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COLLAPSE OF N VORTICES

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Abstract

Numerical approximations of finite systems of vortices in the plane, collapsing to a point in finite time are studied. A fast method leading to highly accurate approximations is presented. The method is illustrated with an example of an approximate collapsing system of seven vortices. Special attention is given to the behaviour of the system near critical time (close to the collapse).

Key words: collapsing vortex configurations

INTRODUCTION

The importance of the point vortices in applications is due to the dominant role of the coherent vortical structures in many two-dimensional turbulent flows. Point-vortex dynamics is responsible for bringing the vortices together and in this way it determines what kind of a merging process will occur. The number of vortices in a flow can be quite large; this makes the complicated dynamical description intractable. On the other hand, few-vortex systems can be investigated in much more detail. The evolution of such systems has been studied for more than 130 years.

In 1883 Kirchhoff showed that the motion can be put into the Hamilton framework. Systems of three vortices are integrable since they possess enough Poisson commuting invariants. It was quite surprising that there exist three-vortices systems whose evolution leads to a collapse to a point in finite time. This phenomenon can be treated as a change of scale, characteristic for turbulent motion. A deeper insight into the way systems of vortices can collapse could thus lead to a better understanding of turbulence itself.

Collapsing systems of three vortices were first described by Groebli in 1877 [3] and rediscovered by Aref (see [1]) and independently by Novikov and Sedov [4] around 1979. The motion of four vortices is no longer integrable in general. Nevertheless in 1979 Novikov and Sedov gave some special, explicit examples of collapsing systems of four and five vortices.

In 1987 O'Neil proved ([5]) the existence of collapsing systems of n vortices for arbitrary n . The proof makes use of a system of algebraic equations that self-similar collapsing configurations should satisfy. O'Neil proved that for some circulations the set of solutions is a non-empty algebraic curve in the configuration space. Thus solutions exist, although

the proof does not indicate how they should be found. In fact [5] does not provide any examples for the interesting case $n \geq 6$.

We solve numerically the above mentioned (non-linear) algebraic system of equations and obtain collapsing configurations for many circulations for six or more vortices. To the best of our knowledge no such examples have appeared in the literature so far. A precise description of the algorithm used is given below. The solutions we found are taken as initial conditions to the differential evolution equation. Standard numerical procedures give then trajectories whose collapsing property can be directly seen. A sample collapsing system of seven vortices is shown in Fig. 3, 4 and 5. Note that in order to obtain a system whose diameter lessens sufficiently during evolution, it is necessary to use extremely high precision both when calculating the initial state and when solving the evolution equation.

It may be worth mentioning that although the O'Neil's results on existence of collapsing configurations give some moral support in seeking such configurations, nevertheless our numerical results do not rely on his theorems. The examples we found prove independently that collapsing configurations of n vortices do exist, obviously only for those values of n , for which the calculations have been performed. In particular we have examples of collapsing configurations for such circulations, for which O'Neil's theorems do not work.

BASIC NOTIONS AND FACTS

We start with a review of some basic notions and facts, which will help us put our results in a proper perspective. For this basic material one can consult e.g. [5], [2].

A two-dimensional fluid motion can be discretized by dividing the vorticity field into regions U_i and replacing each of them by a point vortex z_i given a suitable circulation Γ_i . The idea can be traced back to Helmholtz and Kirchhoff. The equations of motion of the system $z(t) = (z_1, \dots, z_n) \in \mathbb{C}^n$ of n vortices are

$$\frac{dz_k(t)}{dt} = V_k(z(t)) = \frac{i}{2\pi} \sum_{l=1, l \neq k}^n \Gamma_l \frac{z_k - z_l}{|z_k - z_l|^2}$$

with nonzero real numbers Γ_l . Let's observe that our dynamical system is a hamiltonian system with respect to the symplectic form

$$\Omega(z, w) = -Im \sum_{k=1}^n \Gamma_k z_k \bar{w}_k = \sum_{k=1}^n \Gamma_k dx_k dy_k.$$

This means that for some hamiltonian total energy function H the vector field V should be dual to dH with respect to Ω . By definition we need $i_V \Omega = dH$ or equivalently $\Omega(V, U) = UH$ (the directional derivative of H along U) for any vector field U . In fact

$$H = -\frac{1}{2} \sum_{k < l} \Gamma_k \Gamma_l \log |z_k - z_l|$$

would do.

