XX Polish Fluid Mechanics Conference,
Gliwice, 17-20 September 2012

WEIGHTING FUNCTION APPROXIMATION IN TRANSIENT PIPE FLOW

URBANOWICZ KAMIL
West Pomeranian University of Technology, Szczecin, Faculty of Mechanical Engineering and Mechatronics, Department of Mechanics and Machine Elements, Poland, Al. Piastów 19, 70-310 Szczecin
E-mail: kamil.urbanowicz@zut.edu.pl

Abstract

In transient liquid pipe flow analysis a very important problem is accurate and effective modeling of hydraulic resistances. In order to simulate unsteady resistances one need to solve in a numerical way so called integral convolution of the mean local acceleration of liquid and a weighting function. In effective numerical calculations a necessary condition is that the weighting function needs to be defined as a finite sum of exponential terms. In a laminar flow that function keep a constant shape, in turbulent its shape is changing and is dependent of the instantaneous Reynolds number. In this article an easy method is present that enable to determine proper weighting function very straight forward way in a quick time. Also comparison of determined functions and prototype ones in a frequency domain will be presented.

Key words: numerical fluid mechanics, transient flow, hydraulic resistance, weighting function

INTRODUCTION

Modelling hydraulic resistance occurring during transient flow of liquids through pressure lines is important. Failing to consider the maximum or minimum possible pressures in a hydraulic system at the design phase can lead to a major system damage or, for long transmission pipelines, even injuries. For an ordinary water hammer phenomena without cavitation, Joukowski relationship is a simple dependence helpful in determining the maximum pressures. On the other hand, if cavitation is given, pressure fluctuation can be significantly larger, which calls for numerical modelling of such systems.
It has been known for some time that wall shear stress exerted on the pipe wall is a sum of quasi-steady τ_q component and a parameter τ_u related to flow unsteadiness:

$$\tau = \tau_q + \tau_u \tag{1}$$

This approach was pioneered by Zielke [1] who demonstrated that, for laminar flow, parameter τ_u can be correctly described analytically in the form of convolution integral of the product of momentary liquid velocity variation and a weighting function (that has fixed shape in the case of laminar flow):

$$\tau_u = \frac{2\cdot\mu}{R}\cdot\int_0^t w(t-u)\cdot\frac{\partial v}{\partial t}(u)\,du \tag{2}$$

where: μ – dynamic viscosity; R – inner radius of pipe; v – instantaneous mean flow velocity; t – time; u – time, used in convolution integral; w(t) – weighting function

Similarly, formula (2) can be also used for determining parameter τ_u for turbulent flow, which was demonstrated in the works by Zarzycki [2-4] and Vardy & Brown [5-10], but with one difference: the weighting function dedicated to turbulent flow should be used, the shape of which depends on the momentary value of the Reynolds number.

Literature offers two methods for resolving the convolution integral (2): the classic one (inefficient) presented by Zielke [1] (slightly improved by Vardy-Brown [11] in 2010) and the efficient one presented by Trikha [12] (later on improved by Kagawa et al. [13] and Schohl [14]). The efficient solutions require that the weighting function is written in the form of a finite sum of exponential expressions:

$$\sum_{i=1}^{k} m_i e^{-n_i \cdot \hat{t}} \tag{3}$$

where: $\hat{t} = \nu \cdot t / R^2$ – diensionless time, m_i and n_i – coefficients of weighting function

Approximating the classical weighting functions (Zielke [1] for laminar flow and Vardy & Brown [7] or Zarzycki [3] for turbulent flow) is not easy to accomplish. The last four decades brought many works, the authors of which dealt with estimating coefficients of efficient weighting functions both for laminar and turbulent flows. In time, as computerization progressed, the number of exponential expressions making the efficient function increased in cycles (with the exception of the weighting function by Kagawa et al. that was recalled after many years, as the Japanese original article remained unknown for long to the world), which can be seen from Table 1.

Table 1: Matrix of works concerning efficient weighting functions for laminar and turbulent flows

LAMINAR FLOW						
Author:	Trikha [12] **1975**	Schohl [14] **1993**	Vardy & Brown [9] **2004**	Kagawa at al. [13] **1983**	Vitkovsky at al. [15] **2004**	Urbanowicz [16] **2009**
Number of exponential terms:	**3**	**5**	**9**	**10**	**10**	**26**
TURBULENT FLOW						
Author:	Vitkovsky et al. [15] **2004**	Urbanowicz [16] **2009**	Vardy & Brown [10] **2007**	Zarzycki & Kudźma [4] **2004**	Kudźma [17] **2005**	Urbanowicz [16] **2009**
	basing on classic Vardy & Brown [7] weighting function			basing on classic Zarzycki [3] weighting function		
Number of exponential terms:	**10**	**16**	**17**	**6**	**8**	**22**

In 2004, Vardy & Brown [9] presented a certain numerical method for estimating coefficients of the efficient weighting functions. However, because this method is used to determine the sought values of coefficients based on a system of equations, the number of which depends on the number of exponential expressions sought, the method is inefficient and complex. The following work will present another method that will enable much simpler estimation of the values of coefficients m_i and n_i representing an efficient weighting functions.

UNSTEADY WALL SHEAR STRESS COMPONENT

Convolution Integral

The major shortcoming in the classic numerical solution of the convolution integral presented by Zielke (2) is the fact that the sought parameter τ_u is computed from a longer sum in each time step (because the sum takes account of all velocity fluctuations from the beginning of the transient state) [1]:

$$\tau_u = \frac{2\mu}{R}\sum_{j=1}^{k-1}\left(v_{i,k-j+1} - v_{i,k-j}\right)\cdot w\left(j\Delta\hat{t} - \frac{\Delta\hat{t}}{2}\right) \tag{4}$$

where: k – current numerical time step, $\Delta\hat{t} = \Delta t\frac{\nu}{R^2} = \frac{L\cdot\nu}{f\cdot c\cdot R^2}$ – dimensionless time increase, L – length of the pressure line, f – number of analyzed cross pipe section, c – pressure wave velocity.

