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ON THE GENERALISED HERSCHEL MODEL APPLIED TO BLOOD FLOW MODELLING

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Abstract

This paper introduces a new rheological model of blood as a certain generalisation of the standard Herschel-Bulkley model (Herschel and Bulkley, 1926). This model is a rheological constitutive equation and belongs to the group of the so-called generalised Newtonian fluids. Experimental data (Yeleswarapu et al, 1998) is compared with results, obtained from the new model, to demonstrate that it allows for the best agreement together with Luo-Kuang model (Luo and Kuang, 1992; Easthope and Brooks, 1980). The new model may be easily implemented into commercial CFD codes, which is not that obvious for more complicated models such as differential, integral and rate type fluids (Astarita and Marrucci, 1974; Tesch, 2012). What is more, it allows for modelling such phenomena as the shear thinning, yield stress and constant viscosity values at high shear rates.

Key words: non-Newtonian fluids, rheology

INTRODUCTION

Blood may be treated as a non-Newtonian fluid. This means that another constitutive equation is needed to close the system of equations describing blood motion. The non-Newtonian blood nature arises due to presence of red blood cells in the plasma. The mentioned equation belongs to the class of the the so-called mechanical (rheological) constitutive equations.

The ideal rheological constitutive equation allowing for proper and complete blood flow behaviour modelling should take under consideration flexibility and aggregation of the red blood cells, influence of temperature on viscosity, yield stress and thixotropy. It is hardly possible to satisfy all of these features. The more attributes are satisfied the better the model is. One has to keep in mind that then such a model is also more complicated.

MODELS

Typically, we can divide rheological constitutive equations into categories of Newtonian-, generalised Newtonian-, differential-, integral- and rate type fluids. For any case the stress tensor $\boldsymbol{\sigma}$ is decomposed into reversible and irreversible (viscous) part $\boldsymbol{\tau}$. If the density is constant we have

$$\boldsymbol{\sigma} = -p\boldsymbol{\delta} + \boldsymbol{\tau}. \tag{1}$$

For Newtonian fluids we have linear relationship between viscous part of the stress tensor $\boldsymbol{\tau}$ and strain rate tensor $\mathbf{D}$

$$\boldsymbol{\tau} = 2\mu\mathbf{D}, \tag{2}$$

where the dynamic viscosity μ is a factor of proportionality. The definition (2) is a generalisation of the following one-dimensional expression taken from an experiment

$$\tau = \mu\gamma. \tag{3}$$

All the fluids that do not fulfil equations (2) or (3) are known to be non-Newtonian.

Newtonian fluids

For Newtonian fluids the relationship between shear stress and shear rate is linear which also means that viscosity is constant

$$\begin{aligned} \tau &= \mu\gamma, \\ \mu &= \text{const}. \end{aligned} \tag{4}$$

The above constitutive equation will not allow for correct description of blood behaviour such as shear thinning and yield stress and many others. The only advantage of this model is that it keeps constant viscosity at high shear rates.

Generalised Newtonian fluids

Generalised Newtonian fluids satisfy the following rheological equation

$$\tau = \mu(\gamma)\gamma, \tag{5}$$

where the viscosity depends on shear rate. Despite the name these fluids are non-Newtonian. The Newtonian rheological equation (4) may be always obtained as a certain simplification of the selected generalised Newtonian fluid. The most popular generalised Newtonian fluids, easily applied to blood flow modelling, are Bingham, Ostwald-de Waele, Herschel-Bulkley, Casson and Luo-Kuang models.

The Bingham model (Bingham, 1916) expresses the shear rate and dynamic viscosity in the following manner

$$\begin{aligned} \tau &= \tau_0 + k\gamma, \\ \mu &= \frac{\tau_0}{|\gamma|} + k. \end{aligned} \tag{6}$$

We have an additional term responsible for yield stress τ_0 in comparison with Newtonian model (4). If $\tau < \tau_0$ the Bingham fluid behaves as a solid otherwise it behaves as a fluid. Except for ability of yield stress modelling it is not the best model for blood flow description. This is simply because it cannot mimic shear thinning. If $\tau_0 = 0$ we have the Newtonian fluid (4).