Let's introduce the size $S(z) \in \mathbb{R}$ and the moment of vorticity $M(z) \in \mathbb{C}$ by

$$S(z) = \sum_k \Gamma_k |z_k|^2, \quad M(z) = \sum_k \Gamma_k z_k.$$

Observe that our hamiltonian system is rotationally and translatory invariant and therefore, except for the hamiltonian H itself, it has three additional integrals of motion (that is real functions conserved along the trajectories): S and the real and imaginary part of M . It follows that the dynamical system of three vortices is integrable.

Two systems of vortices z and w are said to form the same configuration if they are similar i.e., $w_k = az_k + b$ for some complex a, b with $a \neq 0$. The configuration space for n vortices has dimension $2n - 4$ since the group of similarities $z \rightarrow az + b$ is four-dimensional. It is easy to see that $w_k = az_k + b$ implies $V(w) = \frac{1}{a}V(z)$ and therefore any trajectory $t \rightarrow z(t)$ gives a trajectory $t \rightarrow az\left(\frac{t}{|a|^2}\right) + b$ with initial conditions $az(0) + b$. We express this by saying that similar systems are dynamically equivalent. A system $z(0)$ is said to collapse if for some $t_0 > 0$ its trajectory $z(t)$ converges to a point. All systems similar to $z(0)$ also collapse; thus the collapsing families are at least four-dimensional. In fact, as O'Neil's work shows, quite often they are five-dimensional. It is more convenient to speak of collapsing configurations instead of collapsing systems: the families described by O'Neil are one-dimensional (form algebraic curves in the configuration space).

A system is called self-similar if it remains similar to the initial state during evolution. This amounts to say that the trajectory $z(t)$ is a fixed point in the configuration space. One also says its configuration is stationary. It is believed that any collapsing system must be self-similar. Therefore all collapsing systems considered in this paper are assumed to be self-similar.

Lemma 1. *Let w be a system of vortices and $z(t)$ the trajectory starting at w . The following conditions are equivalent.*

- (a) *The system w is self-similar.*
- (b) $V_k(w) - V_l(w) = \omega(w_k - w_l)$ for some $\omega \in \mathbb{C}$.
- (c) $V_k(w) = \omega w_k - p$ for some $\omega, p \in \mathbb{C}$.
- (d) *The system w belongs to one of the following five classes.*
 - (i) *stationary:* $V_k(w) = 0$ or equivalently $z(t) = w$,
 - (ii) *translatory:* $V_k(w) = v \neq 0$ or equivalently $z(t) = w + tv$,
 - (iii) *rotational:* $V_k(w) = i\lambda(w_k - p)$, $\lambda \neq 0$, $z(t) = p + e^{i\lambda t}(w - p)$,
 - (iv) *collapsing:* $V_k(w) = \omega(w_k - p)$, $Re(\omega) < 0$,

$$z(t) = p + \sqrt{2Re(\omega)t + 1} e^{i\frac{Im(\omega)}{2Re(\omega)} \ln(2Re(\omega)t + 1)} (w - p),$$

- (v) *expanding:* $V_k(w) = \omega(w_k - p)$, $Re(\omega) > 0$, $z(t)$ given by the same formula as for the collapsing system.

Note that a collapsing trajectory is defined for $t \in (-\infty, t_0)$, $t_0 = \frac{-1}{2Re(\omega)} > 0$, and that in fact $\lim_{t \rightarrow t_0} |z_k(t) - p| = 0$. Note also that ω in (b), (c) and (d) is the same. The zero value of ω corresponds to (d-i..ii). A system is collapsing, expanding or rotational (d-iii..v) iff it satisfies (b) or (c) with $\omega \neq 0$.

NECESSARY CONDITIONS FOR COLLAPSING

In this section we shall specify and reformulate algebraic conditions that collapsing configurations and their circulations should satisfy. These equations will be solved numerically in the following sections.

