INVERSE METHOD FOR AXISYMMETRIC AND BLADE-TO-BLADE VISCOUS FLOW MODELS IN CYLINDRICAL COORDINATES USING STREAM-FUNCTION TRANSFORMATION

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May 9, 2012

ABSTRACT

A new, inverse method for viscous 2D, laminar and turbulent duct flows will be presented. The method is based on incompressible Navier-Stokes equations transformed to the stream function coordinate system (von Mises coordinates). The method formulation is described for axisymmetric ($\partial/\partial\phi = 0$) and cylindrical models ($u_r = 0$). Then the design problem with appropriate boundary conditions is solved numerically in stream-function coordinates domain. Next, the geometrical shape of the boundary walls is obtained through the integration along streamlines. The singularity problem at the wall and its special treatment are considered. The numerical implementation and validation of the model are described. Potential applications to design problem of fluid-flow machines are considered and developments towards 3D method are suggested.

Keywords: design method, inverse problem, stream function coordinates

NOMENCLATURE

u	velocity component
p	pressure
Re	Reynolds number
k, ε	turbulent kinetic energy and its dissipation rate
β	artificial compressibility coefficient
r, ϕ, z	radial, angular and axial coordinates
ψ	stream function, stream function coordinate
τ	pseudo-time in marching technique
J	Jacobian of the transformation
c	series expansion constant
U_r, U_ϕ, U_z	radial, angular and axial reference velocity scales
$R, R\Phi, Z$	radial, angular and axial reference length scales
ϵ	estimated error of the method

SUBSCRIPTS AND SUPERSCRIPTS

Ω	the solution domain
$\partial\Omega$	the boundary of solution domain
in	at the inlet of the domain
out	at the inlet of the domain
w	at the wall boundary
z, ϕ, r	axial direction, angular and radial direction
$'$	component in stream function coordinates

1 INTRODUCTION

Many engineering problems involve design of a geometrical shape (e.g., the nozzle wall or blading system) which should satisfy prescribed conditions (Figure 1). Some of these conditions may not refer directly to geometric quantities, but to some flow parameters (for example: the pressure distribution along wall). Yet, since the geometry is not known at this stage, the design process is brought to the field of inverse methods. Many of such methods in fluid dynamics rely on some model simplifications, which affects accuracy. Inverse methods were developed using such concepts as 2D blade-to-blade or through-flow models [1, 2]. Mixed direct-inverse techniques of moving boundary/transpiration concepts have been proposed even for viscous flows [3, 4] in recent years. However, those methods need to handle time-consuming remeshing. With the growth of computer power, the optimisation techniques [5, 6] (in general, less restrictive) have gained attention. However, performing full optimisation process on a daily basis is still prohibitive for most applications. Consequently, there is a need of inverse methods to circumvent those limitations.

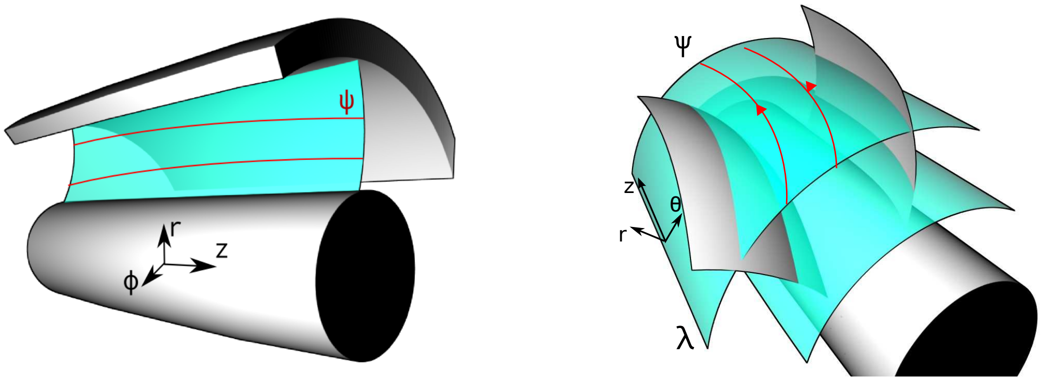

Figure 1: Axisymmetric and cylindrical (blade-to-blade) channel design problems.

