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CONVOLUTION INTEGRAL IN TRANSIENT PIPE FLOW

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Abstract

This paper is devoted to modelling hydraulic losses during transient flow of liquids in pressure lines. Unsteady pipe wall shear stress was presented in the form of convolution integral of liquid acceleration and a weighting function. The weighting function depends on dimensionless time and the Reynolds number. In its first revision (Zielke [21], 1968) it had complex and inefficient mathematical structure (featured power growth of computational time). Therefore, further work aimed to develop so-called efficient models for correct estimation of hydraulic resistance with simultaneous linear loading of the computer's operating memory. The work compared the methods of numerical solving convolution integral known from literature (classic by Zielke [21] and Zielke-Vardy-Brown [10] and efficient by Trikha [6], Kagawa et al. [3] and Schohl [5]). The comparison highlighted the level of usefulness of the analyzed models in simulating water hammer and revealed demand for further research for the improvement of efficiency of the solutions.

Key words: numerical fluid mechanics, transient flow, hydraulic resistance, convolution integral

INTRODUCTION

Many studies of unsteady flows of liquids through pressure lines assume that hydraulic losses are quasi-steady. The models provide correct results only for low frequencies or slow velocity variation, i.e., for quasi-steady flows. The approach visible in many contemporary works is that instantaneous pipe wall shear stress τ can be presented in the form of a sum of quasi-steady quantities τ_q and quantity τ_u [1,3-21] variable in time:

$$\tau = \tau_q + \tau_u \tag{1}$$

Quantity τ_q is determined based on the transformed Darcy-Weisbach formula:

$$\tau_q = \frac{1}{8}\lambda\rho v|v| \tag{2}$$

where: λ – friction factor, ρ – liquid density, v – instantaneous flow velocity.

It is known that during **laminar flow** liquid molecules fill porous pipe cavities, creating smooth "sliding surface". Many experiments have conformed this liquid behaviour. In this scenario it is assumed that hydraulic resistance is independent of pipeline wall porosity and depends on the value of the Reynolds number only. The flow remains laminar until the critical Reynolds number value (approx. 2320), is exceeded. Friction factor in laminar flow is calculated using the Hagen-Poiseuille law:

$$\lambda = \frac{64}{\mathrm{Re}} \tag{3}$$

Once the critical value of the Reynolds number is exceeded, the flow becomes **turbulent** and the friction loss coefficient for coarse pipes can be computed from the Colebrook-White dependence:

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \lg\left(\frac{2.51}{Re \cdot \sqrt{\lambda}} + \frac{\varepsilon}{3.7 \cdot D}\right) \tag{4}$$

where: ε/D – relative roughness of internal pipe walls.

For hydraulically smooth pipes the friction loss coefficient can be computed from the Prandtl-Karman equation:

$$\frac{1}{\sqrt{\lambda}} = 2 \cdot \lg\left(Re \cdot \sqrt{\lambda}\right) - 0.8 \tag{5}$$

Experimental results have shown that the foregoing equation (5) features very good fit for single-phase flow for any large Reynolds number.

MODELLING PIPE WALL SHEAR STRESS

Zielke ([21], 1968) presented an analytical solution enabling determination of unsteady fiction losses (instantaneous pipe wall shear stress in the form of convolution integral from local acceleration of liquid and a weighting function) for laminar flow. The Zielke's model can be easily used in equations describing 1D unsteady flow, including specifically the popular method of characteristics (MOC).

In his deliberations, Zielke referred to the dependence presented in the paper by Brown ([2], 1962), describing the impedance of a hydraulic line as a function of frequency:

$$Z_0(s) = \frac{\dfrac{\rho s}{\pi \cdot R^2}}{1 - \dfrac{2 \cdot J_1\left(jR \cdot \sqrt{\dfrac{s}{\nu}}\right)}{jR \cdot \sqrt{\dfrac{s}{\nu}} \cdot J_0\left(jR \cdot \sqrt{\dfrac{s}{\nu}}\right)}} \tag{6}$$

where: s – Laplace transformation operator; ν – kinematic viscosity coefficient; J_o and J_1 – Bessel functions of the first type of orders 0 and 1; j – imaginary unit; R – internal pipe radius

By reversing the Laplace transformation, he obtained the following dependence for instantaneous pipe wall shear stress [21]:

$$\tau(t) = \underbrace{\frac{4 \cdot \mu}{R} \cdot v}_{\tau_q} + \underbrace{\frac{2 \cdot \mu}{R} \cdot \int_0^t w(t-u) \cdot \frac{\partial v}{\partial t}(u) \cdot du}_{\tau_u} \tag{7}$$

where: w(t) – weighting function, μ – dynamic viscosity coefficient.

The first expression τ_q of the foregoing equation (7) represents quasi-steady quantity (is a result of inserting the expression for linear resistance rate (3) in equation (2) for laminar flow).

The second expression τ_u describes the effect of unsteadiness of flow on wall shear stress. It is convolution integral from instantaneous liquid acceleration and a weighting function:

$$w(\hat{t}) = \sum_{i=1}^{6} m_i \hat{t}^{(i-2)/2} \text{ , for } \hat{t} \le 0.02 \tag{8a}$$

$$w(\hat{t}) = \sum_{i=1}^{5} e^{-n_i \cdot \hat{t}} \text{ , for } \hat{t} > 0.02 \tag{8b}$$

where: $\hat{t} = (\nu / R^2) \cdot t$ – its a dimensionless time and:
m_1 = 0.282095; m_2 = -1.25; m_3 = 1.057855; m_4 = 0.9375; m_5 = 0.396696; m_6 = -0.351563;
n_1 = 26.3744; n_2 = 70.8493; n_3 = 135.0198; n_4 = 218.9216; n_5 = 322.5544.

The variable in time component of instantaneous pipe wall shear stress τ_u can be computed numerically using the differential approximation of the first order [21]:

$$\tau_u = \frac{2\mu}{R} \sum_{j=1,3,...}^{n-2} \left(v_{i,j+2} - v_{i,j}\right) \cdot w\left((n-1-j)\Delta\hat{t}\right) = \frac{2\mu}{R} \sum_{j=1,3,...}^{n-2} \left(v_{i,n-j+1} - v_{i,n-j-1}\right) \cdot w\left(j\Delta\hat{t}\right) \tag{9}$$

where: i – number of subsequent computational pressure pipe cross-section changing from 1 to h; j – number of computational time step changing with the increment of 2 from 1 to n for n≥3; $\Delta\hat{t} = (R^2/\nu) \cdot \Delta t$; Δt – time step in the numerical analysis.