We start with conditions for circulations. For a circulation $\Gamma = (\Gamma_k)_k$ we define the angular momentum L and the total circulation σ by

$$L = \sum_{i < j} \Gamma_i \Gamma_j, \quad \sigma = \sum_i \Gamma_i.$$

Lemma 2. *If a collapsing configuration exists then necessarily*

$$L = 0, \quad \sigma \neq 0.$$

Proof. Suppose $z(t)$ is a trajectory and for some t $z(t)$ is similar to $z(0)$ with the proportionality factor $c \neq 1$. Thus $\log |z_k(t) - z_l(t)| = \log c |z_k(0) - z_l(0)| = \log |z_k(0) - z_l(0)| + \log c$ and $H(z(t)) = H(z(0)) - \frac{1}{2}L \log c$. Since the hamiltonian is constant along the trajectory and $\log c \neq 0$, L must be zero. Furthermore

$$\sigma^2 - 2L = \left(\sum_i \Gamma_i \right)^2 - \sum_{i \neq j} \Gamma_i \Gamma_j = \sum_i \Gamma_i^2 > 0,$$

so that $L = 0$ implies $\sigma \neq 0$. □

From now on we assume that $L = 0$, $\sigma \neq 0$.

Lemma 3. *If $\sigma \neq 0$ then any system z admits a unique translate $w = z - p$ (which means that $w_k = z_k - p$ for some $p \in \mathbb{C}$), such that $M(w) = 0$.*

Proof. The moment of the translate is

$$M(w) = M(z - p) = \sum \Gamma_k (z_k - p) = M(z) - p\sigma.$$

Thus $p = M(w)/\sigma$. □

It follows that each configuration class contains a representative satisfying $M = 0$ and that representative is unique up to a complex factor. In other words the configuration space is in a natural one-to-one correspondence with the complex projective space $\mathbb{C}P(W)$ over the kernel W of the linear map M . This will be of practical importance since now a set of configurations can be specified by a system of homogeneous equations, one of them being $M = 0$.

Lemma 4. *(See [5], Lemma 1.2.4 and Lemma 1.2.7) Suppose that $L = 0$, $\sigma \neq 0$, $M(w) = 0$. Then the following conditions are equivalent*

- (a) w is self-similar and expanding, collapsing or rotational.
- (b) $V_k(w) - V_l(w) = \omega(w_k - w_l)$ for any k, l and some non-zero $\omega \in \mathbb{C}$.
- (c) $V_k(w) = \omega w_k$ for any k and some non-zero $\omega \in \mathbb{C}$.
- (d) at least one $V_j(w)$ is non-zero and for some fixed k the equality $w_l V_k(w) = w_k V_l(w)$ is satisfied for all indices l .

- (e) at least one $V_j(w)$ is non-zero, $S(w)=0$ and furthermore for some fixed k the equality $w_l V_k(w) = w_k V_l(w)$ is satisfied for at least $n-3$ indices l different from k .

Proof. The previous lemma tells us that (a) and (b) are equivalent. Obviously (c) implies (b). If (b) is satisfied then $p = V_k - \omega w_k$ is independent of k and

$$p\sigma = \sum_k \Gamma_k (V_k - \omega w_k) = \sum_k \Gamma_k V_k - \omega M(w) = \sum_k \Gamma_k V_k = 0.$$

Thus $p = 0$. Clearly (c) and (d) are equivalent. Assume (c). Then

$$\omega S = \sum_k \Gamma_k \omega w_k \bar{w}_k = \sum_k \Gamma_k V_k \bar{w}_k = iL = 0$$

and (e) follows. Conversely (see [5], Lemma 1.2.7), assume $k = 1$ and let $D_l = V_1 w_l - V_l w_1$ be zero for $l = 4..n$. We have

$$\begin{aligned} \Gamma_2 D_2 + \Gamma_3 D_3 &= \sum_l \Gamma_l (V_1 w_l - V_l w_1) = V_1 M = 0, \\ \Gamma_2 \bar{w}_2 D_2 + \Gamma_3 \bar{w}_3 D_3 &= \sum_l \Gamma_l w_l (V_1 w_l - V_l w_1) = V_1 S - w_1 (iL) = 0. \end{aligned}$$

This set of equations has non-zero determinant as $w_2 \neq w_3$, and therefore $D_2 = D_3 = 0$. \square

SOME ANALYTICAL RESULTS ON COLLAPSING

Collapsing systems of three vortices

Collapsing systems of three vortices were first described by Groebli in 1877. The configuration space is two-dimensional. The equations given in the previous section amount to

$$M = \sum \Gamma_k z_k = 0, \quad S = \sum \Gamma_k |z_k| = 0.$$

For dimensional reasons we expect the solution set to be one-dimensional. In fact the above equations can be easily transformed to one quadratic equation in two variables which determines a circle in the plane.