According to the foregoing equation (4), the last velocity change, that is multiplied by the value of the weighting function determined for the smallest dimensionless time $w(\Delta\hat{t}/2)$, has the strongest effect on parameter τ_u. As it is known, the weighting function takes large values for small dimensionless times and small values for relatively larger times (Fig. 1).

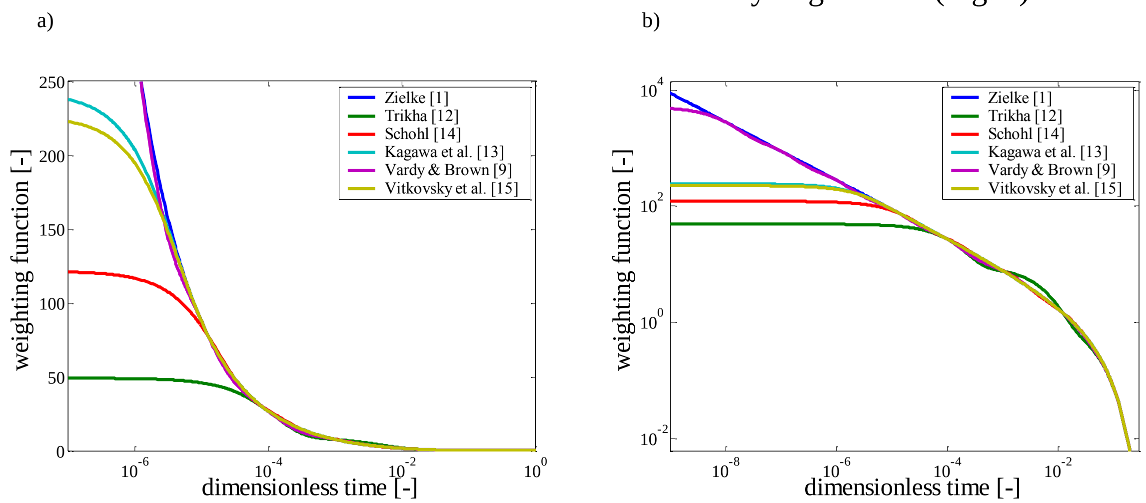

Fig. 1 Pattern of laminar weighting functions

Trikha [12] was the first to present certain efficient solution of the convolution integral in 1975 but Kagawa et al. [13] improved the solution in 1983 as it had been based on too many simplifying assumptions:

$$\tau_u = \frac{2\mu}{R}\sum_{i=1}^{k}\left(\underbrace{y_i(t)\cdot e^{-n_i\cdot\Delta\hat{t}} + m_i\cdot e^{-n_i\cdot\frac{\Delta\hat{t}}{2}}\cdot\left[v(t+\Delta t) - v(t)\right]}_{y_i(t+\Delta t)}\right) \tag{5}$$

where: $y_i(t)$ – parameter computed for the previous time step (during the occurrence of the transient state, i.e., for the first time step of numerical analysis $y_i(0)=0$).

Yet another efficient form of convolution integral solution was proposed by Schohl [14] in 1993:

$$\tau_u = \frac{2\mu}{R}\sum_{i=1}^{k}\left(\underbrace{y_i(t)\cdot e^{-n_i\cdot\Delta\hat{t}} + \frac{m_i\left(1-e^{-n_i\cdot\Delta\hat{t}}\right)}{n_i\cdot\Delta\hat{t}}\cdot\left[v(t+\Delta t) - v(t)\right]}_{y_i(t+\Delta t)}\right) \tag{6}$$

The foregoing efficient solutions (5) and (6) require that the weighting function is written in the form of a finite sum of exponential expressions

Classic Forms of the Weighting Function

In addition to presenting the classic convolution integral solutions (2) and (4), Zielke proposed a correct form of the weighting function for <u>laminar flow</u> in his work of 1968 [1]:

$$w(\hat{t}) = \sum_{i=1}^{6} m_i \hat{t}^{(i-2)/2}\text{, for } \hat{t} \le 0.02 \tag{7a}$$

$$w(\hat{t}) = \sum_{i=1}^{5} e^{-n_i \cdot \hat{t}}\text{, for } \hat{t} > 0.02 \tag{7b}$$

where: $m_1 = 0.282095$; $m_2 = -1.25$; $m_3 = 1.057855$; $m_4 = 0.9375$; $m_5 = 0.396696$; $m_6 = -0.351563$; $n_1 = 26.3744$; $n_2 = 70.8493$; $n_3 = 135.0198$; $n_4 = 218.9216$; $n_5 = 322.5544$.

Literature offers two weighting functions models for <u>turbulent flow</u>:

- Vardy & Brown [7] $$w(\hat{t}, Re) = \frac{A^* e^{-B^* \hat{t}}}{\sqrt{\hat{t}}} \tag{8}$$

where: $A^* = \sqrt{1/4\pi}$ and $B^* = Re^{\kappa}/12.86$; $\kappa = \log_{10}(15.29/Re^{0.0567})$.

- Zarzycki [3] $$w(\hat{t}, Re) = \frac{C}{\sqrt{\hat{t}}} \cdot Re^n \tag{9}$$

where: $C=0.299635$; $n= -0.005535$.

SIMPLE METHOD OF APPROXIMATING THE WEIGHTING FUNCTION

A single exponential expression will not provide for the correct mapping of the classic weighting function onto the required range of dimensionless time. This means that a function approximating the classic weighting function should be a finite sum of such expressions:

$$w_{apr.}(\hat{t}) = \sum_{i=1}^{k} m_i e^{-n_i \cdot \hat{t}} \tag{10}$$

The process of computing coefficients m_i and n_i describing subsequent exponential expressions is not as easy as it could seem. This is demonstrated, last but not least, by the first efficient weighting functions proposed in literature, which featured great simplicity (few exponential expressions) but poor mapping of the approximated function (consider, for instance, the weighting function proposed by Trikha [12] and Schohl [14] for laminar flow or by Zarzycki-Kudzma [4] for turbulent flow).