In the method of characteristics based on rectangular grid the foregoing equation (9) can be written as follows [4]:

$$\tau_u = \frac{2\mu}{R} \sum_{j=1}^{n-1} \left(v_{i,j+1} - v_{i,j}\right) \cdot w\left((n-j)\Delta\hat{t} - \frac{\Delta\hat{t}}{2}\right) = \frac{2\mu}{R} \sum_{j=1}^{n-1} \left(v_{i,n-j+1} - v_{i,n-j}\right) \cdot w\left(j\Delta\hat{t} - \frac{\Delta\hat{t}}{2}\right) \tag{10}$$

where: j – number of computational time step changing with the increment of 1 from 1 to n for n≥2.

An analysis of the two last equations explains why the solution of convolution integral by Zielke is inefficient. This is because the number of expressions representing the instantaneous value of wall shear stress increases as part of the numerical process with each successive time step “j”.

In time, it was demonstrated [11-15, 19-20] that dependence (7) can be also used for transient turbulent flows. However, the weighting function in turbulent flow has no fixed pattern, as for laminar flow. Its shape varies depending on conditions: namely the value of the Reynolds number.

Based on the 2D (axial-symmetric) Reynolds equation, Boussinesq hypothesis and experimental data (concerning turbulent viscosity coefficient in the pipe cross-section), Vardy-Brown and Zarzycki proposed their own weighting functions for turbulent flow:

- **Vardy–Brown model [13]**

$$w\left(\hat{t}, Re\right) = \frac{A^* e^{-B^* \hat{t}}}{\sqrt{\hat{t}}} \tag{11}$$

where: $A^* = \sqrt{1/4\pi}$ and $B^* = Re^{\kappa}/12.86$; $\kappa = \log_{10}(15.29/Re^{0.0567})$

- **Zarzycki model [20]**

$$w(\hat{t}, Re) = C \cdot \frac{1}{\sqrt{\hat{t}}} \cdot Re^n \tag{12}$$

where: C=0.299635; n= - 0.005535.

The foregoing dependences (11, 12) for the weighting function can be used within the $2000 \le \mathrm{Re} \le 10^8$ range of the Reynolds number.

Vardy and Brown ([10], 2010) proposed an adjustment to the classic solution by Zielke (10) consisting of computing the integral from the weighting function. After adopting this approach, numerical simulations start to reflect the actual change of wall shear stress more accurately (among others, they avoid the error in determining hydraulic resistance for accelerated flow; see [10]).

The integral derived from equation (8a) for zero-dimensional time $\hat{t}$ is:

$$I_1 = \left(\frac{R^2}{\nu}\right) \cdot \left[2 \cdot m_1 \cdot \hat{t}^{0.5} + m_2 \cdot \hat{t}^1 + \left(\frac{2}{3}\right) m_3 \cdot \hat{t}^{1.5} + \left(\frac{1}{2}\right) m_4 \cdot \hat{t}^2 + \left(\frac{2}{5}\right) m_5 \cdot \hat{t}^{2.5} + \left(\frac{1}{3}\right) m_6 \cdot \hat{t}^3\right] \quad (13a)$$

Whereas, for equation (8b), the integral is:

$$I_2 = \left(\frac{R^2}{\nu}\right) \cdot \sum_{i=1}^{5} \left(-\frac{1}{n_i}\right) e^{-n_i \cdot \hat{t}} \quad (13b)$$

Modified solution proposed by Vardy-Brown:

$$\tau_n = \frac{2\mu}{R} \sum_{j=1}^{n-1} \frac{\left(v_{i,j+1} - v_{i,j}\right)}{\Delta t} \cdot I\Big|_{(n-j-1)\cdot\Delta\hat{t}}^{(n-j)\cdot\Delta\hat{t}} \quad (14)$$

where: $I = I_1$ when $(n - j) \cdot \Delta\hat{t} \le 0.02$ and $I = I_2$ when $(n - j) \cdot \Delta\hat{t} > 0.02$.

EFFICIENT SOLUTION OF CONVOLUTION INTEGRAL

Trikha [6] was the first to present an efficient numerical solution of convolution integral (8) in 1975:

$$\tau_u(t + \Delta t) \approx \frac{2 \cdot \mu}{R} \cdot \sum_{i=1}^{j} \left[\underbrace{y_i(t) \cdot e^{-n_i \cdot \frac{\nu}{R^2} \cdot \Delta t} + m_i \cdot \left[v_{(t+\Delta t)} - v_t\right]}_{y_i(t+\Delta t)} \right] \quad (15)$$

To obtain the foregoing solution, it was necessary to write the weighting function in the form of a finite sum of exponential expressions:

$$w(\hat{t}) = \sum_{i=1}^{j} m_i e^{-n_i \cdot \hat{t}} \qquad \text{where: } i=1,2\ldots j \quad (16)$$

this is because only this form of the function enables efficient solution of convolution integral. Because Trikha made too many simplifications while deriving his equations for the efficient solution of convolution integral (15, 16), Kagawa et al. [3], and then Schohl [5], proposed more accurate solutions.

The Schohl's solution is slightly different from that by Kagawa et al. See the following for the derivation of the solutions:

$$\tau_u(t) = \frac{2\mu}{R} \int_0^t w_{apr}(t-u) \cdot \frac{\partial v}{\partial u}(u) du = \frac{2\mu}{R} \int_0^t \sum_{i=1}^{j} w_i(t-u) \cdot \frac{\partial v}{\partial u}(u) du = \frac{2\mu}{R} \sum_{i=1}^{j} \int_0^t w_i(t-u) \cdot \frac{\partial v}{\partial u}(u) du \quad (17)$$

$$\tau_u(t) = \frac{2\mu}{R} \sum_{i=1}^{j} y_i(t) \quad (18)$$

$$y_i(t)=\int_0^t w_i(t-u)\cdot\frac{\partial v}{\partial u}(u)du=\int_0^t m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t-u)}\cdot\frac{\partial v}{\partial u}(u)du$$
$$y_i(t)=m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot t}\cdot\int_0^t e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du \tag{19}$$

Using the method of characteristics to solve the system of partial differential equations describing transient flow requires that the computation is performed for certain predefined time steps Δt . The notation for the subsequent time step can be as follows:

$$y_i(t+\Delta t)=\int_0^{t+\Delta t} w_i(t+\Delta t-u)\cdot\frac{\partial v}{\partial u}(u)du=$$
$$=\int_0^t w_i(t+\Delta t-u)\cdot\frac{\partial v}{\partial u}(u)du+\int_t^{t+\Delta t} w_i(t+\Delta t-u)\cdot\frac{\partial v}{\partial u}(u)du= \tag{20}$$
$$=\int_0^t m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t-u)}\cdot\frac{\partial v}{\partial u}(u)du+\int_t^{t+\Delta t} m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t-u)}\cdot\frac{\partial v}{\partial u}(u)du=$$
$$=e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}\cdot\underbrace{\int_0^t m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t-u)}\cdot\frac{\partial v}{\partial u}(u)du}_{y_i(t)}+\underbrace{m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}\cdot\int_t^{t+\Delta t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du}_{\Delta y_i(t)}$$