Collapsing systems of four and five vortices

The motion of four vortices is no longer integrable. Nevertheless in 1979 Novikov and Sedov gave explicit examples of collapsing systems of four and five vortices. For four vortices their approach works basically for only one specific circulation and doesn't give all possible collapsing systems even for that circulation. From our point of view the specific circulation has the property that the polynomial system of equations factorizes so that one component of the solution set (in the configuration space) forms a circle (or ellipse) in a plane contained in the projective configuration space. These examples can be therefore thought of as prototypes of a general situation described by O'Neil.

O'Neil's results

We shall briefly sketch those results obtained by O'Neil in 1987 which are of importance for this paper. In his setting the existence of collapsing configurations for arbitrary n follows from two theorems.

Theorem 1. (O’Neil [5] Theorem 7.1.1) Suppose $\Gamma_l > 0$ for $l = 1, \dots, n-1$, $\Gamma_n < 0$, and $L = 0$. Then there are at least $s(n-2)!$ collinear rotational configurations, where s is the number of pairs (l, n) such that $\Gamma_l + \Gamma_n > 0$.

Theorem 2. (O’Neil [5] Theorem 7.4.1) Let $n > 3$. For each complex Γ satisfying $\Gamma_1 = 1$ and $L = 0$, except for a subvariety of codimension 1, every collinear rotational configuration lies on a one-dimensional family of collapsing configurations. Each family is a submanifold except at a finite number of points.

APPROXIMATIONS OF COLLAPSING CONFIGURATIONS

In this section we shall describe an approach leading to high-accuracy numerical approximations of collapsing configurations. We shall illustrate the approach with a specific example of a collapsing system of seven vortices.

In the preceding sections we reviewed some algebraic conditions for a system of vortices to be collapsing. Write such a system of equations as

$$f(z) = 0, \quad f: \mathbb{C}^n \rightarrow \mathbb{R}^m.$$

We find a numerical solution of this system of equations using a basically very simple approach: take an arbitrary point in \mathbb{C}^n and follow the integral curve of the vector field $-\nabla g$, $g(z) = |f(z)|^2$, as the method of steepest descent tells us to do, until we obtain a local minimum of g . Accept this point as a zero of f if the value $f(z)$ is small enough.

Several issues require some care. First of all it is not clear what range for the initial points should be taken. We were lucky enough to obtain solutions for many randomly taken initial points. Secondly it is not clear how small the value of $g(z)$ should be in order to be accepted. Besides the steepest descent method works fine for a vigorously changing function, but is much less efficient when f is close to zero, since the derivatives of g are close to zero then. We decide to accept a point for which the steepest descent method gives a value of $g(z)$ of several orders smaller than the initial value, and change the method to one working like the Newton’s method then. This enables us to obtain solutions of accuracy of several dozen (or even several hundred) decimal digits. The importance of high accuracy will be made clear later.

Another issue is the unique specification of the solution. The O’Neil’s results suggest that the collapsing configurations should form one-dimensional families (smooth curves). Such a curve projects by a submersion onto at least one coordinate axis. It follows that by fixing a value of this coordinate we should obtain a system of equations with discrete solutions. This means that an initial point close to a solution should specify that solution as a unique solution in some explicitly given neighbourhood of that point.

Finally let’s mention an important question of whether the approximate solution we found is really close to a true (precise) solution of the system. This has been taken care of by an approach using the implicit function theorem. Without going into details note that to this end it is enough to bound second derivatives of f in a neighbourhood of the point we found and show that both the value of f and the norm of the inverse of the first derivative matrix at the point is small enough. Note also that the rigorous proof of the existence

of collapsing systems thus obtained does not rely on O'Neil's results. In particular our approach works for diverse circulations, not necessarily meeting the conditions given in the first O'Neil's theorem. Moreover the collapsing configurations we obtain have no direct relation to any collinear rotational configuration, as the second O'Neil's theorem requires.