The authors of many functions made ancillary use of complex statistical and fine-tuning procedures [4,13,14,15] while computing their coefficients:

- **Schohl** [14] applied a fine-tuning procedure based on the least squares method (so he did manage to match 5 exponential expressions to 136 points describing the pattern of the classic weighting function by Zielke). Vitkovsky et al. [15], Kudzma & Zarzycki [4] and others estimated their functions similarly.
- **Kagawa et al.** [13] followed the classic function by Zielke (from the smallest values on the right) and fitted in new exponential expressions in real time. Vardy [9] himself appreciated the potential of this approach by concluding that this method could be used

for determining the smallest number of expressions required for the approximation while preserving a predefined level of accuracy. Urbanowicz [16] used a similar method in his work.

- **Vardy & Brown** [9] proposed a numerical procedure in which parameter n_i values are adopted for known dimensionless times and then an appropriate system of equations is construed and resolved to find individual coefficients m_i.

The following article presents a simple alternative method based on determination of subsequent exponential expressions in steps (as in Kagawa et al.) and adjusting the weighting function so that the trace crosses certain points selected using the classic weighting function (as in Vardy-Brown).

The range of applicability of efficient weighting functions should be sufficient enough to ensure correct simulation of actual turbulent flows. Vardy & Brown suggest [9] that the range of applicability of the new functions should depend on the time step $< 10^{-2} \cdot \Delta\hat{t};\ 10^{3} \cdot \Delta\hat{t} >$ adopted for the numerical analysis. And this remark seems to be right because it implies that the range of applicability of the function should indeed depend on the tested hydraulic system.

For an approximation of a turbulent classic weighting function, it is best to perform the approximation for as small Reynolds number as possible (for instance $Re=2\cdot10^{3}$). This approach serves its purpose because, for such numbers, the shape of the turbulent weighting function resembles the shape of the laminar weighting function and, what is important, provides near-zero values (smaller than 10^{-4}) for dimensionless times $\hat{t} > 5.47\cdot10^{-2}$. On the other hand, if large Reynolds numbers are used (such as $Re=10^{6}$) the classic turbulent weighting function provides near-zero values much sooner: as early as for dimensionless times $\hat{t} > 1.2\cdot10^{-3}$. Accordingly, approximating the new function will be more difficult for so small values of the weighting function.

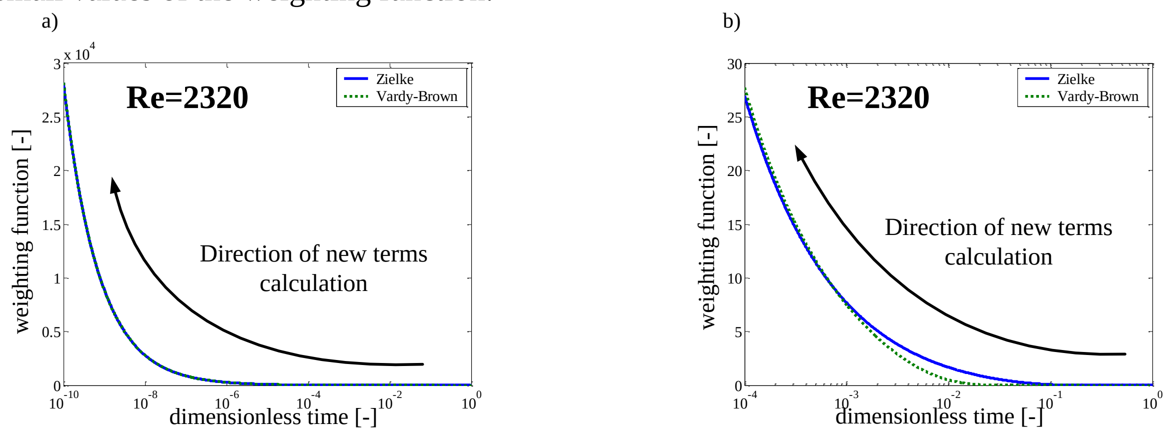

Fig. 2 Direction of determination of new exponential expressions

As stated, the new procedure is a stepped one in which new values of coefficients m_i and n_i describing a single exponential expression $m_i \cdot \exp(-n_i \cdot \hat{t})$ will be determined in each subsequent step. The determination of exponential expressions starts from large values of dimensionless times (times $\hat{t} \approx 10^{0}$ can be regarded as such) for which the classic weighting function provides smallest values (near-zero) (Fig. 2). The following Fig. 3 presents a block diagram of subsequent steps of developing an efficient function representing a sum of five exponential expressions $w_{apr.}(\hat{t}) = \sum_{i=1}^{5} m_i e^{-n_i \cdot \hat{t}}$.