The Ostwald-de Waele (Ostwald, 1925; de Waele, 1923) or so-called the power-law model is given by

$$\begin{aligned} \tau &= k\gamma^n, \\ \mu &= k|\gamma|^{n-1}, \end{aligned} \tag{7}$$

where k is a flow consistency index and n is a flow index. It is probably the simplest model allowing for the shear thinning phenomenon. This is because of the dimensionless flow index n presents in equation (7). However, it suffers lack of yield stress. The disadvantages of this model is the problem of correct prediction of the viscosity at low and high stresses. For $n = 1$ we obtain the Newtonian fluid (4).

The Herschel-Bulkley model (Herschel and Bulkley, 1926) combines the two previous models, i.e. the Bingham and Herschel-Bulkley model. This results in

$$\begin{aligned} \tau &= \tau_0 + k\gamma^n, \\ \mu &= \frac{\tau_0}{|\gamma|} + k|\gamma|^{n-1}. \end{aligned} \tag{8}$$

This also means that it is now possible to model shear thinning behaviour and the yield stress. One has to keep in mind that the Herschel-Bulkley model inherits the disadvantages of the Ostwald-de Waele model, i.e. it cannot correctly predict blood behaviour at high and low shear stresses. For $\tau_0 = 0$ and $n = 1$ we can obtain the Newtonian fluid (4). It is also possible to obtain both the Ostwald-de Waele and Bingham models.

The Casson model (Mill, 1959) follows the definitions

$$\sqrt{\tau} = \sqrt{\tau_0} + \sqrt{k\gamma}, \qquad \sqrt{\mu} = \sqrt{\frac{\tau_0}{|\gamma|}} + \sqrt{k}. \tag{9}$$

It cannot be derived from the three above models. The advantages, however, are exactly the same as previously. It allows for shear thinning and the yield stress modelling. What is even more important this model is able to keep constant viscosity at high shear rates and is commonly used for blood flow modelling. For $\tau_0 = 0$ we have the Newtonian fluid (4). There is also the generalised Casson model. The only difference is that we do not have the power 0.5 ($\sqrt{\ }$) but an optional power m instead. This makes it even more flexible in the sense of shear thinnig level calibration.

The Luo-Kuang model (Luo and Kuang, 1992) is the best discussed blood model so far. It is defined by means of the following equations

$$\tau = \tau_0 + k\sqrt{\gamma} + \mu_\infty \gamma, \qquad \mu = \frac{\tau_0}{|\gamma|} + \frac{k}{\sqrt{|\gamma|}} + \mu_\infty, \tag{10}$$

where μ_∞ stands for constant viscosity at high shear rates. It cannot be derived from any previously discussed models. It allows for shear thinning behaviour modelling and for yield stress as well as for the correct prediction of viscosity at high stresses. It allows for the best blood behaviour modelling (Easthope et al., 1980) at least within the frame of generalised Newtonian fluids. For $\tau_0 = 0$ and $k = 0$ we have the Newtonian fluid (4).

The generalised Herschel model, introduced here, is given by the following definition

$$\tau = \tau_0 + k\gamma^n + \mu_\infty \gamma, \qquad \mu = \frac{\tau_0}{|\gamma|} + k|\gamma|^{n-1} + \mu_\infty. \tag{11}$$

There is a new rheological parameters n introduced here. It allows for better flexibility in comparison with the Luo-Kuang model (10). There are three components of the viscosity in equations (10) and (11) that may be easily distinguished. The first one $\tau_0|\gamma|^{-1}$ is responsible for yield stress. The second one $k|\gamma|^{n-1}$ is responsible for non-Newtonian behaviour and the last one μ_∞ which allows for constant viscosity at high shear rates. The combination of these components make these two models the best among models discussed in this paper. Similarly as for the Luo-Kuang model for $\tau_0 = 0$ and $k = 0$ we have the Newtonian fluid (4). Obviously for $n = 0.5$ we obtain the Luo-Kuang model (10).