We shall illustrate the above procedure for seven vortices and for the circulation $\Gamma = 2\pi (2, 2, -4, -4, -4, -4, 3)$. We have $L = 0$ and $\sigma = -9 \cdot 2\pi \neq 0$, so that the necessary condition on Γ for the existence of collapsing configurations is fulfilled. We know from one of previous sections that collapsing configurations of n vortices can be represented by solutions of the following system of algebraic equations:

$$M(z) = 0, \quad S(z) = 0, \quad V_1 z_k = V_k z_1 \quad (2 \leq k \leq n-2).$$

For $n = 7$ it corresponds to eleven real equations. These equations are complex homogeneous and therefore we can fix any non-zero complex coordinate of the solution. By random search we find that the point $P = (-1.31 + 10.00i, 4.51 + 5.39i, -1.27 + 7.59i, 1.21 + 2.06i, -0.58 - 1.00i, 2.99 - 0.95i, 1.00 + 0.00i)$ may be close enough to a solution. Therefore we choose to complete our set of equations with $z_7 = 1$ and $y_1 = \text{Im}(z_1) = 10$. Now we have fourteen equations; we expect this restricts the set of solutions to a finite set. Thus our aim is to solve the equation $f(z) = 0$ for

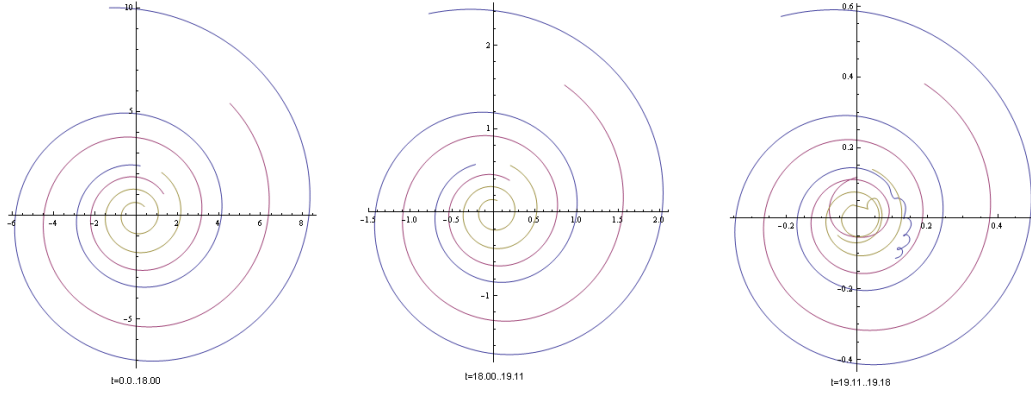
$$\mathbb{C}^7 \ni z \rightarrow (V_1 z_2 - V_2 z_1, V_1 z_3 - V_3 z_1, V_1 z_4 - V_4 z_1, V_1 z_5 - V_5 z_1, \\ z_7 - 1, \sum_k \Gamma_k z_k, \sum_k \Gamma_k |z_k|^2, \text{Im}(z_1) - 10) \in \mathbb{C}^6 \times \mathbb{R}^2 \approx \mathbb{C}^7.$$

By the method of steepest descent we follow the integral curve of the vector field $-\nabla g$ for $g(z) = |f(z)|^2$, starting from P (given above). The method allows us to arrive at a point Q where the value $g(Q) = 1.65 \cdot 10^{-6}$, which is smaller than $g(P) = 7.29$ by a factor of $2.26 \cdot 10^7$. The values y_1 and z_7 are kept fixed and the other coordinates change by less than 0.01. A measure of quality of the obtained point is the value $\omega = (V_k - V_l)/(z_k - z_l)$, which should be independent of k, l and have negative real part. At Q the real part of ω varies between -0.02625 and -0.02595 , and $\text{Im}(\omega)$ is in $[-0.23185, -0.23118]$.