Ultimately:

$$y_i(t+\Delta t)=y_i(t)\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}+\Delta y_i(t) \tag{21}$$

where:

$$\Delta y_i(t)=m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}\cdot\int_t^{t+\Delta t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du \tag{22}$$

Assuming for the foregoing expression that function v(u) is linear function [v(u)=au+b] within range <t, t+Δt>, its derivative after time ∂v(u)/∂u can be considered as a constant, the value of which is computed as follows:

$$\frac{[v_{(t+\Delta t)}-v_t]}{\Delta t} \tag{23}$$

Given this assumption, $\Delta y_i(t)$ can be written as follows, as in Schohl [5]:

$$\Delta y_i(t)\approx m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}\cdot\frac{[v_{(t+\Delta t)}-v_t]}{\Delta t}\cdot\int_t^{t+\Delta t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}du=$$
$$=m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}\cdot\frac{[v_{(t+\Delta t)}-v_t]}{\Delta t}\cdot\frac{R^2}{n_i\cdot\nu}\cdot\left[e^{n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}-e^{n_i\cdot\frac{\nu}{R^2}\cdot t}\right] \tag{24}$$
$$=\frac{m_i\cdot R^2}{\Delta t\cdot n_i\cdot\nu}\cdot\left[1-e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}\right]\cdot[v_{(t+\Delta t)}-v_t]$$

Or as follows, as in Kagawa et al. [3]:

$$
\begin{aligned}
\Delta y_i(t) &\approx m_i \cdot e^{-n_i \cdot \frac{\nu}{R^2}\cdot(t+\Delta t)} \cdot \frac{\left[v_{(t+\Delta t)} - v_t\right]}{\Delta t} \cdot \int_t^{t+\Delta t} e^{n_i \cdot \frac{\nu}{R^2}\cdot u} du = \\
&= m_i \cdot e^{-n_i \cdot \frac{\nu}{R^2}\cdot(t+\Delta t)} \cdot \frac{\left[v_{(t+\Delta t)} - v_t\right]}{\Delta t} \cdot e^{n_i \cdot \frac{\nu}{R^2}\cdot\left(t+\frac{\Delta t}{2}\right)} \cdot \int_t^{t+\Delta t} du \\
&= m_i \cdot e^{-n_i \cdot \frac{\nu}{R^2}\cdot\left(\frac{\Delta t}{2}\right)} \cdot \frac{\left[v_{(t+\Delta t)} - v_t\right]}{\Delta t} \cdot (t+\Delta t - t) \\
&= m_i \cdot e^{-n_i \cdot \frac{\nu}{R^2}\cdot\left(\frac{\Delta t}{2}\right)} \cdot \left[v_{(t+\Delta t)} - v_t\right]
\end{aligned} \tag{25}
$$

The final efficient numerical solution of convolution integral by Schohl is as follows:

$$
\tau_u(t+\Delta t) \approx \frac{2\cdot\mu}{R} \cdot \sum_{i=1}^{j} \left[\underbrace{ y_i(t)\cdot e^{-n_i \cdot \frac{\nu}{R^2}\cdot\Delta t} + \frac{m_i \cdot R^2}{\Delta t \cdot n_i \cdot \nu} \cdot \left[1 - e^{-n_i \cdot \frac{\nu}{R^2}\cdot\Delta t}\right] \cdot \left[v_{(t+\Delta t)} - v_t\right] }_{y_i(t+\Delta t)} \right] \tag{26}
$$

While the solution by Kagawa et al. is the following:

$$
\tau_u(t+\Delta t) \approx \frac{2\cdot\mu}{R} \cdot \sum_{i=1}^{j} \left[\underbrace{ y_i(t)\cdot e^{-n_i \cdot \frac{\nu}{R^2}\cdot\Delta t} + m_i \cdot e^{-n_i \cdot \frac{\nu}{R^2}\cdot\left(\frac{\Delta t}{2}\right)} \cdot \left[v_{(t+\Delta t)} - v_t\right] }_{y_i(t+\Delta t)} \right] \tag{27}
$$

Because simulation starts from steady flow (v = const.), wall shear stress τ_u parameter occurring during transient flow and the values of all components $y_i(t)$ is equal to 0 in the first computational time step. In each subsequent time step the values of components change according to equation (21).

SIMULATION RESULTS

The following presents results of illustrative simulations of fluctuations of parameter τ_u using the solutions of convolution integral (3 efficient and 2 inefficient ones) discussed in the two preceding sections. The simulation results were obtained for a known experimental pattern (Fig. 1) of variation of the mean liquid velocity (occurring during simple water hammering in the center of the cross-section of a pressure pipe – Re=1111, v_o=0.066 m/s, L=98.11 m, R=0.008 m, $\nu=9.493\cdot10^{-7}$ m^2/s i c=1305 m/s [1]). The experiment consisted of sudden closure of the terminal valve of a pipe transporting liquid from a constant pressure tank.

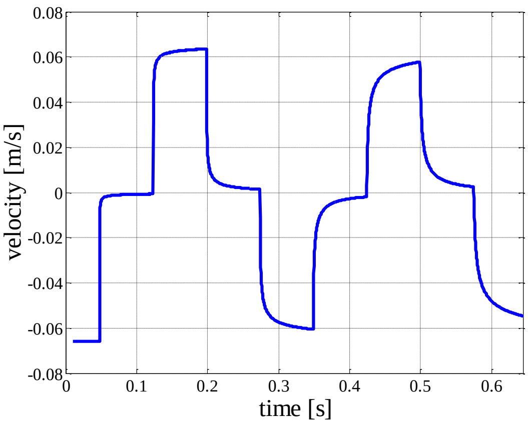


Fig. 1: Mean velocity profile – pipe midpoint

As can be seen in the foregoing drawing (Fig. 1), the work analyzed the effect of velocity variation on the pattern of parameter τ_u only for the two first water hammer effect periods (within t = 0.644 s from the occurrence of the transient state).

