TO UNIFIED SYSTEM OF EQUATIONS OF CONTINUUM MECHANICS AND SOME MATHEMATICAL PROBLEMS FOR THIN-WALLED STRUCTURES

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Summary Proposed the unified mathematical models of continuum mechanics containing in particular Navier-Stokes, Euler Differential equations, systems of PDEs of Solid mechanics. In the second part the method of constructing 2D nonlinear models of von Kármán-Mindlin-Reissner type for binary mixture of porous, piezo and viscous elastic thin-walled structures with variable thickness is given.

Unified form for some nonlinear problems of continuum mechanics

Within the Newtonian mechanics and Noll's axiomatic we propose a unified dynamic system of pseudo-differential equations. Such form allows us to prove that the nonlinear phenomena observed in problems of solid mechanics can also be detected in Navier-Stokes type equations, and vice versa. For this case the basic system of PDE has the following form:

$$\rho\frac{D_\Gamma^2 u}{Dt^2}=f-(1-\Gamma)\nabla p+\nabla\left[(1+\nabla u)\tau\right],\qquad \frac{D_\Gamma^2 u}{Dt^2}=\begin{cases}\partial^2 u/\partial t^2,\Gamma=1\\ Dv/Dt,\Gamma=0\end{cases},\tag{1}$$

where ρ is a density, p is pressure, f is known volume forces, D/Dt is total or convective derivative, τ is stress tensor, $u=(u_1,u_2,u_3)^T$ and $v=(v_1,v_2,v_3)^T$ denote displacement and velocity vectors.

Newton's type law for viscous flow and Hooke's generalized law for solid structures may be recovered from the following expression:

$$\tau=\left[(1-\Gamma)\frac{\partial}{\partial t}+\Gamma\right]A_\Gamma\cdot\varepsilon\,.\tag{2}$$

Here symmetric matrix A_Γ corresponds to fluid if Γ = 0 and to solid media if Γ=1. The strain tensor is $\varepsilon=(\varepsilon_{11},\varepsilon_{22},\varepsilon_{33},\varepsilon_{23},\varepsilon_{13},\varepsilon_{12})^{\mathrm{T}}$, where $2\varepsilon_{ij}=\partial_i u_j+\partial_j u_i+u_{k,i}u_{k,j}$.

For conditions of conservation of mass or equations of continuity we have:

$$[(1-\Gamma)\partial_t+\Gamma]B_\Gamma[\varepsilon]=0\,,\tag{3}$$

where $B_0[\rho,\varepsilon]=\partial_t\rho+\nabla(\rho v)$. $B_1[\varepsilon]=(B_{11},B_{12},B_{13},B_{14},B_{15},B_{16})^T$ describes the St.Venant-Beltrami conditions. In the classical case, i.e. $B_{11}=2(u_{3,12})^2-2u_{3,11}u_{3,22}$.

For completeness, in addition to (1)-(3), the energy conservation (energy balance) equalities, which have evidently the same form with (3), must be considered.

From (1) Euler and Navier–Stokes' PDEs (see [1]) can be recovered if

$$\Gamma=0,\quad \nabla u=0\,,\ \ \tau_{ij}=-\frac{2}{3}\mu\delta_{ij}\,div v+\mu(v_{i,j}+v_{j,i})\,.$$

If $\Gamma=1$ and $\nabla(1+\nabla u)\tau=\left(\partial_j(\tau_{1j}+\tau_{kj}u_{1,k}),\partial_j(\tau_{2j}+\tau_{kj}u_{2,k}),\partial_j(\tau_{3j}+\tau_{kj}u_{3,k})\right)^T$ from (1) follows system of nonlinear PDEs of spatial theory of elasticity (see for example. [2]: If $\Gamma=0$ and $\nabla u\neq 0$, (1) represents Navier – Stokes type (new class) PDEs.

It's also possible to consider the Maxwell's electromagnetic dynamical system (see [3] Ch II, p.9).