Now we change the method of seeking a solution to one of Newton's type, which allows us to obtain a solution practically with as high accuracy as we wish. With the standard machine precision of around 18 digits we can get immediately a point R with $g(R) \leq 10^{-34}$ and with 17 accurate digits in $\text{Re}(\omega)$. The expected collapse time is $t_\infty = \frac{-1}{2\text{Re}\omega} \approx 19.1806$.

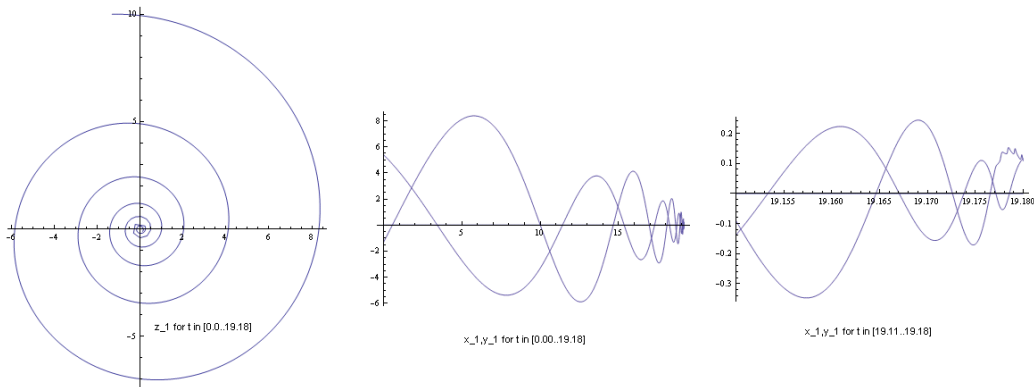
The three pictures in Fig. 1 contain the trajectories of three sample vortices z_1, z_2 and z_4 for time in the range $[0.0, 18.0]$, $[18.00, 19.11]$ and $[19.11, 19.18]$ respectively. Note the scale change in the pictures. As the distance $|z_1(t) - z_2(t)|$ changes from 7.423 at $t = 0$ to 0.1866 at $t = 19.168$, the collapsing scale is almost 40. The trajectories would seem to truly collapse to a point in a single low-resolution picture. In the third picture suitable magnification shows that for time close to 19.18 the self-similarity of the system during evolution is lost. An attempt to more deeply understand what happens near the critical (collapsing) time is deferred to the next section. Figure 2 shows the trajectory of z_1 in its full time-range. We also show the plot of the real and imaginary part of z_1 , separately for

FIGURE 1. Trajectories of z_1 , z_2 and z_4 for time in different ranges.



$t \in [0.0, 19.18]$, and separately in the critical range $[19.15, 19.18]$. Again the loss of self-similarity is clearly seen for t close to 19.18. Figures 3 to 5 show the streamlines of the

