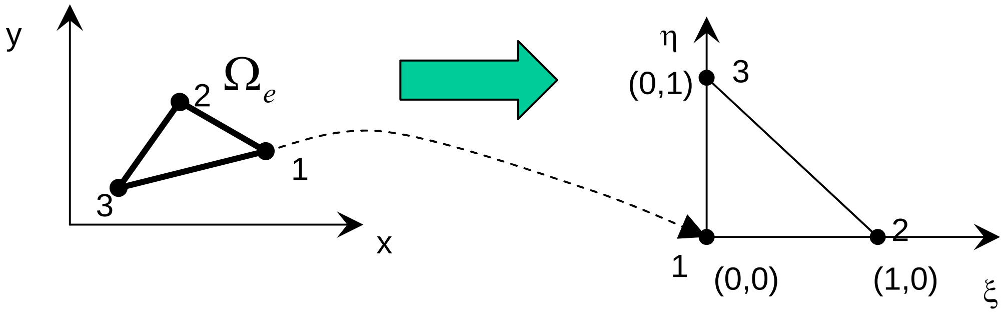
$$L_i = \begin{cases} 1 & \text{at the node } (x_i, y_i) \\ 0 & \text{at other nodes} \end{cases}$$

- Shape functions are the area coordinates

$$N_i = L_i, \quad i = 0,1,2$$

Coordinate mapping for the linear triangular element

$$N_0 = L_0 = 1 - \xi - \eta$$
$$N_1 = L_1 = \xi$$
$$N_2 = L_2 = \eta$$



$$x(\xi,\eta) = x_0 + (x_1 - x_0)\xi + (x_2 - x_0)\eta$$
$$y(\xi,\eta) = y_0 + (y_1 - y_0)\xi + (y_2 - y_0)\eta$$

$$\left|\mathbf{J}\right| = 2A \qquad \frac{\partial N_i}{\partial x} = \text{const}, \ \frac{\partial N_i}{\partial y} = \text{const}$$

The finite volume method in 2D

- Derivation of the balance equation is similiar to that for 1D

- Invoking the divergence theorem

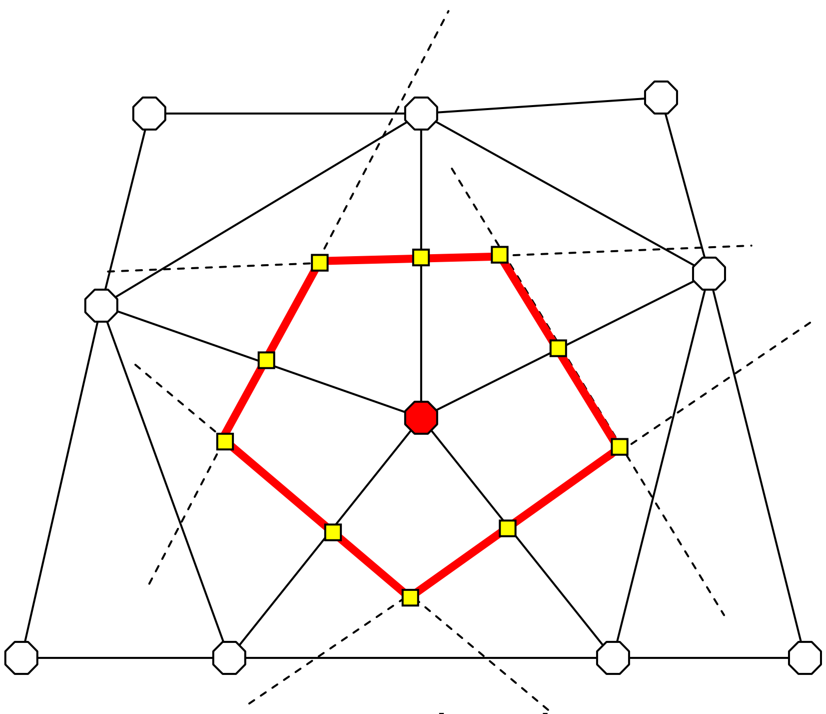
$$\int_{\Omega}\left[\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial u}{\partial y}\right)\right]d\Omega=\int_{\Gamma}k\left(\frac{\partial u}{\partial x}n_x+\frac{\partial u}{\partial x}n_y\right)ds=\int_{\Gamma}\left(-q_n\right)ds$$