The simulation tested the following:

1) The effect of the number of time steps "n" (zero-dimensional time step $\Delta\hat{t}$) to the pattern of parameter τ_u. Three cases were analyzed:

– **CASE I** (n_1=96 time steps, $\Delta\hat{t}_1 = 1 \cdot 10^{-4}$)
– **CASE II** (n_2=266 time steps, $\Delta\hat{t}_2 = 3.6 \cdot 10^{-5}$)
– **CASE III** (n_3=2561 time steps, $\Delta\hat{t}_3 = 3.7 \cdot 10^{-6}$)

2) The effect of quantity of exponential terms describing the efficient weighting function on the pattern of parameter τ_u. Also three cases were analyzed (Fig. 2):

– Function consisting of **16 exponential terms**
– Function consisting of **22 exponential terms**
– Function consisting of **26 exponential terms**

See paper [8] for details of coefficients used in the weighting function.

3) The quality of matching of the results obtained using efficient solutions of convolution integral [derived] compared to the fit of results obtained using classic (or inefficient) solutions.

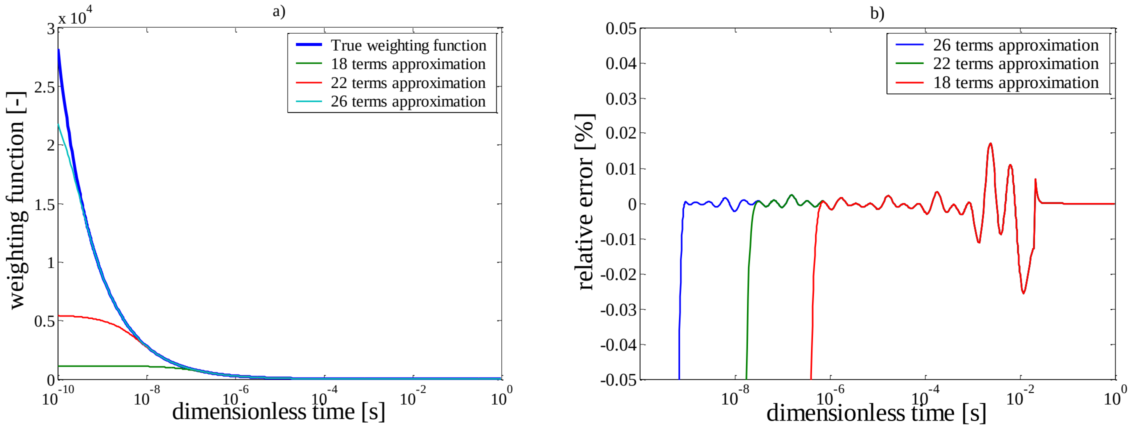
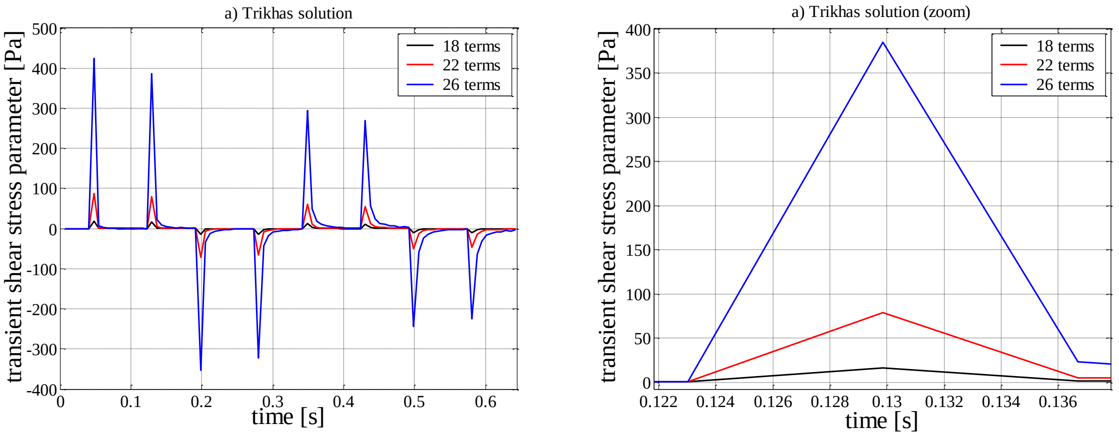


Fig. 2 Weighting function

CASE I (n_1=96 time steps – $\Delta\hat{t}_1 = 1 \cdot 10^{-4}$)