The history of stream function coordinates (SFC, also known as von Mises coordinates) goes back to the boundary layer analysis in von Mises work [7] in 1927. For direct (analysis) problems, the SFC method was used for potential and viscous flows in curved geometries [8, 9, 10, 11] and even for multiphase flows [12]. Up to nowadays, the SFC technique has been adapted also to inverse methods of various 2D and 3D design problems [8, 10, 13, 14, 15, 16, 17]. However, mostly inviscid flows were the subject of the inverse design with stream function coordinates, due to singularity of transformation along no-slip walls. Keller [16] presented an inverse method with possible application to 3D viscous flows. However, Scascighini [17] proved that such an approach is applicable only to the lamellar flows. This is due to the use of natural coordinates [18] which exist if and only if the velocity field satisfies the condition $\mathbf{u}\cdot(\nabla \times \mathbf{u}) = 0$, where $\mathbf{u}$ is the velocity vector. Yet, the assumption about the flow being lamellar is quite restrictive. It is fully satisfied for 2D flows or irrotational 3D flows. Also, much of work was done in the matter of approximate treatment of viscous effects in 2D and quasi-3D flows [17, 19] using the Integral Boundary Layer approach, suitable for inertia-dominated (high Reynolds number) flows.

A new, fully inverse method (no solution of direct problem in the design domain) for 2D incompressible viscous flows, without explicit remeshing, is presented in the paper. In the direct problem (flow analysis), the system of governing equations is solved to obtain fields of flow variables (velocity, pressure, etc.) for a given shape of the boundary, initial and boundary conditions (e.g., no-slip wall, inlet, outlet, etc.) Yet, in inverse problems at least a part of geometry is unknown. One can replace some of geometrical variables by a fluid flow variable. That will affect the boundary condition which must be changed, too. One of the possibilities is the replacement through the transformation of equations to the stream function coordinates. The transformation gives the ability to eliminate one geometrical variable (here, r for axisymmetric flow or ϕ in cylindrical case) by use of a flow variable (here, the stream function ψ). The essence of the approach and details of mathematical model for both cases are described in Section 2.

In Section 3, implementation details are presented. The solution algorithm based on pseudo-time marching technique was applied. The artificial compressibility method was used to obtain pressure. The discretisation scheme are described for each of the model equations. The influence of wall singularity problem on numerical solution is considered in detail.

The numerical results of axisymmetric and cylindrical models are presented in Section 4. The design of straight channel was used as a validation case. The analytical solution of this problem (Poiseuille flow) is compared with that numerically obtained by the stream function inverse method. The convergence of the solution process is analysed. Next, a more sophisticated nozzle design problem is considered. Next, the approach for cylindrical model and the application to turbulent flows are considered (in the meaning of the Reynolds-averaged Navier-Stokes equations with corresponding turbulence model).

2 TWO-DIMENSIONAL MODELS OF INCOMPRESSIBLE FLOW

The steady three-dimensional incompressible, viscous flow is governed by equations in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial r} + \frac{\partial u_z}{\partial z} = 0, \tag{1}$$

$$u_r\frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\phi^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{Re}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \phi^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi} - \frac{u_r}{r^2}\right), \tag{2}$$

$$u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + u_z\frac{\partial u_\phi}{\partial z} + \frac{u_r u_\phi}{r} = -\frac{1}{r}\frac{\partial p}{\partial \phi} + \frac{1}{Re}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u_\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{\partial^2 u_\phi}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2}\right), \tag{3}$$

$$u_r\frac{\partial u_z}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi} + u_z\frac{\partial u_z}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \phi^2} + \frac{\partial^2 u_z}{\partial z^2}\right). \tag{4}$$

Using above system we will consider two different formulations of two-dimensional models. Both are important in turbomachinery applications: axisymmetric flow without swirl (where $\frac{\partial}{\partial \phi} = 0$, $u_\phi = 0$) and cylindrical flow (where $u_r = 0$). However, the method is applicable to general models without this simplifications.

2.1 Axisymmetric flow

Let us consider the channel flow as in Fig. 1. The axisymmetric flow with assumption of no tangential velocity in non-dimensional form is governed by:

$$\frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{\partial u_z}{\partial z} = 0, \tag{5}$$

$$u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}=-\frac{\partial p}{\partial r}+\frac{1}{Re}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u_r}{\partial r}+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}\right), \tag{6}$$

$$u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{Re}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u_z}{\partial r}+\frac{\partial^2 u_z}{\partial z^2}\right). \tag{7}$$

However, in typical channel flow cases one of the flow directions is dominant. Assuming that length scales satisfy $R \ll Z$ and velocity scales behave as $U_R \ll U_Z$, the momentum Eq. (6) and (7) can be reduced to:

$$u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}=-\frac{\partial p}{\partial r}+\frac{1}{Re}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right), \tag{8}$$

$$u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{Re}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u_z}{\partial r}\right). \tag{9}$$

Performing this step is not necessary to solve the inverse problem, however it simplifies the implementation of the method. The accuracy of the solution is not affected significantly as stated in [20].