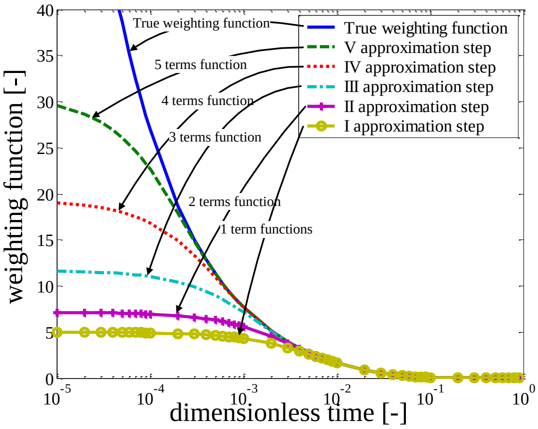

Fig. 3 Determination of subsequent exponential expressions for the new efficient weighting function

The determination of new exponential expressions will be a result of the assumption that the new weighting function is supposed to cross evenly spaced points of the logarithmic scale. The points are the values of the classic weighting function $w_{cl.}$ (estimated using Zielke weighting function $w_{cl.,Z}(\hat{t})$ for laminar flow or the Vardy-Brown weighting function $w_{cl.,V\text{-}B}(\hat{t},Re)$ for turbulent flow). The crossing of the new weighting function by the two points (Fig. 4) relates to the meeting of the following system of equations (used each time to compute one exponential expression):

$$\begin{cases} w_{cl.}(\hat{t}_1) = m_{i+1}\cdot \exp(-n_{i+1}\cdot \hat{t}_1) + \sum_{r=1}^{i} m_r \cdot \exp(-n_r \cdot \hat{t}_1) \\ w_{cl.}(\hat{t}_2) = m_{i+1}\cdot \exp(-n_{i+1}\cdot \hat{t}_2) + \sum_{r=1}^{i} m_r \cdot \exp(-n_r \cdot \hat{t}_2) \\ \hat{t}_1 = E1(i) = 10^{(s-2\cdot i\cdot \Delta s)}; \qquad \hat{t}_2 = E2(i) = 10^{(s-2\cdot i\cdot \Delta s + k\cdot \Delta s)} \end{cases} \tag{11}$$

Where: $w_{cl.}$ – Zielke weighting function value $w_{cl.,Z}(\hat{t})$ for laminar flow or Vardy-Brown weighting function value $w_{cl.,V\text{-}B}(\hat{t},Re)$ for turbulent flow; s – starting exponent; i – step (i=1,2,…,h for laminar flow leaving the first original exponential expressions ($m_1=1$, $n_1=26.3744$); in turbulent flow: i=0,1,2,…,h); Δs – exponent increment; k – increment multiplier (this work tested the values of the parameter within the <0.0001;1> range)

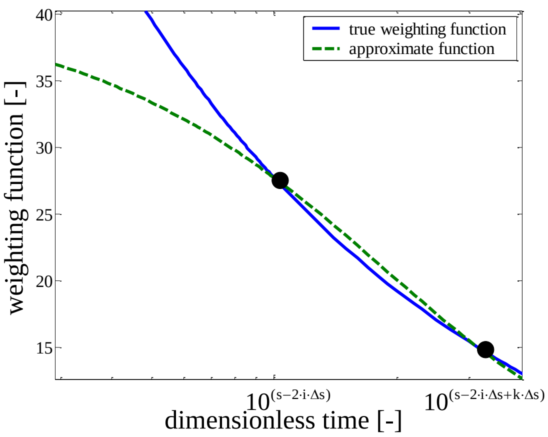

Fig. 4 Transition of the new estimated efficient weighting function through two points

Using the following notation:

$$\begin{cases} w_{cl.}(\hat{t}_1) = C1 \\ \sum_{r=1}^{i} m_r \cdot \exp(-n_r \cdot \hat{t}_1) = C2 \\ w_{cl.}(\hat{t}_2) = C3 \\ \sum_{r=1}^{i} m_r \cdot \exp(-n_r \cdot \hat{t}_2) = C4 \end{cases} \tag{12}$$

Will provide the following system of equations:

$$\begin{cases} C1 = m_{i+1} \cdot \exp(-n_{i+1} \cdot E1) + C2 \\ C3 = m_{i+1} \cdot \exp(-n_{i+1} \cdot E2) + C4 \end{cases} \tag{13}$$

where: in case of turbulent flow if i=0: C2=0 and C4=0; while for laminar flow (leaving the first exponential expression of the classic Zielke weighting function) if i=1: $C2 = \exp(-26.3744 \cdot \hat{t}_1)$ and $C4 = \exp(-26.3744 \cdot \hat{t}_2)$

Transforming the foregoing system of equations (13) the following equation can be produced:

$$\frac{C1 - C2}{\exp(-n_{i+1} \cdot E1)} - \frac{C3 - C4}{\exp(-n_{i+1} \cdot E2)} = 0 \tag{14}$$

The foregoing equation enables determination of the unknown **“n_{i+1}”** numerically, using the **FZERO** function representing a module of MATLAB or, for instance, using the **BISECTION** method. Once **“n_{i+1}”** is found, **“m_{i+1}”** is computed using one of the following equations:

$$\frac{C1 - C2}{\exp(-n_{i+1} \cdot E1)} = m_{i+1} \qquad \text{or} \qquad \frac{C3 - C4}{\exp(-n_{i+1} \cdot E2)} = m_{i+1} \tag{15}$$

Complete numerical procedure used to determine new expressions of the weight function for laminar and turbulent flows is presented schematically in Appendix A.

More accurate functions (featuring better matching, or representing smaller relative percentage error) are obtained by applying smaller values of parameter k (increment multiplier). This is because the reduction of this parameter can be followed by a reduction of parameter Δs. Numerous simulation tests using the foregoing method demonstrated that parameter Δs had certain limits. The lower limit for laminar flow was Δs=0.24 (at k=0.0001). For this value the approximating weighting function features the best match to the Zielke function. After adjusting the values of estimated parameters „m_i” by multiplying the values by correction factor z_l=0.999615 ($m_{icl} = z_l \cdot m_i$ – the role of the correction factor is to spread evenly the distribution of the relative percentage error to minimize the absolute percentage error) the relative error in the domain of time was within the ± 0.04 % range.

For turbulent flow, on the other hand, the approximating function was most accurate for Δs=0.235 (at k=0.0001). The relative percentage error for the function remained within the ± 0.032 % range (with correction factor z_t=0.99964).

Using smaller values of parameter Δs is not possible because such values would make the proposed procedure unstable and produce estimation errors.

In addition to good matching in the time domain, the estimated functions feature good matching in the frequency domain which is conformed in the following Fig. 5 showing the matching of the new estimated laminar function.