Figure 1 shows relationships between all the generalised Newtonian models discussed in this paper except for the Szulman model. This model is a certain generalisation of the generalised Casson and Herschel-Bulkley models. There is no direct connection between the Szulman and Generalised Herschel models. The level of complexity of these two is more or less the same. Theoretically, the Szulman model allows for more than Casson model. However, one has to be careful fitting values of the constants because it is quite easy to end up with values that do not allow for constant viscosity at high shear rates.

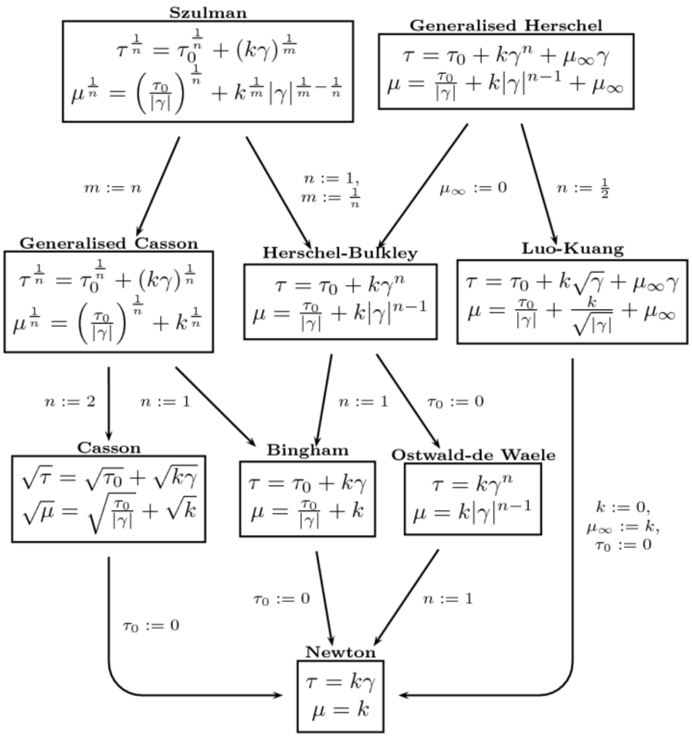

Fig. 1. Various models dependency

Differential type fluids

The viscous part of the stress tensor is expressed explicitly as a function of other tensors and theirs derivatives. Both being of kinematic nature. These may be Rivlin-Ericksen (Rivlin and Ericksen, 1955) tenors $\mathbf{A}_i$

$$\boldsymbol{\tau} = f\left(\mathbf{A}_1, \mathbf{A}_2, \ldots\right) \tag{12}$$

defined by means of the following recurrent equation

$$\mathbf{A}_{i+1} = \frac{d\mathbf{A}_i}{dt} + \mathbf{A}_i \cdot \frac{\partial \vec{U}}{\partial \vec{r}} + \nabla \vec{U} \cdot \mathbf{A}_i ; \quad i = 1,2,\ldots . \tag{13}$$

Integral type fluids

The viscous part of the stress tensor is expressed explicitly as a function of one ore more integrals of other tensors of kinematic character.

$$\boldsymbol{\tau} = \int_{-\infty}^{t} f\left(t-\tau\right)\left(\boldsymbol{\delta} - \mathbf{C}_t(\tau)\right) d\tau . \tag{14}$$

Rate type fluids

Equations describing rate type fluids are not explicit for the stress tensors. This simply means that the constitutive equations involves both, the viscous part of the stress tensor and its derivatives.

$$\boldsymbol{\tau} = f\left(\boldsymbol{\tau}, \mathbf{D}, \dot{\mathbf{D}}\right). \tag{15}$$

The most popular variants of rate type fluids are Maxwell and Oldroyd equations. The former follows the definition

$$\boldsymbol{\tau} + \lambda_1 \boldsymbol{\tau} = 2\mu \mathbf{D} . \tag{16}$$

It generalises Newtonian hypothesis by means of an additional term containing time derivative of the stress tensor. This term is responsible for the so-called fluid memory. λ_1

stands for relaxation time. The latter adds another term containing time derivative of the strain rate tensor

$$\boldsymbol{\tau} + \lambda_1 \dot{\boldsymbol{\tau}} = 2\mu\left(\mathbf{D} + \lambda_2 \dot{\mathbf{D}}\right) \tag{17}$$

and λ_2 is retardation time. It can be thought of the time needed to strain relaxation when the stress is removed.