The stream function ψ in axisymmetric coordinates is defined as:

$$ru_r=-\frac{\partial \psi}{\partial z}, \tag{10}$$

$$ru_z=\frac{\partial \psi}{\partial r}. \tag{11}$$

Let us consider the following coordinate transformation:

$$(r,z)\rightarrow(\psi,z') \tag{12}$$

where z' has the same direction vector as z. To transform derivatives to stream function coordinates one may use the chain rule:

$$\frac{\partial}{\partial r}=\frac{\partial \psi}{\partial r}\frac{\partial}{\partial \psi}+\frac{\partial z'}{\partial r}\frac{\partial}{\partial z'}, \tag{13}$$

$$\frac{\partial}{\partial z}=\frac{\partial \psi}{\partial z}\frac{\partial}{\partial \psi}+\frac{\partial z'}{\partial z}\frac{\partial}{\partial z'}. \tag{14}$$

This leads us to the formula:

$$\frac{\partial}{\partial r}=ru_z\frac{\partial}{\partial \psi}, \tag{15}$$

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial z'}-ru_r\frac{\partial}{\partial \psi}. \tag{16}$$

Please note that the new coordinate system is dependent on r coordinate implicitly via velocity u_r and u_z. This implies that $\partial z'/\partial z=1$ and z is independent of r direction, that is $\partial z'/\partial r = 0$. The Jacobian determinant of the transformation from cylindrical coordinates to stream function coordinates is:

$$J(r,z) = \begin{bmatrix} \frac{\partial\psi}{\partial r} & \frac{\partial\psi}{\partial z} \\ \frac{\partial z'}{\partial r} & \frac{\partial z'}{\partial z} \end{bmatrix} = \begin{bmatrix} ru_z & -ru_r \\ 0 & 1 \end{bmatrix} = ru_z. \tag{17}$$

Please note that the transformation is not directly invertible when $ru_z = 0$. The back transformation may be obtained from the stream function definition:

$$\frac{dr}{dz} = \frac{u_r}{u_z}. \tag{18}$$

Then we can obtain stream line shape through integration from inlet to outlet. However, both velocity components u_r and u_z vanish at the wall due to the no-slip and no-transpiration boundary conditions. This leads to indeterminate form $0/0$. To overcome this issue we propose to use in the near wall region the formula derived from Eq. (11):

$$\int_{r_{\Delta\psi}}^{r_w} r\,dr = \int_{\Delta\psi}^{0} \frac{d\psi}{u_z}. \tag{19}$$

Then we have:

$$r_w = \sqrt{2\int_{\Delta\psi}^{0} \frac{d\psi}{u_z} + (r_{\Delta\psi})^2}. \tag{20}$$

The above integral (20) does not behave well at the wall, either. Nonetheless, the integration of functions with boundary singularity is well established. It may be integrated easily down to the wall (here $\psi = \psi_w = 0$) from the nearest available stream line $\psi = \Delta\psi$ (which was obtained through equation (18)).

The mass Eq. (5) and momentum conservation Eqs. (8), (9) may now be transformed to the stream function coordinates using formulas (15) and (16):

$$u_z \frac{\partial r u_r}{\partial \psi} + \frac{\partial u_z}{\partial z'} - r u_r \frac{\partial u_z}{\partial \psi} = 0, \tag{21}$$

$$u_z \frac{\partial u_r}{\partial z'} = -r u_z \frac{\partial p}{\partial \psi} + \frac{1}{Re}\left(u_z \frac{\partial}{\partial \psi} r^2 u_z \frac{\partial u_r}{\partial \psi} - \frac{u_r}{r^2} \right), \tag{22}$$

$$u_z \frac{\partial u_z}{\partial z'} = -\frac{\partial p}{\partial z'} + r u_r \frac{\partial p}{\partial \psi} + \frac{1}{Re}\left(u_z \frac{\partial}{\partial \psi} r^2 u_z \frac{\partial u_z}{\partial \psi} \right), \tag{23}$$

which are coupled with Eqs. (18) and (20) for evaluation of r.