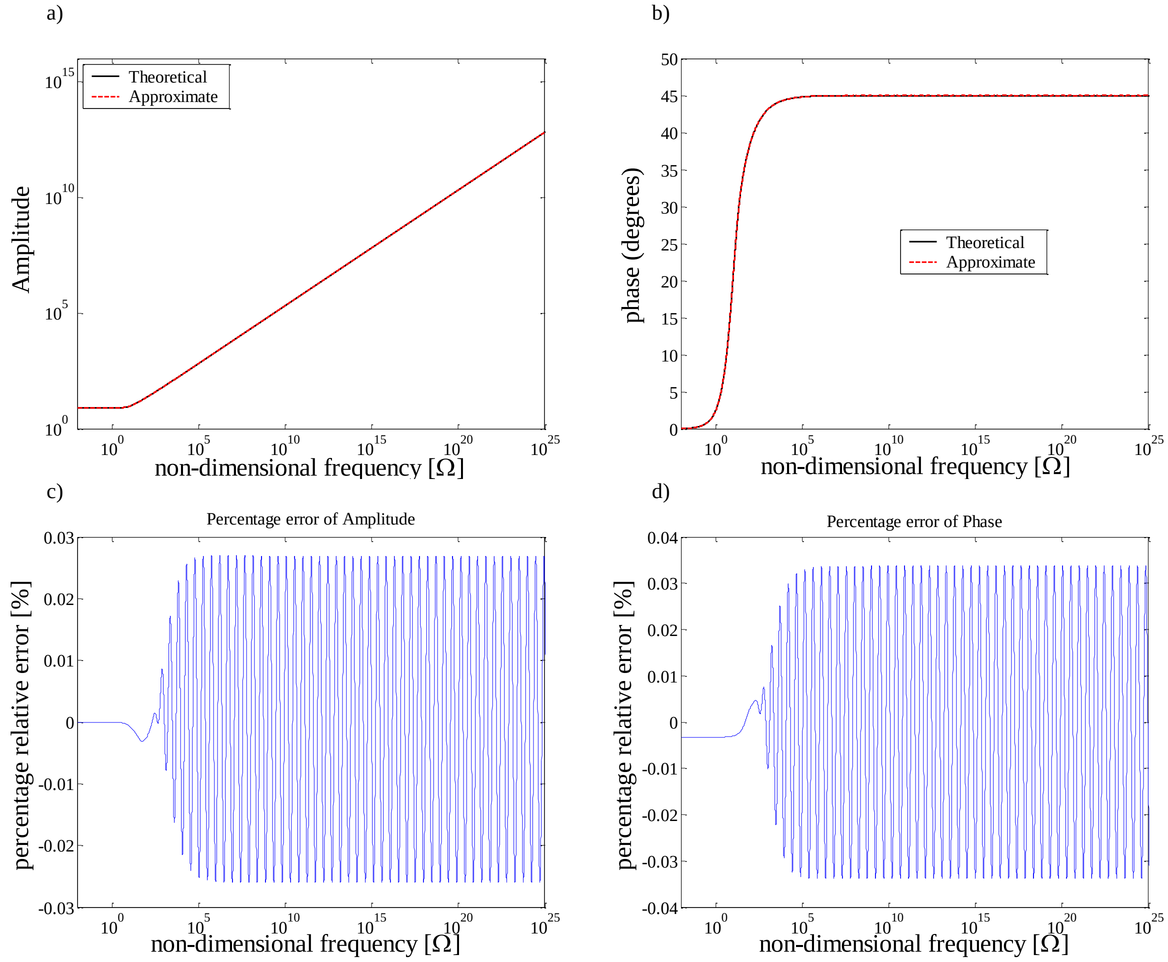

Fig. 5. Analytical laminar weighting function and estimated approximation

Note that selecting correct starting parameters is important for numerical resolution of non-linear equations, such as equation (14). See Appendix B for a broader coverage of the starting value of parameter **“$\mathbf{n_{i+1}}$”**.

CONCLUSION

The following paper presents a simple method for rapid determination of new weighting functions written in the form of a finite sum of exponential expressions. This notation in the form of the finite sum will enable efficient determination of unsteady friction losses (using the efficient solution of convolution integral presented by Kagawa et al. [12] or Schohl [13]). The proposed method will make simulating transient states in complex hydraulic, water supply or heating networks much simpler.

Main Points:

1. Given k=0.0001, the new weighting function is much better matched (with smaller relative percentage error) than for k=1 because reducing parameter k enables reduction of the lower limit of increment of exponent Δs.

2. There exist certain bottom and upper limits of parameter Δs (exponent increment) between which new exponential expressions can be determined. Once the limits are exceeded, the approximation of new expressions can produce errors (e.g., estimating subsequent parameters with values smaller than the previous ones: $m_{i+1}<m_i$ or $n_{i+1}<n_i$) or even can be impossible to complete.

3. The lower (minimum) limit of parameter Δs (exponent increment) required for correct estimation of coefficients describing the weighting function is different for laminar and turbulent flows. We could say it depends on the pattern of the classic weighting function.

4. The number of estimated exponential expressions "h" should depend on the actually simulated transient state. We can follow the recommendation from Vardy & Brown formulated in their paper of 2004 [9] that proposes to adopt the value of the time step Δt from the $<10^{-2}\Delta t;\ 10^{3}\Delta t>$ range as the determinant for identifying the number of expressions. For $t>10^{3}\Delta t$ the values of the weight function should be assumed as null.

5. Note that each change of the form of the classic turbulent weighting function reflects on the minimum value of Δs that can be used for the foregoing procedure. This is because the turbulent function is partly based on experimental data and – considering inputs from ongoing, increasingly more accurate, experimental research – the form of the function will evolve by slightly changing its pattern.

The future work will be oriented towards developing a similar simple method (for determination of coefficients of efficient weight functions) directly in the domain of frequency.