COMPARISON WITH EXPERIMENTAL DATA

This paragraph shows comparison between experimental data and predictions of the generalised Newtonian models discussed earlier. The more complex models (integral, differential and rate type fluids) are not discussed here. This is simply because they cannot be easily implemented into commercial CFD codes.

Experimental velocity profiles (Yeleswarapu et al, 1998) were measured by means of Doppler velocimetry inside a straight plexiglass tube (0.25 inch diameter and 6 feet long). The investigated fluid was composed of porcine blood and 10% sodium citrate. The viscosity of the fluid was measured by means of Couette type and capillary viscometers. All experiments were performed at room temperature within six hours of blood collection.

Viscosity

Figure 2 shows comparison of various models vs experimental data for blood (Yeleswarapu et al, 1998). Values of constants such as τ_0, k and n were obtained by means of the least squares method. It is obvious that the Newtonian fluid model is the worst. The best agreement is achieved for the Luo-Kuang and generalised Herschel models. Models such as Ostwald-de Waele and Hereschel-Bulkley are not able to predict viscosity at high shear rates whereas for middle values of shear rates they fit perfectly. Surprisingly, the Casson model does not fit experimental data well except for constant values (overestimated) at high shear rates.

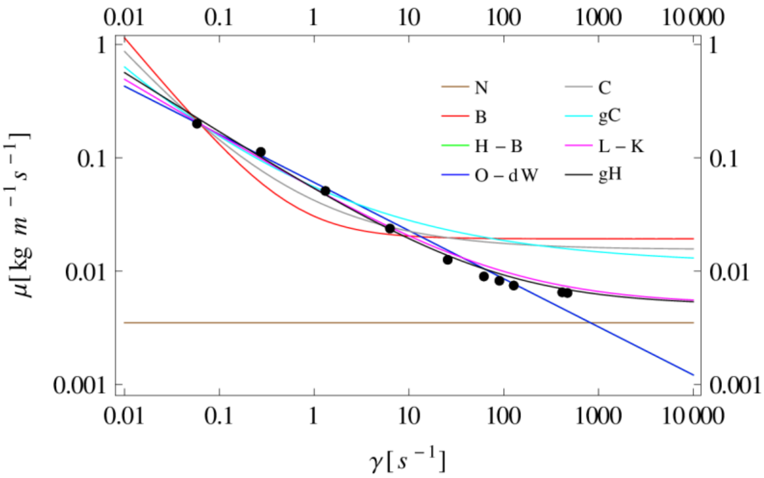

Fig. 2. Various models vs experimental data
(N – Newtonian model, B – Bingham, H-B – Herschel-Bulkley, O-dW – Ostwald-de Waele, C – Casson, gC – generalised Casson, L-K – Luo Kuang, gH – generalised Herschel)

Velocity

Figures 3-9 show comparison among various models and experimental data for three different volumetric flow rates. Velocity profiles may be either solved numerically or analytically (Tesch, 2012). Again, there is no surprise that the Newtonian and Bingham models do not predict the velocity profiles well. As for the rest they predict quite well. The best agreement was obtained for the Luo-Kuang and generalised Hereschel models.