Please note that the convective term was reduced in stream functions coordinates.

2.2 Cylindrical flow

Similarly as in previous case, let us consider the flow on a cylindrical surface (like in blade-to-blade view). The radial velocity is equal to zero. Assuming that length scales satisfy $R\Phi \ll Z$ and velocity scales behave as $U_\Phi \ll U_Z$, the governing equation system is:

$$\frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0, \tag{24}$$

$$\frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + u_z \frac{\partial u_\phi}{\partial z} = -\frac{1}{r}\frac{\partial p}{\partial \phi} + \frac{1}{Re}\left(\frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \phi^2} - \frac{u_\phi}{r^2}\right), \tag{25}$$

$$\frac{u_\phi}{r}\frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re}\left(\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \phi^2}\right), \tag{26}$$

The stream function ψ in this coordinate system is defined as:

$$u_\phi = -\frac{\partial \psi}{\partial z}, \tag{27}$$

$$r u_z = \frac{\partial \psi}{\partial \phi}, \tag{28}$$

which is almost identical to axisymmetric case. However, r is constant in the flow domain. Let us consider the following coordinate transformation:

$$(\phi, z) \to (\psi, z') \tag{29}$$

where z' has the same direction vector as z. Using the same procedure we get the formulae of the transformation:

$$\frac{\partial}{\partial \phi} = r u_z \frac{\partial}{\partial \psi}, \tag{30}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'} - u_\phi \frac{\partial}{\partial \psi}. \tag{31}$$

The Jacobian determinant of the transformation has the same properties as previously:

$$J(\phi, z) = \begin{bmatrix} \frac{\partial \psi}{\partial \phi} & \frac{\partial \psi}{\partial z} \\ \frac{\partial z'}{\partial \phi} & \frac{\partial z'}{\partial z} \end{bmatrix} = \begin{bmatrix} r u_z & -u_\phi \\ 0 & 1 \end{bmatrix} = r u_z. \tag{32}$$

So the back transformation may be obtained from the stream function definition via:

$$\frac{d\phi}{dz} = \frac{u_\phi}{r u_z}. \tag{33}$$

As in axisymmetric case, we can obtain stream line shape through integration from inlet to outlet. The treatment of the near wall boundary region is being done by the formula derived from Eq. (28):

$$\phi_w = \frac{1}{r}\int_{\Delta\psi}^{0} \frac{d\psi}{u_z} \tag{34}$$

The mass Eq. (24) and momentum conservation Eqs. (25), (26) may now be transformed to the stream function coordinates:

$$u_z \frac{\partial u_\phi}{\partial \psi} + \frac{\partial u_z}{\partial z'} - u_\phi \frac{\partial u_z}{\partial \psi} = 0, \tag{35}$$

$$u_z \frac{\partial u_\phi}{\partial z'} = -u_z \frac{\partial p}{\partial \psi} + \frac{1}{Re} \left(u_z \frac{\partial}{\partial \psi} u_z \frac{\partial u_\phi}{\partial \psi} - \frac{u_\phi}{r^2} \right), \tag{36}$$

$$u_z \frac{\partial u_z}{\partial z'} = -\frac{\partial p}{\partial z'} + u_\phi \frac{\partial p}{\partial \psi} + \frac{1}{Re} \left(u_z \frac{\partial}{\partial \psi} u_z \frac{\partial u_z}{\partial \psi} \right), \tag{37}$$

which are coupled with Eqs. (33) and (34) for evaluation of ϕ.

2.3 Boundary conditions

A standard Navier-Stokes boundary conditions are applicable to the equation system. At the inlet one may set:

$$u_r(z' = z'_{in}, \psi) = u_{r,in}(z'), \qquad \textit{(axisymmetric)} \tag{38}$$
$$u_\phi(z' = z'_{in}, \psi) = u_{\phi,in}(z'), \qquad \textit{(cylindrical)} \tag{39}$$
$$u_z(z' = z'_{in}, \psi) = u_{z,in}(z'), \tag{40}$$
$$p(z', \psi = \psi_w) = p_{in}(z'), \tag{41}$$

and for outlet:

$$\frac{\partial u_r(z' = z'_{out}, \psi}{\partial z'} = 0, \qquad \textit{(axisymmetric)} \tag{42}$$
$$\frac{\partial u_\phi(z' = z'_{out}, \psi}{\partial z'} = 0, \qquad \textit{(cylindrical)} \tag{43}$$
$$\frac{\partial u_z(z' = z'_{out}, \psi)}{\partial z'} = 0. \tag{44}$$
$$p(z' = z'_{out}, \psi) = p_{out}(z'). \tag{45}$$