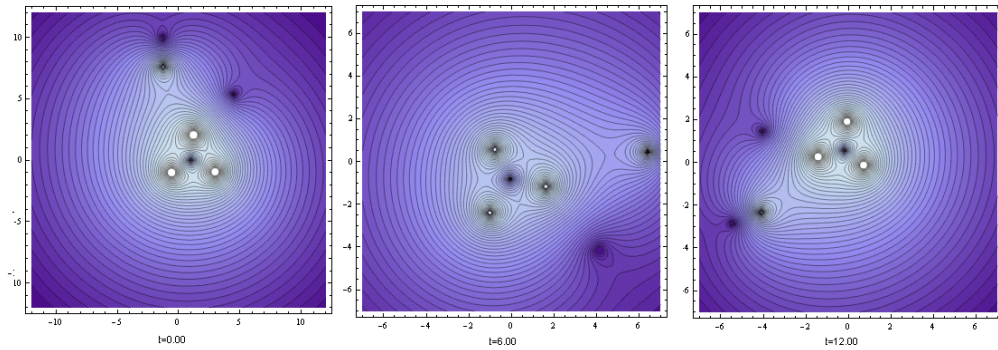
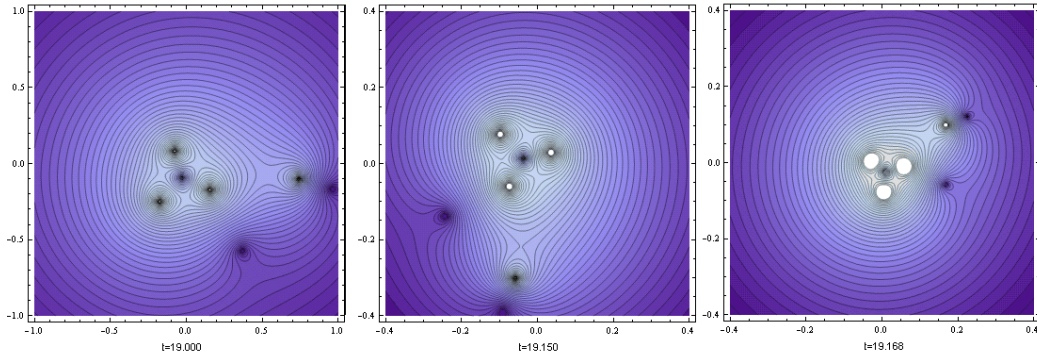
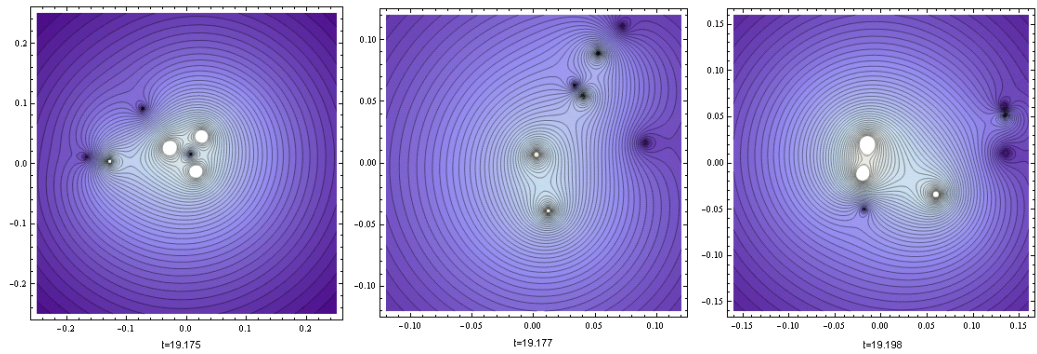
FIGURE 2. Behavior of z_1 and $x_1 = \text{Re}(z_1)$, $y_1 = \text{Im}(z_1)$ for $t \in [0.0, 19.18]$.



flow at different moments of time. Note that the scale changes as the system collapses. For time in $[0.0, 19.175]$ the system retains self-similarity - the only visible change is rescaling and rotation. The state of the system in two last pictures, for $t = 19.177$ and $t = 19.198$, clearly is not similar to the previous states. Moreover in the last picture the system begins to grow. Note also that if the scale used for $t = 0.0$ was applied to the last state ($t = 19.177$), it would be impossible to visibly distinguish the system from a single point vortex.

COLLAPSING NEAR CRITICAL TIME

In this last section we want to address a question which in our opinion is quite exciting and deserves further investigation.

FIGURE 3. The streamlines of the flow for $t = 0.0, 12.0$ FIGURE 4. The streamlines of the flow for $t = 19.000, 19.168$ FIGURE 5. The streamlines of the flow for $t = 19.175, 19.198$ 

Numerical examples of collapsing configurations are given with some definite accuracy and the best one can expect is that their evolution resembles true collapse only in some interval $[0, T]$ with $T \approx t_\infty$. When time nears t_∞ then the system should stay self-similar with diminishing diameter, but after passing T the motion becomes chaotic and the diameter goes up. It turns out that in order to obtain a system whose diameter lessens by a

factor of 10^3 it may be necessary to know the initial state of the system with the accuracy of several dozen decimal digits. This behavior is a feature of the dynamical system and not caused by the inaccuracy of numerical computations. We illustrate this issue by taking different approximations of the collapsing system studied above and presenting the behavior of those approximate solutions during evolution for times close to the collapsing time t_∞ of the system. We pay special attention to the collapsing scale, that is the ratio of the diameter of the system at the initial state and the minimal diameter of the system during evolution. We want to observe the dependence of the collapsing scale on the accuracy of the initial state.