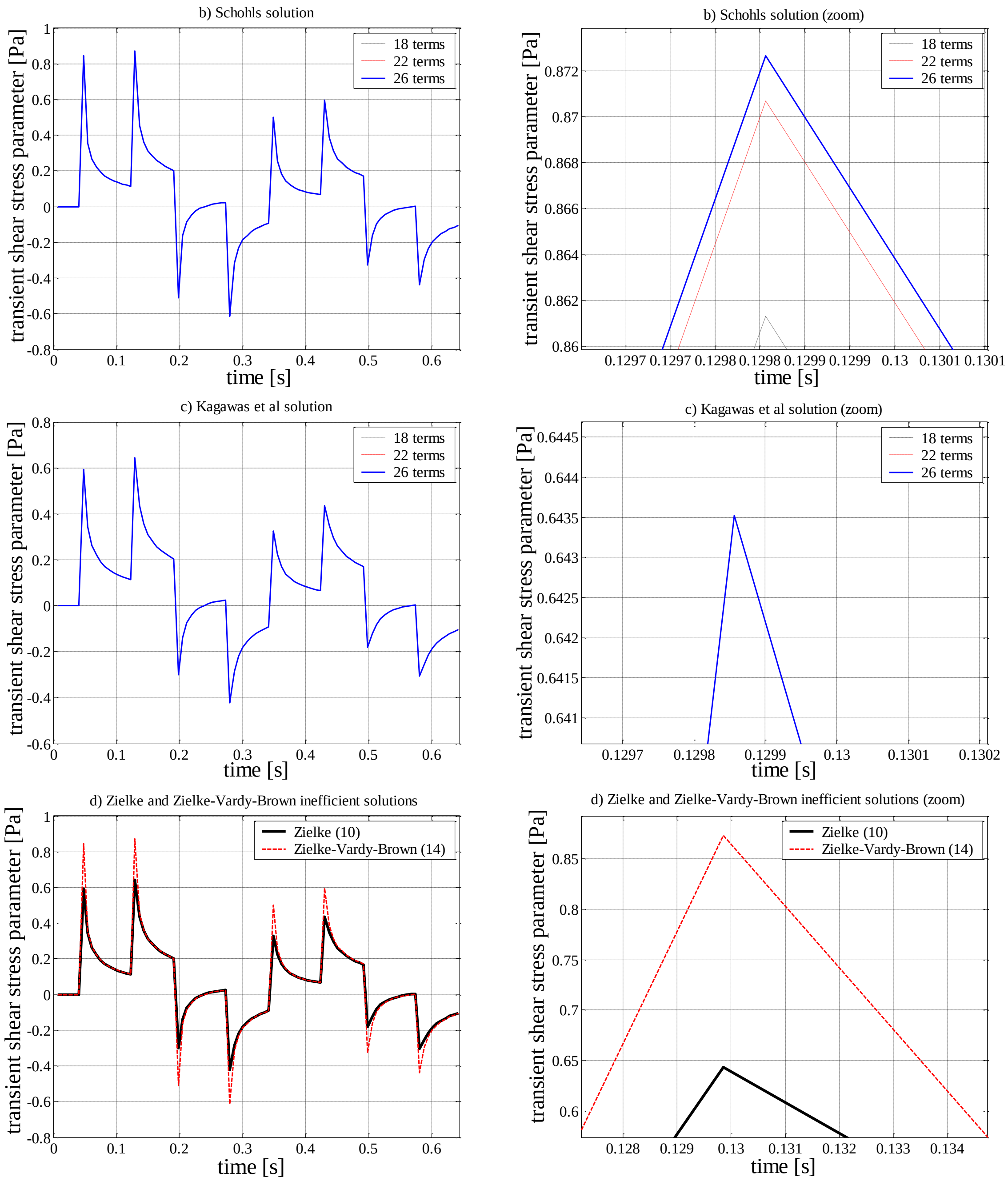


Fig. 3 CASE I - results of simulated τ_u transient shear stress parameter runs using:
a) Trikhas efficient solution [6], b) Schohls efficient solution [5], c) Kagawas et al. efficient solution [3],
d) Zielke [21] and Zielke–Vardy–Brown [10] inefficient solution

The foregoing diagrams show clearly what errors can result from using the efficient solution of convolution integral by Trikha (15) for modelling unsteady flow. This is because unsteady wall shear stresses τ_u simulated using the solution depend mostly on the adopted weighting function. The more expressions the ultimate form of the function contains, the worse are the results. A good visual example is comparing the results shown in Fig. 3a (26 expressions) with those in Fig. 3d (Zielke-Vardy-Brown solution). It is clear that the results of the simulation using the solution by Trikha are approx. 500 times larger than the results provided by the exact adjusted classic solution by Zielke-Vardy-Brown (14). Therefore, using the solution of convolution integral by Trikha should be avoided in numerical computations of unsteady hydraulic resistance. Similarly wrong results were obtained for this solution in the next two cases (CASE II and CASE III). Because of its incompatibility with the classic solutions, the one by Trikha will not be compared or considered in the following parts of this work.

Fig. 3b (zoom) shows that the efficient solution by Schohl (26) is slightly dependent on the quantity of exponential expressions making up the weighting function. The larger quantity of exponential expressions, the higher consistency with the results provided by the classic adjusted solution by Zielke-Vardy-Brown (14) .

On the other hand, the effect of the quantity of exponential expressions making up the weighting function is not observed for the efficient solution by Kagawa et al. (Fig. 3c).

Fig. 3d shows that the results of simulated parameter τ_u are understated for the classic solution by Zielke (10). This means that simulated hydraulic resistance is understated if this solution is used.

CASE II (n_2=266 time steps – $\Delta\hat{t}_2 = 3.6 \cdot 10^{-5}$)

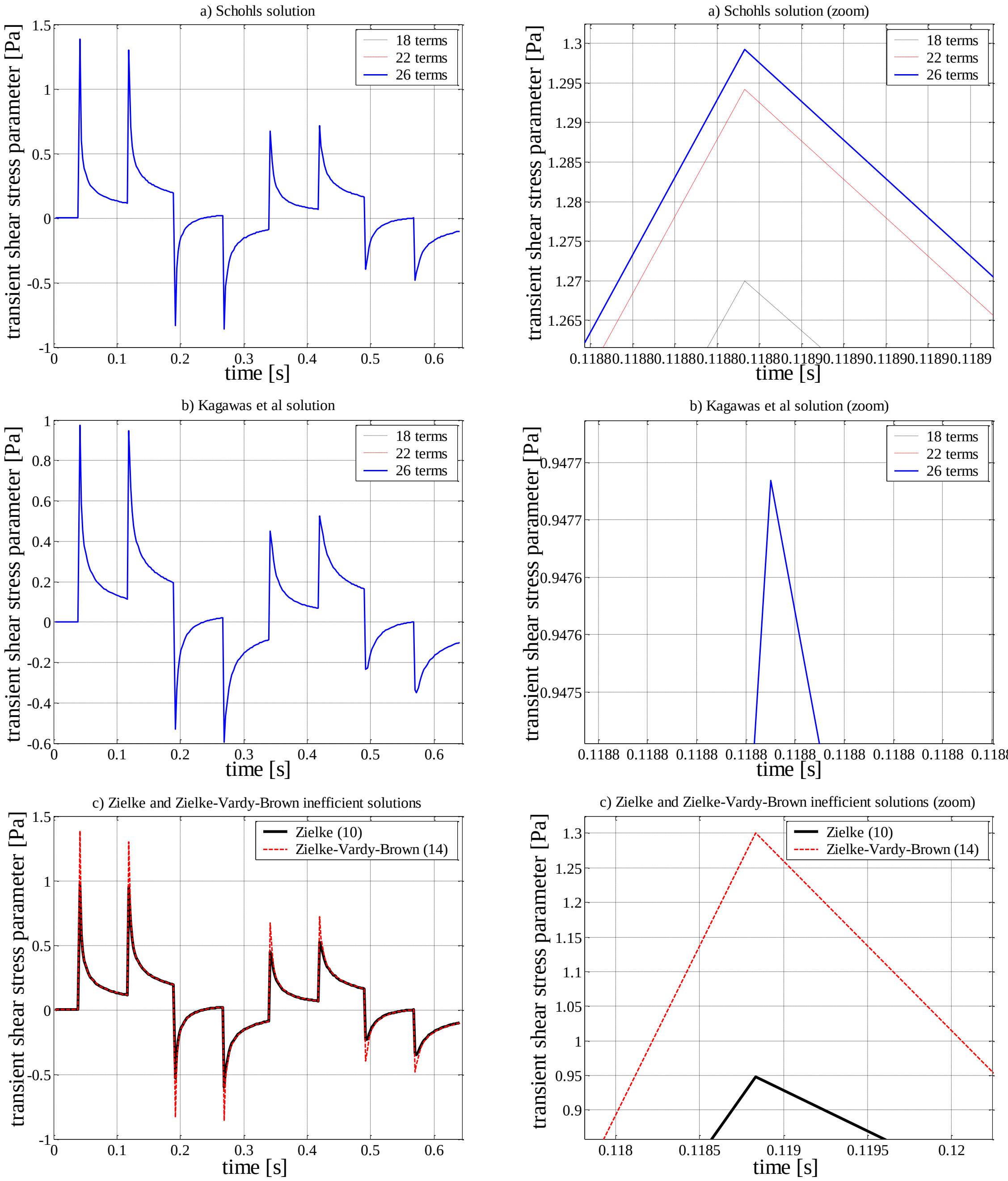


Fig. 4 CASE II - results of simulated τ_u transient shear stress parameter runs using:
a) Schohls efficient solution [5], b) Kagawas et al. efficient solution [3],
c) Zielke [21] and Zielke–Vardy–Brown [10] inefficient solution