At the wall no-slip and no transpiration boundary conditions are applicable. However, to solve inverse problem one needs to apply Dirichlet condition for pressure. So, along the designed channel wall there is:

$$u_r(z', \psi = \psi_w) = 0, \qquad \textit{(axisymmetric)} \tag{46}$$
$$u_\phi(z', \psi = \psi_w) = 0, \qquad \textit{(cylindrical)} \tag{47}$$
$$u_z(z', \psi = \psi_w) = 0, \tag{48}$$
$$p(z', \psi = \psi_w) = p_w(z'). \tag{49}$$

3 NUMERICAL DISCRETISATION

The resulting equation system is highly non-linear with the structure similar to the original Navier-Stokes equations. For the sake of simplicity, the finite difference approach was adopted here for discretisation in space and the pseudo-time marching technique was used to obtain steady state solution. The artificial compressibility method was chosen to satisfy the continuity equation by adding an additional term to (21):

$$\frac{1}{\beta}\frac{\partial p}{\partial \tau} + u_z\frac{\partial r u_r}{\partial \psi} + \frac{\partial u_z}{\partial z'} - r u_r \frac{\partial u_z}{\partial \psi} = 0. \tag{50}$$

Consequently, Eqs. (22) and (23) were extended with pseudo-time derivative for u_r and u_z. This brings us to a problem of three pseudo-time-dependent equations coupled with Eqs. (18) and (20) (or in cylindrical system: (33), (34)) for evaluation of r (ϕ in cylindrical case). The equations are integrated numerically until a steady state is reached. The convergence criterion is used that:

$$\max\left(\frac{\partial u_r}{\partial \tau}, \frac{\partial u_z}{\partial \tau}, \frac{\partial p}{\partial \tau}\right)|_{\Omega} < \epsilon \tag{51}$$

The explicit Euler scheme was used for integration in time. The second order central scheme was used for discretisation of derivatives in space. The uniform grid was used for the sake of simplicity. The second order upwind scheme was used for convective terms in momentum equations.

Equations (18) and (33) are approximated with simple first order accurate formula. To integrate Eqs. (20) and (34) one must approximate u_z function near the wall. One way is to expand velocity profile in the standard way. In axisymmetric coordinates the velocity profile along the wall may be expanded into polynomial series:

$$u_z(r) = c_0 + c_1 r + c_2 r^2 + ... \tag{52}$$

or for cylindrical case:

$$u_z(r) = c_0 + c_1\phi + c_2\phi^2 + ... \tag{53}$$

However, the most significant term near the wall ψ_w is the linear part $c_0 + c_1 r$ (and corresponding $c_0 + c_1\phi$). We may transform the monomial $u_z(r) = cr$ to the stream function coordinates using formula:

$$\psi = \int r u_z dr \qquad \textit{(axisymmetric)}, \tag{54}$$

$$\psi = r\int u_z d\phi \qquad \textit{(cylindrical)}, \tag{55}$$

which after some algebraic manipulations will lead to:

$$u(\psi) = c'\sqrt[3]{\psi} \qquad \textit{(axisymmetric)}, \tag{56}$$

$$u(\psi) = c'\sqrt{\psi} \qquad \textit{(cylindrical)}. \tag{57}$$

On the other hand expanding binomial $u_z(r) = c_0 + c_1 r + c_2 r^2$ will bring us to the function with the following (Puiseux) expansion series:

$$u(\psi) = c_0 + c_1' \sqrt[3]{\psi} + c_2' (\sqrt[3]{\psi})^2 + c_3' \psi + ... \qquad \textit{(axisymmetric)}, \tag{58}$$

$$u(\psi) = c_0 + c_1' \sqrt{\psi} + c_2' \psi + c_3' \psi \sqrt{\psi} + ... \qquad \textit{(cylindrical)}. \tag{59}$$

On this basis, one may state that it is convenient to expand the velocity profile near the wall in the same manner for a better accuracy. Such an approach was firstly proposed by Richard von Mises, in his first work on the stream function coordinates [7]. In our implementation we cut the expansions (58) and (59) after the second term. So, we have demonstrated how to avoid the near-wall singularity, which was one of the concerns in the paper of Dulikravich [14].

