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COLLAPSE OF N VORTICES

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The importance of the point vortices in applications is due to the dominant role of the coherent vortical structures in many two-dimensional turbulent flows. Point-vortex dynamics is responsible for bringing the vortices together and in this way it determines what kind of a merging process will occur. The number of vortices in a flow can be quite large; this makes the complicated dynamical description intractable. On the other hand, few-vortex systems can be investigated in much more detail. The evolution of such systems has been studied for more than 130 years. The equations of motion of a system of n vortices $z(t) = (z_1, \dots, z_n) \in C^n$ are

$$\frac{dz_k(t)}{dt} = V_k\big(z(t)\big) = \frac{i}{2\pi}\sum\nolimits_{l\neq k}\frac{\Gamma_l}{\overline{z_k}-\overline{z_l}}.$$

In 1883 Kirchhoff showed that the motion can be put into the Hamilton framework. Systems of three vortices are integrable since they possess enough Poisson commuting invariants. It was quite surprising that there exist three-vortices systems whose evolution leads to a collapse to a point in finite time. This phenomenon can be treated as a change of scale, characteristic for turbulent motion. A deeper insight into the way systems of vortices can collapse could thus lead to a better understanding of turbulence itself.

Collapsing systems of three vortices were first described by Groebli in 1877 and rediscovered by Aref and independently by Novikov and Sedov around 1979. The motion of four vortices is no longer integrable in general. Nevertheless in 1979 Novikov and Sedov gave some special, explicit examples of collapsing systems of four and five vortices.

It's useful to say that two systems of vortices z and w form the same configuration if they are similar i.e., $w_k = az_k + b$ for some complex a, b with $a \neq 0$. Note that similar systems have the same dynamics. The configuration space for n vortices has dimension $2n-4$. A system is called self-similar if it remains similar to the initial state during evolution (it is a fixed point in the configuration space).

In 1987 O'Neil proved the existence of collapsing systems of n vortices for arbitrary n. His proof is highly non-constructive. It is based on the fact that if the circulation Γ satisfies $L = \sum_{i<j}\Gamma_i\Gamma_j = 0$, $\sigma = \sum_i \Gamma_j \neq 0$, then any self-similar collapsing configuration can be represented by a solution $Z \in C^n$ of the following system of equations

$$V_1 Z_k = V_k Z_1 \ (k = 2..n-2), \sum \Gamma_l\,|Z_l|^2 = 0, \sum \Gamma_l\, Z_l = 0.$$

Then $V_1 Z_k = V_k Z_1$ for any k and furthermore there exist complex numbers $W, \omega \neq 0$, such that $V_k(Z) = \omega(Z_k - W)$ for any k. It follows that the evolution equation with initial conditions $z(0) = Z$ can be solved directly

$$z(t) = W + (Z-W)\sqrt{2Re(\omega)t+1}\,e^{i\frac{Im(\omega)}{2Re(\omega)}\ln(2Re(\omega)t+1)}.$$

Thus if $Re(\omega) < 0$ the system collapses to W when $t \to t_0 = \frac{-1}{2Re(\omega)} > 0$. O'Neil was able to prove that at least for some circulations the set of solutions is a non-empty algebraic curve in the configuration space. Thus solutions exist, although the proof does not indicate how they should be found. In fact O'Neil did not provide any examples for the interesting case $n \geq 6$.

We solve numerically the above non-linear algebraic system of equations and obtain collapsing configurations for many circulations and for six or more vortices. To the best of our knowledge no such examples have appeared in the literature so far. A precise description of the algorithm used will be given in our presentation. The solutions we found are initial conditions to the differential evolution equation. Standard numerical procedures give then trajectories whose collapsing property can be directly seen. A sample collapsing system of seven vortices is shown in Fig. 1. Note that in order to obtain a system whose diameter lessens sufficiently during evolution, it is necessary to use extremely high precision both when calculating the initial state and when solving the evolution equation.

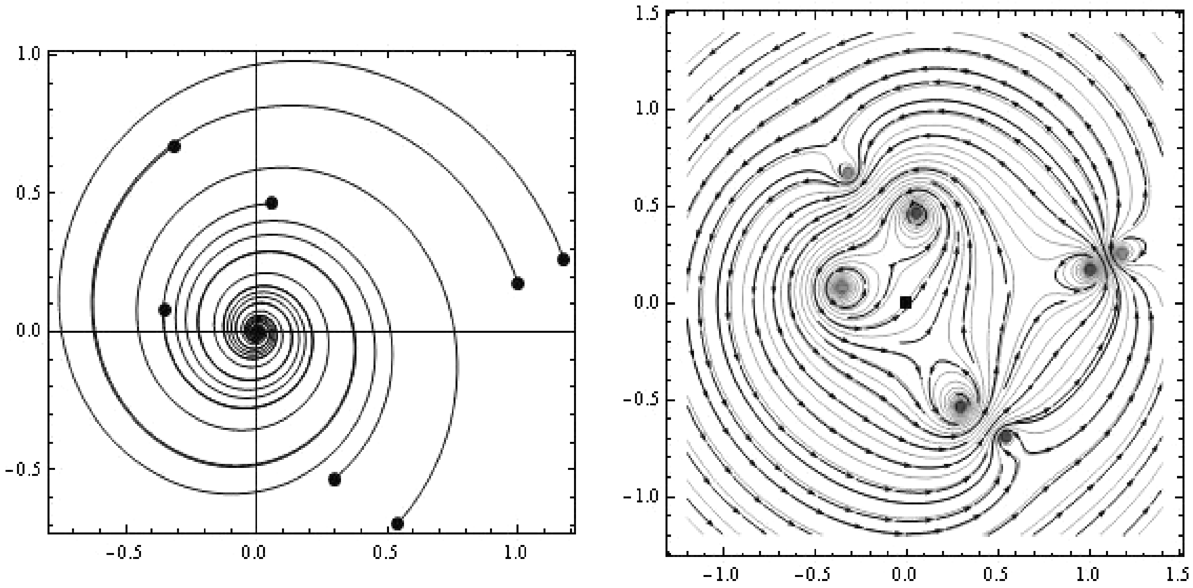

Fig. 1. Collapsing trajectory of a seven-vortex system (left) and streamlines of the system at initial time t=0 (right). The vortices collapse to the mass center, which is marked by a black rectangle. The intensities of the vortices are $\Gamma = (1, 1, -2, -2, -2, 1.5, -2)$.

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