A review of the results shown in Fig. 4a confirmed the trend noted for the results obtained for the previous case (CASE I). The more expressions the efficient weighting function contains, the more accurate simulation is for the solution by Schohl (this is related to the fact that the efficient function containing more expressions is matched to the classic weighting function by Zielke (8) within a broader range of dimensionless time).
Again, the number of expressions did not affect the results obtained using the solution by Kagawa et al. (Fig. 4b).
Fig. 4c shows the same dependence that was observed for the previous case (CASE I): the results obtained using the classic solution by Zielke (10) were understated vs. the results provided by the corrected model by Zielke-Vardy-Brown (14).

CASE III (n_3=2561 time steps – $\Delta\hat{t}_3 = 3.7 \cdot 10^{-6}$)

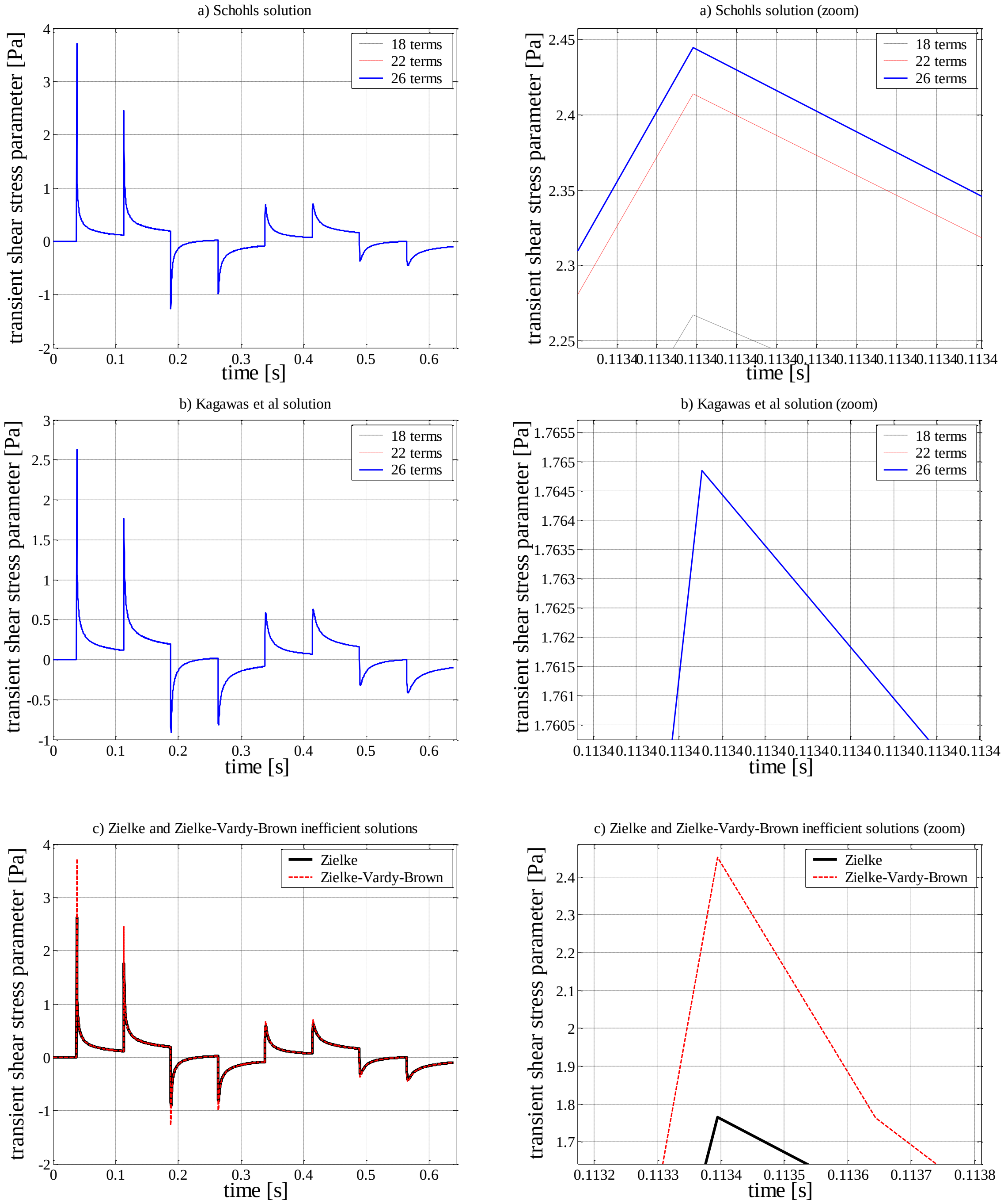


Fig. 5 CASE III - results of simulated τ_u transient shear stress parameter runs using:
a) Schohls efficient solution [5], b) Kagawas et al. efficient solution [3],
c) Zielke [21] and Zielke–Vardy–Brown [10] inefficient solution

The foregoing illustrative comparisons (for all cases: CASE I, II and III) show clearly that the efficient solution of convolution integral by Kagawa et al. (27) corresponds to the classic solution by Zielke (10). However, as Vardy and Brown [10] correctly noted, the classic solution by Zielke is unable to provide correct simulation (as shown by Vardy and Brown in the example of accelerated flow) because of the simplification consisting of not computing the integral from the weighting function.

The solution by Schohl (26) is the efficient solution that computes the integral from the weighting function. And, as shown by the qualitative analysis of the foregoing results, the solution corresponds to the adjusted classic solution by Zielke-Vardy-Brown (14) with good fit.

Also, the analysis of all the results answered the question about the effect of the time step on the simulation results. Namely, it is clear that the maximum values of peaks occurring in the patterns of parameter τ_u grow as the value of the dimensionless time step $\Delta\hat{t}$ decreases. It is a regularity justified by the fact that the values of the weighting function are the larger the smaller is the time step in numerical computations. This means that velocity increments are multiplied by larger values (the solution by Trikha was a marked exception as it was the only solution that displayed different behaviour, which is an argument for definitive need for avoiding this solution in simulations).