4 RESULTS

4.1 Poiseuille flow design

A steady laminar flow through a straight channel was chosen as a test case for both models. Its analytical solution is well known so one can easily estimate the error of the method.

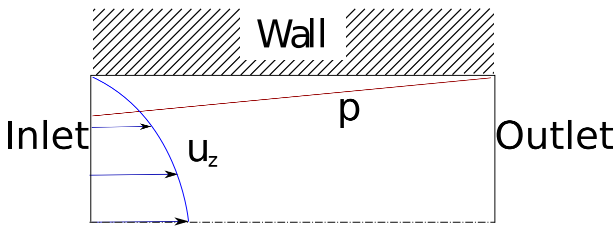

Figure 2: *Boundary conditions for design of the Hagen-Poiseuille channel flow.*

For pipe flow (axisymmetric case), the steady Navier-Stokes equations reduce to the following viscous relation (no convective terms) in the stream function coordinates:

$$\frac{\partial p}{\partial z'} = \frac{1}{Re} \left(u_z \frac{\partial}{\partial \psi} r^2 u_z \frac{\partial u_z}{\partial \psi} \right). \tag{60}$$

For straight blade-to-blade flow (cylindrical case) the corresponding equations are:

$$\frac{\partial p}{\partial z'} = \frac{1}{Re} \left(u_z \frac{\partial}{\partial \psi} u_z \frac{\partial u_z}{\partial \psi} \right). \tag{61}$$

Let us assume that the walls be at the stream function ψ_w and the middle of the channel will be at $\psi_m = 0$ (the rotation axis of the pipe for axisymmetric case). Then one may find solution in the form of velocity distribution:

$$u_z(\psi) = 2\sqrt{\psi_w - \psi}$$
$$u_r(\psi) = 0, \tag{62}$$

and similarly for cylindrical model:

$$u_z(\psi) = 1 - 4\cos^2\left(\frac{5\pi - \arccos(\frac{\psi}{\psi_w})}{3}\right),$$
$$u_\phi(\psi) = 0. \tag{63}$$

In both cases it implies the constant pressure gradient along z' direction. So the pressure distribution along the walls is linear function of z':

$$p(z') = p_{in} - \frac{4}{Re}z. \tag{64}$$

The corresponding stream lines will be straight and parallel to each other as:

$$\frac{dr}{dz} = \frac{u_r}{u_z} = 0. \qquad \textit{(axi)} \tag{65}$$
$$\frac{d\phi}{dz} = \frac{u_\phi}{ru_z} = 0. \qquad \textit{(cyl)} \tag{66}$$

The analytical solutions (62), (63) and (64) will be respectively used as inlet and wall boundary conditions for inverse design problem of straight channel (see Fig. 2).

The Poiseuille flow was designed following the methodology just presented. The velocity and pressure fields were obtained.

Table 1: The inverse problem for the Poiseuille flow: estimated error ϵ, Eq. (68), as a function of the number N of grid nodes in ψ direction.

N	ϵ_{axi}	ϵ_{cyl}
32	0.0212	0.0090
64	0.0149	0.0052
128	0.0109	0.0035
256	0.0084	0.0027

From the analytical solution (66) it is known that the channel walls should be parallel to z direction. The difference between known (theoretical) and numerically designed shape represents an error of the inverse method:

$$\epsilon_{\text{axi}} = \int_{z_{\text{in}}}^{z_{\text{out}}} |r_{\text{w,th}}(z) - r_{\text{w,num}}(z)|\, dz, \tag{67}$$
$$\epsilon_{\text{cyl}} = \int_{z_{\text{in}}}^{z_{\text{out}}} |\phi_{\text{w,th}}(z) - \phi_{\text{w,num}}(z)|\, dz. \tag{68}$$

As seen in Table 1, the total error of inverse method decreases with increasing grid density. The character of the convergence is slightly below linear. This is because the total error of the method is the sum of the errors originating from Eq. (18) and Eq. (20) in axisymetric case; or the sum of Eqs. (33) and (34) in cylindrical model. The amplitude of both constituent errors may change non-linearly, as a result of nonlinear displacement of the node nearest to the wall $\psi = \psi_w - \Delta\psi$ with the increasing grid density. This is the direct consequence of velocity dependent transformation (SFC). However, the maximum estimated error is satisfactorily small. Note that the Poiseuille flow is dominated by diffusion, and a possible error in the convective part is not estimated in this case. Nonetheless, the test case confirms that the method is applicable to the viscous flows.