Using *Mathematica*, we first solved the algebraic system of equations with some precision (the number of decimal digits used) in the range [10..200], and then took the so-obtained approximate collapsing system as the initial data for the differential evolution equation, which was solved with the same precision. We tried to find the minimal value of $|z_1(t) - z_2(t)|^2$. The minimum found and the corresponding value of time are given in the table below. The last column contains the calculated value of collapsing scale (the ratio of the initial diameter and the minimal one).

<i>prec</i>	<i>t</i>	$ z_1(t) - z_2(t) ^2$	<i>scale</i>
10	19.2320	3.4000	30
15	19.1080	0.2000	123
20	19.1390	0.0100	551
30	19.1695	0.0130	483
50	19.1791	0.0016	1377
80	19.1802	$5.5 \cdot 10^{-5}$	7430
90	19.1804	$9.8 \cdot 10^{-5}$	5566
100	19.1804	$2.0 \cdot 10^{-6}$	38965
120	19.1804	$1.0 \cdot 10^{-6}$	55105
150	19.1804	$2.3 \cdot 10^{-9}$	1149018
200	19.1804	$2.6 \cdot 10^{-10}$	3417467

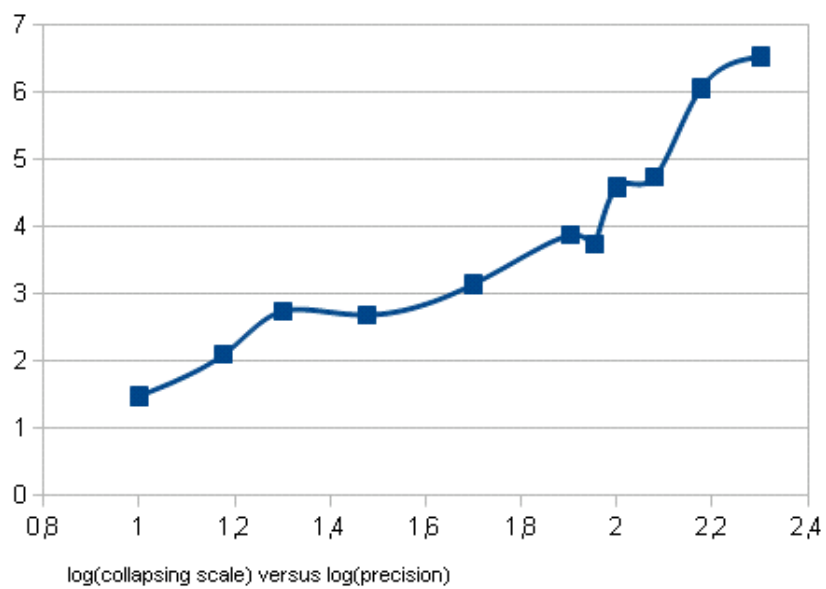
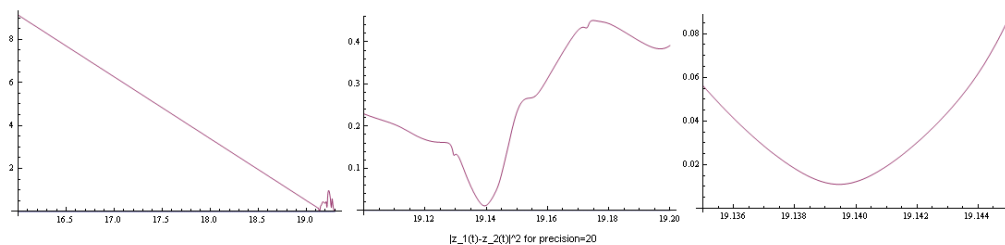
The log-log graph in Figure 6 shows how the collapsing scale depends on precision.

We also show some sample plots of the function $|z_1(t) - z_2(t)|^2$ at the places of interest. Note that for the self-similar collapsing evolution the function should be linear in t . This is clearly seen up to a place close to the critical time, where self-similarity is lost.

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FIGURE 6. The log-log dependency of the collapsing scale on precision

FIGURE 7. $|z_1(t) - z_2(t)|^2$ for precision 20FIGURE 8. $|z_1(t) - z_2(t)|^2$ for precision 200