Quantitative Analysis

Apart from the standard qualitative analysis, the paper contains a quantitative one. The qualitative analysis demonstrates clearly that the efficient solution by Schohl conforms to the adjusted classic solution by Zielke-Vardy-Brown and that the efficient solution by Kagawa et al. conforms to the classic solution by Zielke (considered the most accurate one until recently). Therefore, the following sections compare the results of the simulation performed using the solution by Kagawa et al. to those provided by the classic solution by Zielke and the results provided by the efficient solution by Schohl to those provided by the classic solution adjusted by Zielke-Vardy-Brown.

Only absolute percentage errors of the maximum and minimum values occurring in the simulated patterns of parameter τ_u were analyzed (marked with circles in the following Fig. 5).

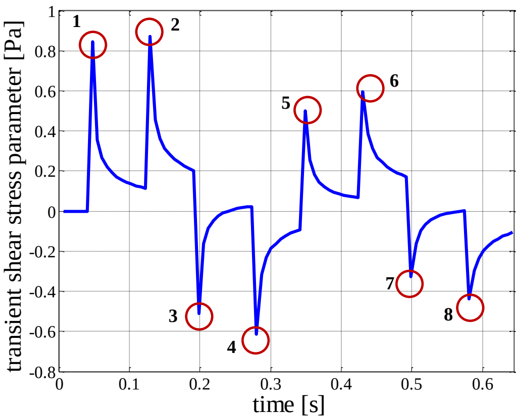


Fig. 6 Analyzed shear stress peakes

After calculating 8 errors from the stimulated patterns, the errors were used to estimate a single parameter, “E”, representing the arithmetic mean of all the errors, using the following equation:

$$E = \frac{\sum_{i=1}^{8}\left|\frac{\tau i_{eff.(max,min)} - \tau i_{ineff.(max,min)}}{\tau i_{ineff.(max,min)}}\right| \cdot 100\%}{8} \tag{28}$$

where: $\tau i_{eff.(max,min)}$ - maximum and minimum values of effective runs (Kagawa and Schohl solution); $\tau i_{ineff.(max,min)}$ - maximum and minimum values of innefective runs (Zielke and Zielke-Vardy-Brown solution)

The following Table 1 shows the results of the proposed quantitative analysis.

Table 1: Mean absolute error of actual results

Case	**Error parameter E [%]**	
	Kagawa vs Zielke	Schohl vs Zielke-Vardy-Brown
CASE I	0.0051	0.075
CASE II	0.0015	0.118
CASE III	0.0019	0.230

It follows clearly from the foregoing table that the fit of the results obtained using the efficient solution by Kagawa is very good and this is why the solution used to be most popular one. However, the one-way tendency to improvement of the fit as the time step in the numerical becomes smaller is missed. A reverse trend (where time step reduction deteriorates the results) can be observed for the fit of the results obtained using the efficient solution by Schohl. Without doubt, this behaviour relates to the incorrect result of integration using the weighting function for the last time step. Event the first drawing shows that the efficient weighting function approach a certain fixed value (namely $\sum_{i=1}^{k} m_i$) rather than infinity for dimensionless time approaching zero. Without doubt, the incorrect calculation of the integral using the weighting function for the last time step is the source of the error (because the last change of velocity is multiplied by the result of integration calculated using the weighting function within the 0 to $\Delta\hat{t}$ range), which can be eliminated by adjusting the efficient solution by Schohl (Appendix A).

CONCLUSION

The following paper analyzes three solution of convolution integral known from literature: Trikha [6], Kagawa et al. [3] and Schohl [5]. The results of the research show that the Trikha's simplifications are responsible for significant errors and this model should be ruled out as a tool for simulating hydraulic resistance.

Also, the results show that the efficient solution by Kagawa et al. (often used in the past by the authors of the paper) features very good correspondence to the classic solution by Zielke. As recently demonstrated [10], the solution is not error-free because it underestimates unsteady hydraulic resistance. Further, comparisons show that the adjusted solution of convolution integral used to calculate exact integral using the weighting function has its efficient counterpart: the solution by Schohl.

The qualitative analysis of the results provided by the efficient solution by Schohl demonstrates also that increasing the number of expressions describing the weighting function improves slightly the fit of simulation results compared to the results obtained using the accurate classic solution by Zielke-Vardy-Brown.

Note also that the quantitative results signal a slight problem that have been solved in the Appendix A: the computation of integral using the weighting function for the last time step in the efficient Schohl solutions generates an error that increases as the time step in the numerical analysis is smaller (Table 1).

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Appendix A

Below we show the revised effective solution by Schohl:

$$\tau_u(t)=\frac{2\mu}{R}\left(\int_0^{t-\Delta t} w_{apr}(t-u)\cdot\frac{\partial v}{\partial u}(u)du+\int_{t-\Delta t}^{t} w_{true}(t-u)\cdot\frac{\partial v}{\partial u}(u)du\right) \quad \text{(A1)}$$

The second term can be writen as:

$$\int_{t-\Delta t}^{t} w_{true}(t-u)\cdot\frac{\partial v}{\partial u}(u)du=\eta\cdot\int_{t-\Delta t}^{t} w_{apr}(t-u)\cdot\frac{\partial v}{\partial u}(u)du \quad \text{(A2)}$$

where: η – the correction coefficient ($\eta>1$)

Therefor:

$$\tau_u(t)=\frac{2\mu}{R}\left(\int_0^{t-\Delta t}\sum_{i=1}^{j} w_i(t-u)\cdot\frac{\partial v}{\partial u}(u)du+\eta\cdot\int_{t-\Delta t}^{t}\sum_{i=1}^{j} w_i(t-u)\cdot\frac{\partial v}{\partial u}(u)du\right)=$$
$$\frac{2\mu}{R}\sum_{i=1}^{j}\left(\int_0^{t-\Delta t} w_i(t-u)\cdot\frac{\partial v}{\partial u}(u)du+\eta\cdot\int_{t-\Delta t}^{t} w_i(t-u)\cdot\frac{\partial v}{\partial u}(u)du\right) \quad \text{(A3)}$$

$$\tau_u(t)=\frac{2\mu}{R}\sum_{i=1}^{j} y_i(t) \quad \text{(A4)}$$

$$y_i(t)=\int_0^{t-\Delta t} w_i(t-u)\cdot\frac{\partial v}{\partial u}(u)du+\eta\cdot\int_{t-\Delta t}^{t} w_i(t-u)\cdot\frac{\partial v}{\partial u}(u)du=$$
$$\int_0^{t-\Delta t} m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t-u)}\cdot\frac{\partial v}{\partial u}(u)du+\eta\cdot\int_{t-\Delta t}^{t} m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t-u)}\cdot\frac{\partial v}{\partial u}(u)du= \quad \text{(A5)}$$
$$m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot t}\cdot\left(\int_0^{t-\Delta t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du+\eta\cdot\int_{t-\Delta t}^{t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du\right)$$