- Integral conservation equation for the Poisson equation is simply

$$\int_{\Gamma}q_n ds=\int_{\Omega}f d\Omega$$

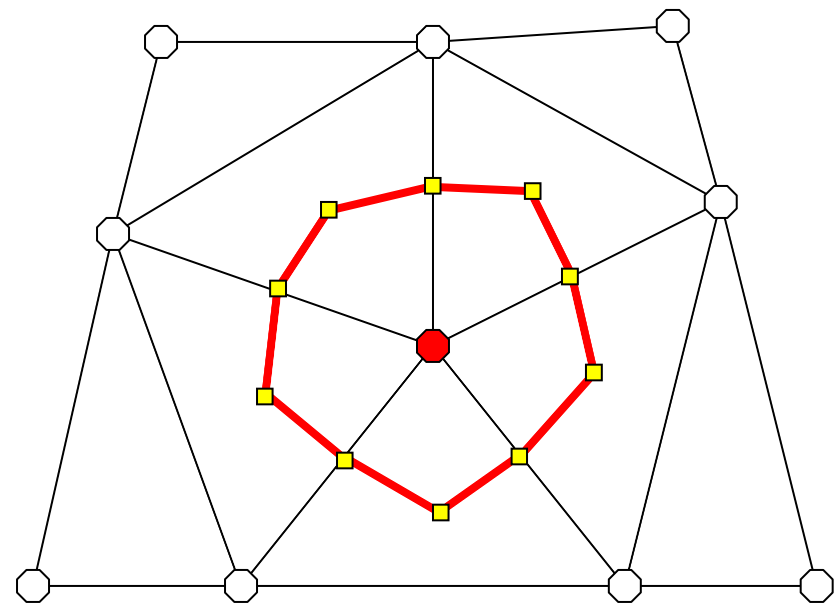
The finite volume method in 2D, cont.

Dual mesh is the mesh of finite volumes around nodes of a mesh of triangles or quadrilaterals



Voronoi region

(midpoints and circumcenters)

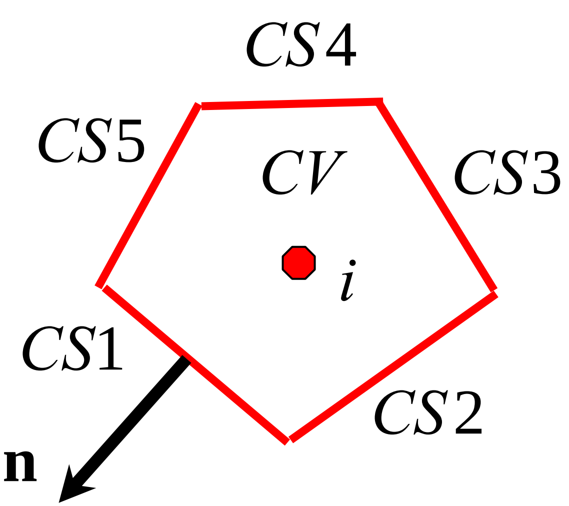


Median dual element

(midpoints and centers of gravity)

The finite volume method in 2D, cont.

Balance equation for a finite volume around node *i*



$$\int_{\partial\Omega_i} q_n ds = \int_{\Omega_i} f d\Omega$$

$$\sum_{i=1,2,\ldots} \int_{CS_i} q_n ds = \int_{CV} f d\Omega$$

$$\sum_{i=1,2,\ldots} \int_{CS_i} -k\left(\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y\right) ds = \int_{CV} f d\Omega$$

The finite volume method in 2D, cont.

- Integrals are evaluated numerically.

$$\int_{CV} f d\Omega \approx f(x_i, y_i) \cdot \mathrm{Volume}(CS)$$

$$\int_{CS} q_n ds \approx q_n(\mathrm{middle}) \cdot \mathrm{Area}(CS)$$

- Interpolation of partial derivatives in the middle of the control surface
  - central finite differences
  - using finite element shape functions