REFERENCES

[1] ZIELKE W., *Frequency–Dependent Friction in Transient Pipe Flow*, Journ. of ASME, 90, March 1968, pp. 109–115.

[2] ZARZYCKI Z., *Opory niestacjonarnego ruchu cieczy w przewodach zamkniętych*, Prace naukowe Politechniki Szczecińskiej, nr 516, Szczecin 1994.

[3] ZARZYCKI Z., *On Weighting Function for Wall Shear Stress During Unsteady Turbulent Flow*, Proc. of 8th International Conference on Pressure Sergues, 12–14 April 2000, The Hague, The Netherlands, BHR Group Conference Series, No 39, pp. 529–534.

[4] ZARZYCKI Z., KUDŹMA S., *Simulations of transient turbulent flow in liquid lines using time – dependent frictional losses*, Proceedings of the 9th International Conference on Pressure Surges, BHR Group, Chester UK, 24–26 March 2004, pp. 439–455.

[5] VARDY A.E., BROWN J.M.B., *Transient, turbulent, smooth pipe friction*, J. Hydraul. Res. 33, 1995, pp. 435–456.

[6] VARDY A.E., BROWN J.M.B., *On turbulent unsteady, smooth – pipe friction,* Proc. of the 7th International Conf. on Pressure Surges – BHR Group, Harrogate, United Kingdom, 1996, pp. 289–311.

[7] VARDY A.E., BROWN J.M.B., *Transient turbulent friction in smooth pipe flows*, Journal of Sound and Vibration, Vol 25**9**, Issue 5, 2003, pp. 1011–1036.

[8] VARDY A.E., BROWN J.M.B., *Transient turbulent friction in fully rough pipe flows,* Journal of Sound and Vibration, Vol. 270, 2004, pp. 233–257.

[9] VARDY A.E., BROWN J.M.B., *Efficient Approximation of Unsteady Friction Weighting Functions*, Journal of Hydraulic Engineering, ASCE, Vol. 130, No. 11, 2004, pp. 1097–1107.

[10] VARDY A.E., BROWN J.M.B., *Approximation of Turbulent Wall Shear Stresses in Highly Transient Pipe Flows*, Journal of Hydraulic Engineering, Vol. 133, No. 11, 2007, pp. 1219-1228.

[11] VARDY A.E., BROWN J.M.B., *Evaluation of Unsteady Wall Shear Stress by Zielke's Method*. Journal of Hydraulic Engineering, Vol. 136, No. 7, 2010, pp. 453-456.
[12] TRIKHA A.K., *An Efficient Method for Simulating Frequency–Dependent Friction in Transient Liquid Flow*. Journ. of Fluids Eng., Trans. ASME, March 1975, pp. 97–105.
[13] KAGAWA T., LEE I., KITAGAWA A., TAKENAKA T., *High speed and accurate computing method of frequency–dependent friction in laminar pipe flow for characteristics method*, Trans. Jpn. Soc. Mech. Eng., Ser. A, 49 (447), 1983, pp. 2638–2644 (in Japanese).
[14] SCHOHL G.A., *Improved Approximate Method for Simulating Frequency – Dependent Friction in Transient Laminar Flow*, Journ.of Fluids Eng., Trans. ASME, Vol. 115, September 1993, pp. 420–424.
[15] VÍTKOVSKÝ J.P., STEPHENS M.L., BERGANT A., SIMPSON A.R., LAMBERT M.F., *Efficient and accurate calculation of Zilke and Vardy–Brown unsteady friction in pipe transients*, 9th International Conference on Pressure Surges, Chester, United Kingdom, 24–26 March 2004, pp. 405–419.
[16] URBANOWICZ K., *Modelowanie przebiegów przejściowych w przewodach ciśnieniowych z uwzględnieniem kawitacji i zmiennych oporów przepływu*, Praca doktorska, Szczecin, 2009.
[17] KUDŹMA S., *Modelowanie i symulacja przebiegów dynamicznych w układach hydraulicznych z uwzględnieniem niestacjonarnego tarcia cieczy w przewodach zamkniętych*, Praca doktorska, Szczecin, 2005.