4.2 Nozzle flow design example

The pressure drop in the Poiseuille flow (cf. Sec. 4.1) is connected to viscous stress. If the pressure change along the wall is different from the one in the pipe flow (for a given inlet velocity profile), the convective term will participate in it. These possibilities bring us to a problem of curvilinear channel flow design. This test case is an example of such a design issue.

We slightly changed the pipe flow case by increasing pressure drop along the wall. The velocity profile at the inlet was left the same (quadratic in r). The sketch of the nozzle design problem is shown in Fig. 3.

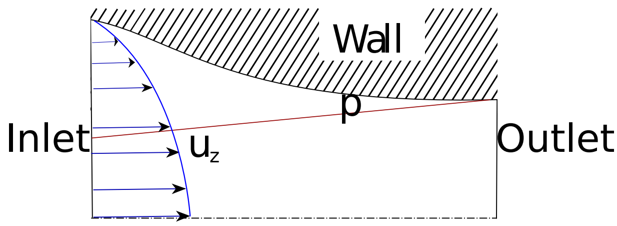

Figure 3: *Boundary conditions for design of the curved nozzle flow.*

In a such nozzle design outflow velocity should be much more uniform than that at the inlet. The geometry was obtained with the method. Next, it was an subject of the flow analysis with external solver. In Fig. 4 the velocity map of obtained geometry is shown. The resulting outflow velocity is more uniform, as predicted.

4.3 Curved blade-to-blade channel

As the last example we show the design problem of the curvilinear channel in cylindrical model. It is a first step to develop the inverse method for blade/airfoil shapes.

In practical applications, the flow regime is turbulent. The method is easily expandable to turbulent flows by the eddy viscosity assumption, with a change of the Reynolds number

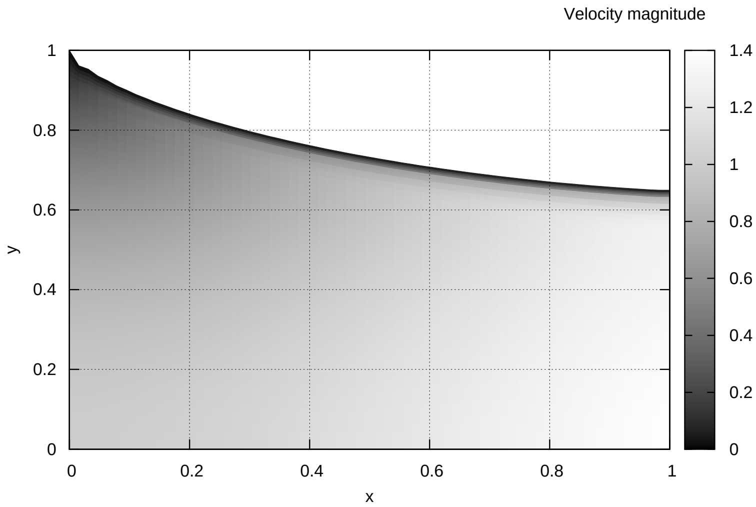

Figure 4: *Velocity vectors of the flow through the channel.*

to the effective Reynolds number:

$$\frac{1}{Re} = \frac{1}{Re_m} + \frac{1}{Re_t}, \tag{69}$$

where Re_m and Re_t are Reynolds numbers with respect to molecular and turbulent viscosities.

To model turbulent flow case, we used standard two-equations $k - \varepsilon$ model for the purpose. After the transformation to stream function coordinates for cylindrical model, it becomes:

$$u\frac{\partial k}{\partial z'} = (\frac{1}{Re_m} + \frac{1}{\sigma_k}\frac{1}{Re_t})\left(u_z\frac{\partial}{\partial \psi}u_z\frac{\partial k}{\partial \psi}\right) + P_k - \varepsilon, \tag{70}$$

$$u\frac{\partial \varepsilon}{\partial z'} = (\frac{1}{Re_m} + \frac{1}{\sigma_\epsilon}\frac{1}{Re_t})\left(u_z\frac{\partial}{\partial \psi}u_z\frac{\partial \varepsilon}{\partial \psi}\right) + \frac{\varepsilon}{k}(C_1 P_k - C_2\varepsilon), \tag{71}$$

where P_k is production term and $\sigma_k, \sigma_\epsilon, C_1, C_2, C_\mu$ are standard model constants. The turbulent Reynolds number is obtained from $1/Re_t = C_\mu k^2/\epsilon$.