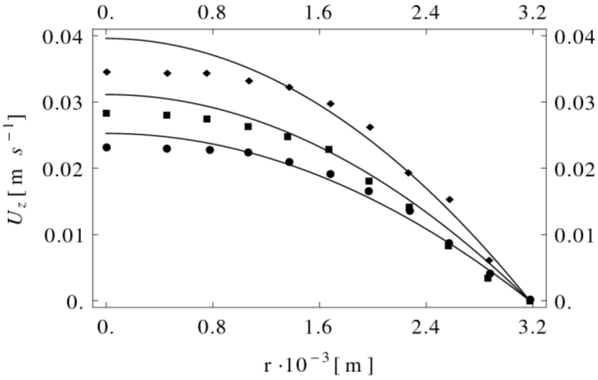

Fig. 3. Newton model

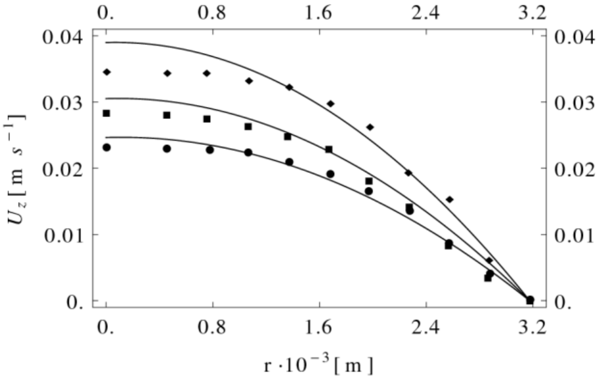

Fig. 4. Bingham model

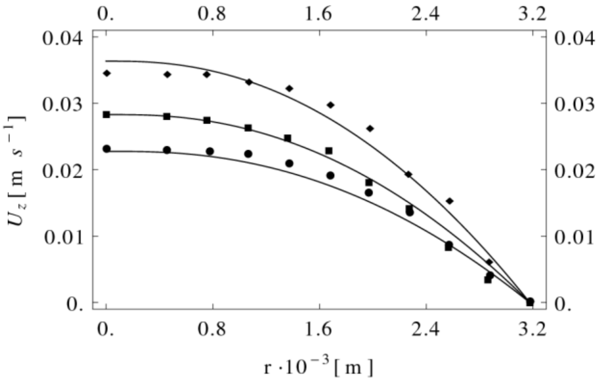

Fig. 5. Casson model

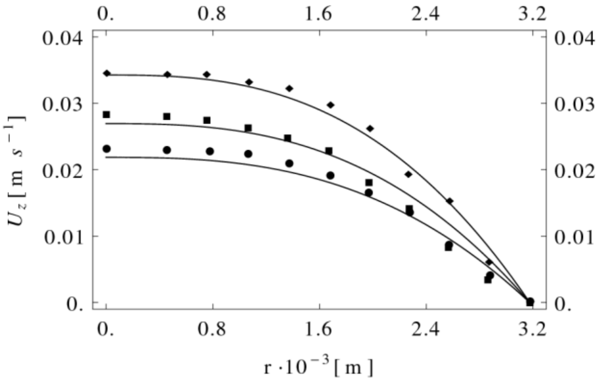

Fig. 6. Ostwald-de Waele model

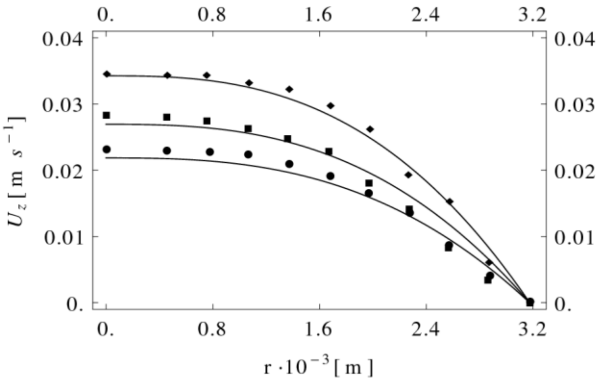

Fig. 7. Herschel-Bulkley model

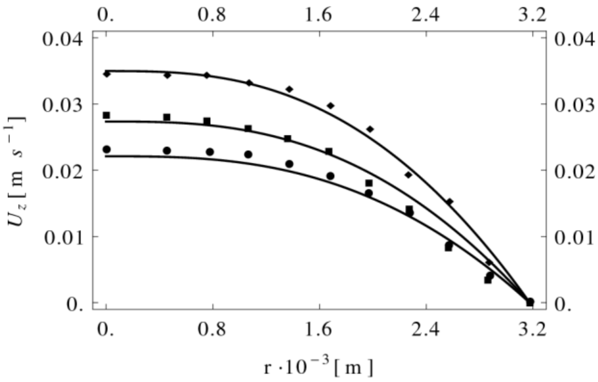

Fig. 8. Generalised Herschel-Bulkley model

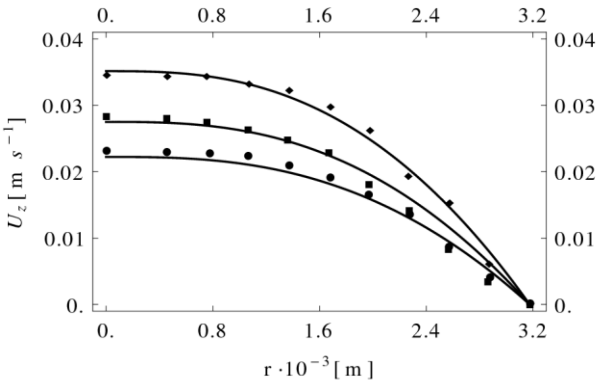

Fig. 9. Luo-Kuang model

CONCLUSIONS

The generalised Newtonian fluids are the simplest and easiest to implement into commercial CFD codes. The Luo-Kuang and generalised Herschel models allow for the best experimental data fitting. This has been shown in this paper. More advanced models such as Oldroyd and Maxwell have potential for even better approximation of blood features but they cannot be directly implemented into commercial CFD codes. Generally speaking, there is not any universal constitutive equation that is able to model all the features of blood. This means that there is still need for further research in this field.

REFERENCES

Astarita, G., Marrucci, G. (1974): *Principles of non-Newtonian Fluid Mechanics*, London: McGraw-Hill

Bingham, E. C. (1916): *An Investigation of the Laws of Plastic Flow*, U.S. Bureau of Standards Bulletin, Vol. 13, pp. 309–353

Mill, C. C. (1959): *Rheology of Disperse Systems*, Pergamon Press

Easthope, P.L., Brooks, D.E. (1980): *A Comparison of Rheological Constitutive Functions for Whole Human Blood*, Biorheology, Vol. 17, pp. 235–247