APPENDIX A

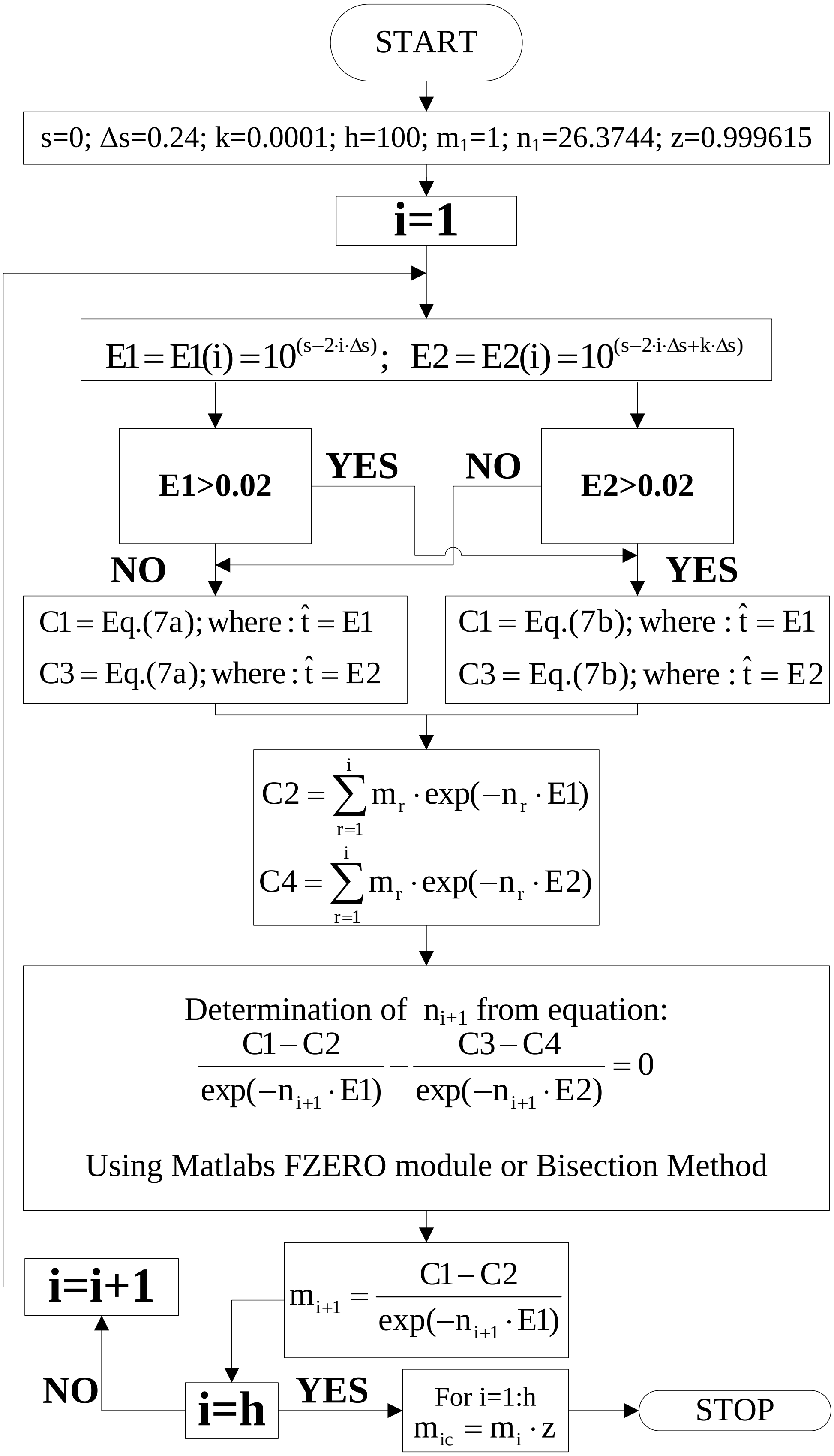

Fig. A1 Simplified block diagram of determination of subsequent exponential expressions for laminar flow

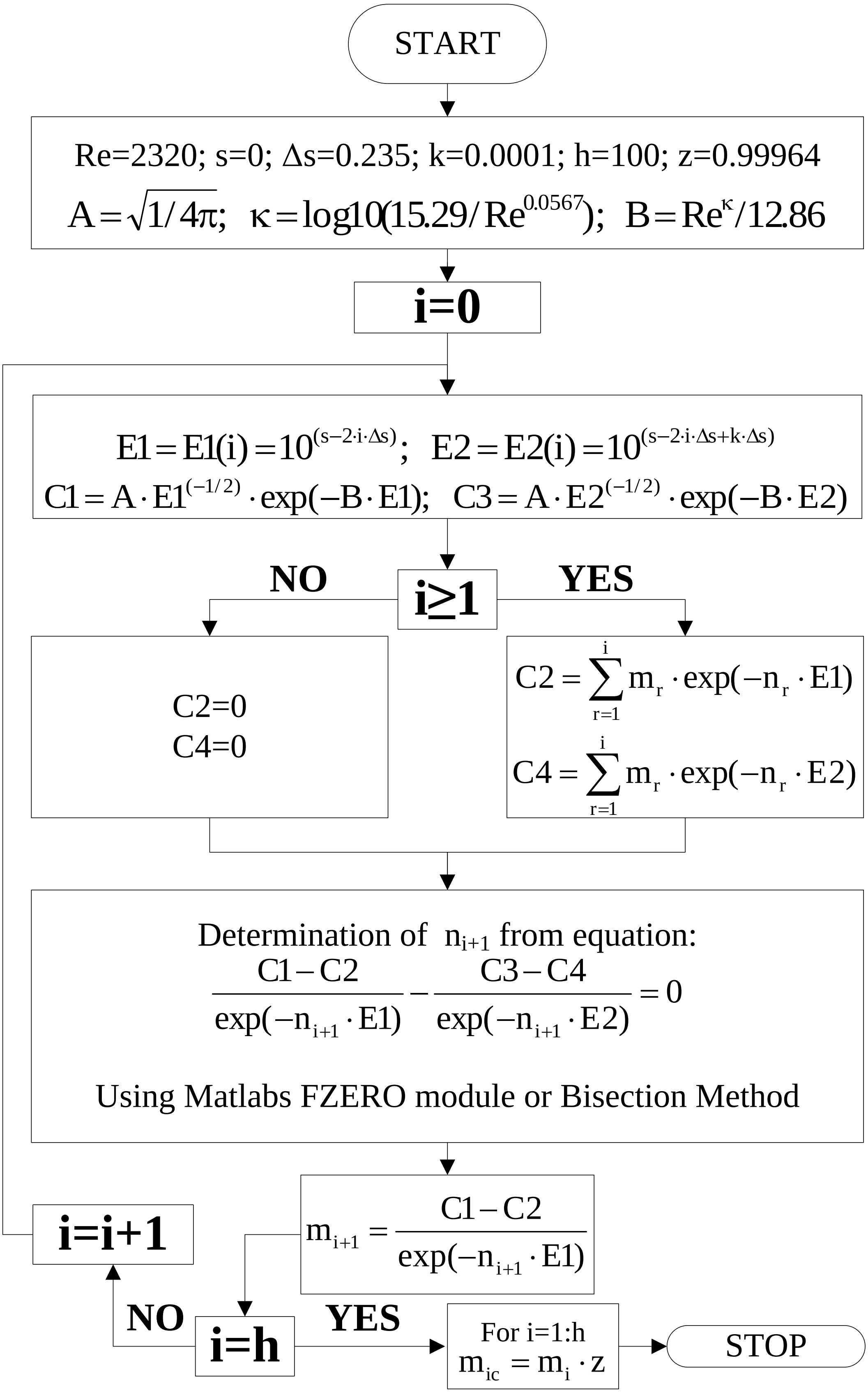

Fig. A2 Simplified block diagram of determination of subsequent exponential expressions for turbulent flow