To construct an example of non-symmetric nozzle, we slightly changed the flow case by increasing pressure drop along one of the walls and setting no pressure drop on the second wall, as contrasted to the Poiseuille flow considered before. The velocity profile at the inlet was left the same (quadratic in ϕ). The sketch of the nozzle design problem is shown in Fig. 5.

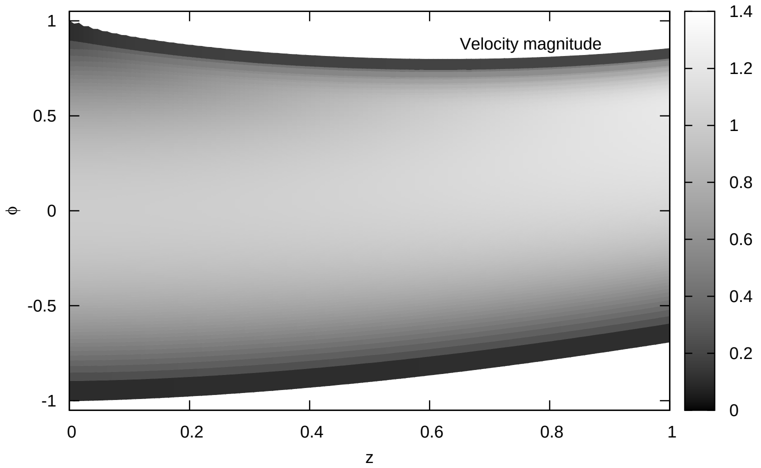

Figure 5: *Velocity vectors of the turbulent flow through the channel.*

5 Conclusions

A new inverse method for 2D viscous flow design was formulated and implemented for axisymmetric and cylindrical models. The method of curvilinear stream function coordinates was used. We excluded recirculating flow regions (normally, reflecting a bad design) and situations where the stream-wise velocity component u_z is zero. However, for the unavoidable treatment of walls (where, obviously, both velocity components are zero) we have proposed a technique to deal with the indeterminacy of backward transformation at the domain boundary as described in Section 2.

The simple pseudo-time stepping technique was adopted. The second order numerical scheme for space discretisation was implemented, however with a caveat of a slow-down convergence in certain cases (Sec. 4.1). It has been demonstrated that the overall error of the method is linked to the limit behaviour of flow velocity at the wall. The usage of Puiseux series for velocity function expansion near the wall was proposed as a solution to this problem. Comparisons with analytically-known case of the Poiseuille channel and the numerical nozzle flow design case have served to validate the approach. The total error of the method is kept at an acceptable level. Convergence of the solution process may be seen indirectly as a proof of existence of the inverse problem solution. Tests of the method show its applicability for design problems with prescribed pressure distribution (such as nozzles).

As shown in the last example, the RANS turbulence model with conjunction with eddy viscosity concept can be successfully applied after a suitable coordinate transformation. The validation of the model for turbulent flows is currently worked on.

An extension of the method to compressible flow problems is straightforward. In such a case we need to change the stream function formulation. The stream function will depend not only on the velocity field but also on the mass density ρ and its reference level ρ_0:

$$\rho r u_r = \rho_0 \frac{\partial \psi}{\partial z}, \quad \rho r u_z = -\rho_0 \frac{\partial \psi}{\partial r}. \tag{72}$$

The full, three-dimensional extension of the method is the subject of current work of the authors. However, in three dimensional space a second stream function must be prescribed [13, 1]. So, the approach can basically be extended to 3D with the methodology similar to [15, 16].

The method has shown its flexibility and other extensions may be attempted as well. Though, the method is unable to solve the inverse problem with reverse flow regions (like boundary layer separation) which may be seen as a disadvantage. In engineering one does not want to design such a geometry, which (due to reverse flow) will imply additional losses, usually to be avoided. Therefore, a variety of geometries which may be designed with the method may encourage to implement a general purpose, inverse solver of the Reynolds-averaged Navier-Stokes equations in the same manner as direct problem solvers appear in modern Computational Fluid Dynamics.

The authors hope that the proposed approach (with further developments) may be useful as a fast tool in turbomachinery design process, specially for the preliminary design where state-of-the-art optimisation methods are still quite expensive.

6 ACKNOWLEDGEMENTS

The work was supported by National Science Committee under grant N° 6694/B/T02/2011/40 for IFFM Polish Academy of Sciences in Gdańsk.

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