The next time step can be writen:

$$y_i(t+\Delta t)=\int_0^{t} w_i(t+\Delta t-u)\cdot\frac{\partial v}{\partial u}(u)du+\eta\cdot\int_{t}^{t+\Delta t} w_i(t+\Delta t-u)\cdot\frac{\partial v}{\partial u}(u)du \quad \text{(A6)}$$

Divid in detail the first expression of that sum:

$$\int_0^{t} w_i(t+\Delta t-u)\cdot\frac{\partial v}{\partial u}(u)du=\int_0^{t-\Delta t} w_i(t+\Delta t-u)\cdot\frac{\partial v}{\partial u}(u)du+\int_{t-\Delta t}^{t} w_i(t+\Delta t-u)\cdot\frac{\partial v}{\partial u}(u)du+$$
$$+\eta\cdot\int_{t-\Delta t}^{t} w_i(t+\Delta t-u)\cdot\frac{\partial v}{\partial u}(u)du-\eta\cdot\int_{t-\Delta t}^{t} w_i(t+\Delta t-u)\cdot\frac{\partial v}{\partial u}(u)du=$$
$$\int_0^{t-\Delta t} m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t-u)}\cdot\frac{\partial v}{\partial u}(u)du+\eta\cdot\int_{t-\Delta t}^{t} m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t-u)}\cdot\frac{\partial v}{\partial u}(u)du-\eta\cdot\int_{t-\Delta t}^{t} m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t-u)}\cdot\frac{\partial v}{\partial u}(u)du+ \quad \text{(A7)}$$
$$\int_{t-\Delta t}^{t} m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t-u)}\cdot\frac{\partial v}{\partial u}(u)du=e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}\cdot\left(\int_0^{t-\Delta t} m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t-u)}\cdot\frac{\partial v}{\partial u}(u)du+\eta\cdot\int_{t-\Delta t}^{t} m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t-u)}\cdot\frac{\partial v}{\partial u}(u)du\right)+$$
$$m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}\cdot\left(\int_{t-\Delta t}^{t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du-\eta\cdot\int_{t-\Delta t}^{t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du\right)$$

Then:

$$y_i(t+\Delta t)=e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}\cdot\underbrace{\left(\int_0^{t-\Delta t} m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t-u)}\cdot\frac{\partial v}{\partial u}(u)du+\eta\cdot\int_{t-\Delta t}^{t} m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t-u)}\cdot\frac{\partial v}{\partial u}(u)du\right)}_{y_i(t)}+$$
$$\underbrace{m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}\cdot\left(\int_{t-\Delta t}^{t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du-\eta\cdot\int_{t-\Delta t}^{t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du\right)+\eta\cdot m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}\int_{t}^{t+\Delta t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du}_{\Delta y_i(t)} \tag{A8}$$

Presented below will concern the transformation of the expression $\Delta y_i(t)$. Assuming that the function v (u) is a linear function [v(u)=au+b] in the compartments <t;t+Δt> and <t-Δt;t>, its time derivative ∂v(u)/∂u may be treated as a constant whose value is calculated from the following expressions:

$$\frac{[v_{(t+\Delta t)}-v_t]}{\Delta t}\quad \text{and}\quad \frac{[v_t-v_{(t-\Delta t)}]}{\Delta t} \tag{A9}$$

Under this assumption, we can write $\Delta y_i(t)$ in the following form:

$$\begin{aligned}
\Delta y_i(t)&\approx m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}\cdot\left(\int_{t-\Delta t}^{t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du-\eta\cdot\int_{t-\Delta t}^{t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du+\eta\cdot\int_{t}^{t+\Delta t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du\right)=\\
&m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}\cdot\left(\frac{[v_t-v_{(t-\Delta t)}]}{\Delta t}\cdot\left(\int_{t-\Delta t}^{t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}du-\eta\cdot\int_{t-\Delta t}^{t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}du\right)+\eta\cdot\frac{[v_{(t+\Delta t)}-v_t]}{\Delta t}\cdot\int_{t}^{t+\Delta t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}du\right)=\\
&m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}\cdot\left(\frac{[v_t-v_{(t-\Delta t)}]}{\Delta t}\cdot\left([1-\eta]\cdot\int_{t-\Delta t}^{t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}du\right)+\eta\cdot\frac{[v_{(t+\Delta t)}-v_t]}{\Delta t}\cdot\int_{t}^{t+\Delta t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}du\right)=\\
&m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}\cdot\left(\frac{[v_t-v_{(t-\Delta t)}]}{\Delta t}\cdot[1-\eta]\cdot\frac{R^2}{n_i\cdot\nu}\cdot\left[e^{n_i\cdot\frac{\nu}{R^2}\cdot(t)}-e^{n_i\cdot\frac{\nu}{R^2}\cdot(t-\Delta t)}\right]+\eta\cdot\frac{[v_{(t+\Delta t)}-v_t]}{\Delta t}\cdot\frac{R^2}{n_i\cdot\nu}\cdot\left[e^{n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}-e^{n_i\cdot\frac{\nu}{R^2}\cdot t}\right]\right)=\\
&\frac{[v_t-v_{(t-\Delta t)}]}{\Delta t}\cdot[1-\eta]\cdot\frac{m_i\cdot R^2}{n_i\cdot\nu}\cdot\left[e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}-e^{-2\cdot n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}\right]+\eta\cdot\frac{[v_{(t+\Delta t)}-v_t]}{\Delta t}\cdot\frac{m_i\cdot R^2}{n_i\cdot\nu}\cdot\left[1-e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}\right]
\end{aligned} \tag{A10}$$

The final revised form of efficient numerical solution of the integral convolution by Schohl is as follows:

$$\tau_u(t+\Delta t)\approx\frac{2\cdot\mu}{R}\cdot\sum_{i=1}^{j}\underbrace{\left[\begin{aligned}&y_i(t)\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}+\eta\cdot\frac{m_i\cdot R^2}{\Delta t\cdot n_i\cdot\nu}\cdot\left[1-e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}\right]\cdot\left[v_{(t+\Delta t)}-v_t\right]+\\&[1-\eta]\cdot\frac{m_i\cdot R^2}{\Delta t\cdot n_i\cdot\nu}\cdot\left[e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}-e^{-2\cdot n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}\right]\cdot[v_t-v_{(t-\Delta t)}]\end{aligned}\right]}_{y_i(t+\Delta t)} \tag{A11}$$