APPENDIX B

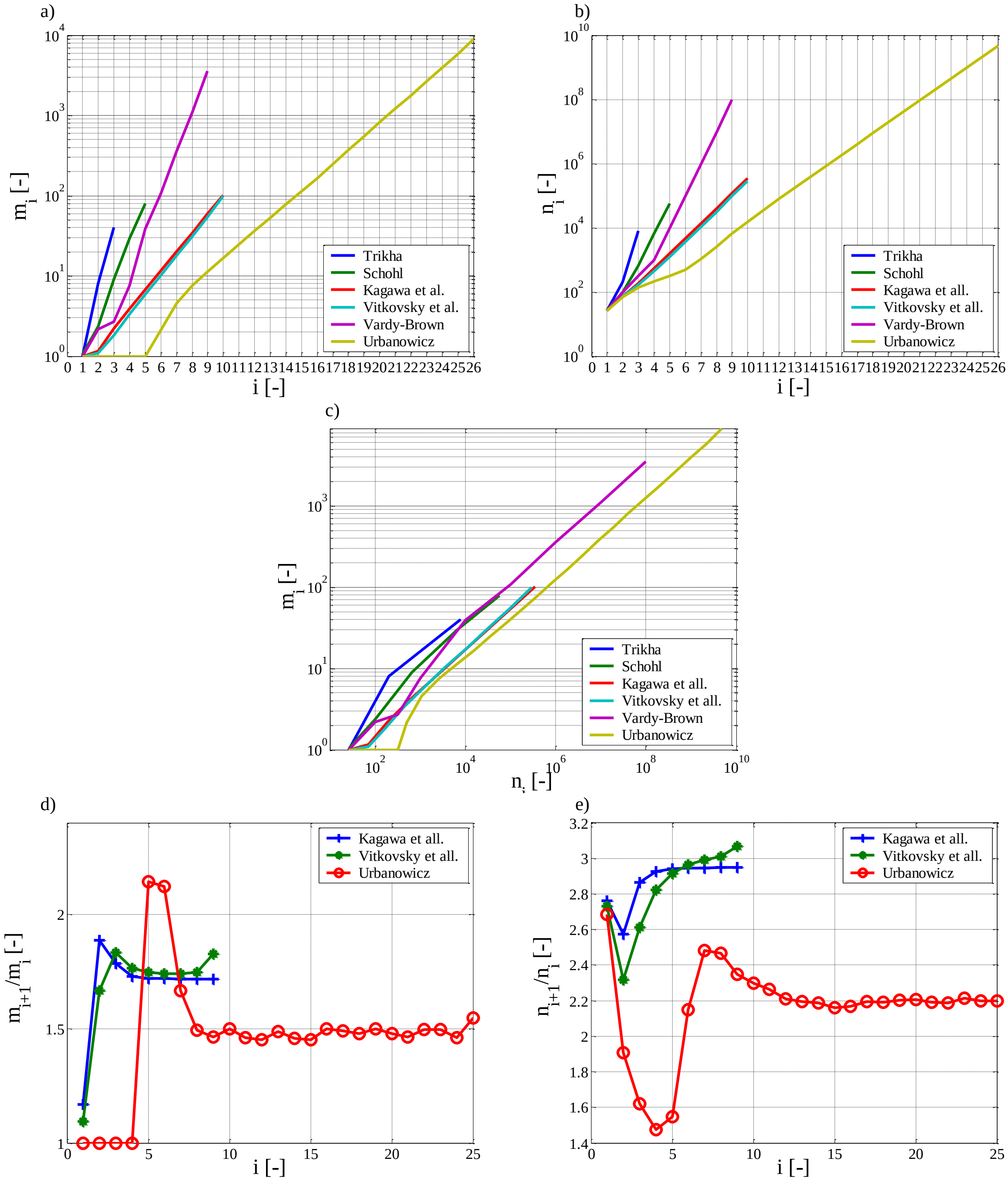

Fig. B1 Review of coefficients describing laminar efficient weighting functions

The known efficient laminar weighting functions (Fig. B1) were analyzed in detail to ensure correct selection of the starting values of coefficients **"n_{i+1}"**. Apart from the starting parameters m_i and n_i, diagrams B1 a), B1 b) and B1 c) show nearly linear relationship of the growth of the parameters sought in the logarithmic scale. Diagram B1 d) shows that the m_{i+1}/m_i ratio varies for the most accurate of the known functions within the 1-2.15 range. Also, the diagram shows that this relationship stabilizes to some extent starting from i=3 for the functions by Kagawa et al. and Vitkovsky et al. and starting from i=8 for the function by Urbanowicz (at 1.72 for Kagawa et al., 1.78 for Vitkovsky et al. and 1.48 for Urbanowicz). Diagram B1 e) confirms the foregoing observation for diagram B1 d). Namely, a similar trend is visible for the ratio of coefficients n_{i+1}/n_i that ranges from 1.46 to 3.1. In this case it is also clear that starting from i=4 for the function by Kagawa et al. and starting from i=12 for the function by Urbanowicz the ratio stabilizes to certain extent (at 2.94 for Kagawa et al. and 2.2 for Urbanowicz). However, this stabilization was not observed for the function by Vitkovsky et al, where the n_{i+1}/n_i ratio initially declined but then showed regular growth trend starting from i=2.

Also, the review of the foregoing diagrams shows clearly that subsequent values of these parameters can be estimated in practice with small error, which enables the research algorithm to estimate the exact values of the parameters without any error.