Herschel, W.H., Bulkley, R. (1926): *Konsistenzmessungen von Gummi-Benzolloesungen*, Kolloid-Zeitschrift, Vol. 39, No. 4, pp. 291–300

Luo, X., Kuang, Z.B. (1992): *A Study on the Constitutive Equation of Blood*, Journal of Biomechanics, Vol. 25, No. 8, pp. 929–934

Ostwald, W. (1925): *Ueber die Geschwindigkeitsfunktion der Viskositaet disperser Systeme I*, Kolloid-Zeitschrift, Vol. 36, pp. 99–117

Rivlin, R. S., Ericksen, J. L. (1955): *Stress-deformation Relations for Isotropic Materials*, Journal of Rational Mechanics and Analyses, Vol. 4, pp. 323–425

Tesch, K. (2012): *Selected Topics of Blood Flows and Microclimate Modelling in Protective Suits (in Polish)*, Gdańsk University of Technology Press, Gdańsk

De Waele, A. (1923): *Viscometry and Plastometry*, J Oil Colour Chem Assoc, Vol. 6, pp. 33–69

Yeleswarapu, K.K., Kameneva, M.V., Rajagopal, K.R., Antaki, J.F. (1998): *The Flow of Blood in Tubes: Theory and Experiment*, Mechanics Research Communications, Vol. 25, No 3, pp. 257–262

NOMENCLATURE

$\mathbf{A}_i$	Rivlin-Ericksen tensor
$\mathbf{C}_t$	Left Cauchy-Green tensor
$\mathbf{D}$	Strain rate tensor
f	Function
k	Flow consistency
n	Flow index
p	Pressure

r	Radius
γ	Shear rate
$\boldsymbol{\delta}$	Kronecker delta
λ_1	Relaxation time
λ_2	Retardation time
μ	Viscosity
μ_∞	Viscosity at high shear rates
$\boldsymbol{\sigma}$	Stress tensor
$\boldsymbol{\tau}$	Viscous part of the stress tensor
τ	Shear stress
τ_0	Yield stress
$\nabla\vec{U}$	Velocity gradient
$\frac{\partial\vec{U}}{\partial\vec{r}}$	Strain rate tensor