Some mathematical problems of thin walled structures

One of the most principal objects in development of mechanics and mathematics is a system of nonlinear differential equations for elastic isotropic plate constructed by *von Kármán.* In 1978 *Truesdell* expressed a doubt: "Physical Soundness"of *von Kármán* system. This circumstance generated the problem of justification of *von Kármán* system. Afterwards this problem is studied by many authors, but with most attention it was investigated by Ciarlet. In particular, he wrote: "The *von Kármán* equations may be given a full justification by means of the leading term of a formal asymptotic expansion" ([2], p. 368). This result obviously is not sufficent for a justification of "Physical Soundness" of this system, because representations by asymptotic expansions is dissimilar and leading terms are only coefficients of power series without any "Physical Soundness."

Based on the [3], the method of constructing such anisotropic nonhomogeneous 2D nonlinear models of *von Kármán-Mindlin-Reissner (KMR)* type for binary mixtures; (poro/visco/piezo-electric/electrically conductive)elastic thin-walled structures with variable thickness is given, by means of which the terms become physically sound. The corresponding variables are quantities with certain physical meaning: averaged components of the displacement vector, bending and twisting moments, shearing forces, rotation of normals, surface efforts. The given method differs from the classical one by the fact that according to the classical method, one of the equations of *von Kármán* system represents one of *Saint-Venant*'s compatibility conditions, i.e. it`s obtained on the basis of geometry and not taking into account the equilibrium equations.

At last we remark that in dynamical cases the corresponding system contains wave processes not only in the vertical, but also in the horizontal direction. The corresponding equations are [3]:

$$\left(D\Delta^2 + 2h\rho\partial_{tt} - 2DE^{-1}(1+\upsilon)\rho\partial_{tt}\Delta\right)w = \left(1 - \frac{h^2(1+2\gamma)(2-\upsilon)}{3(1-\upsilon)}\Delta\right)\left(g_3^{\,+} - g_3^{\,-}\right)$$

$$2h\left(1 - \frac{2h^2(1+2\gamma)}{3(1-\upsilon)}\Delta\right)[w,\varphi] + h\left(g_{\alpha,\alpha}^{\,+} - g_{\alpha,\alpha}^{\,-}\right) - \int_{-h}^{+h}\left(tf_{\alpha,\alpha} - \left(1 - \frac{1}{1-\upsilon}\Delta\left(h^2 - t^2\right)\right)f_3\right)dt \quad , \tag{4}$$

$$\left(\Delta^2 - \frac{1-\nu^2}{E}\rho\Delta\partial_{tt}\right)\varphi = -\frac{E}{2}[w,w] + \frac{\nu}{2}\left(\Delta - \frac{2\rho}{E}\partial_{tt}\right)\left(g_3^+ + g_3^-\right) + \frac{1+\nu}{2h}f_{\alpha,\alpha} \quad . \tag{5}$$

From (4), (5) we get *von Kármán* system (in dynamical case too) if $\gamma = 0.5$, $\rho = g^{\pm} = f_\alpha = 0$. Thus *von Kármán* classical system gives the possibility to use methods of Harmonic Analysis. Since the new dynamical terms are $\Delta\partial_{tt}\varphi$ and also $\partial_{tt}\left(g_3^+ - g_3^-\right)$, therefore the *KMR* type (4)-(5) systems describe new nonlinear wave processes and it's evident that for them it is not possible to apply the Fourier Analysis technique .

The system (4)-(5) represents 2Dim mathematical model having clear practical meaning where it is possible to consider together the methods of Mathematical Analysis (in wide sense) and Optimal Control Theory of Bellman-Pontryagin.

Let us consider the main terms from (4):

$$D'\Delta[w,\varphi] = D'\left([\Delta w,\varphi] + [w,\Delta\varphi] + 2\left[\partial_\alpha w,\partial_\alpha\varphi\right]\right) \quad \left(D' = 4h^3(1+2\gamma)/3(1-\nu)\right), \quad D\Delta^2 w .$$

By using for simplicity the typical relations as $\partial_{11}\varphi = \overline{\sigma}_{22}$, $\partial_{12}\varphi = -\overline{\sigma}_{12}$, $\partial_{22}\varphi = \overline{\sigma}_{11}$, the first term may be rewritten in the following form:

$$\Delta[w,\varphi] = \left(\overline{\sigma}_{11}\partial_{11}\Delta w + 2\overline{\sigma}_{12}\partial_{12}\Delta w + \overline{\sigma}_{22}\partial_{22}\Delta w\right) + \left(\partial_{11}w\Delta\overline{\sigma}_{11} + 2\partial_{12}w\Delta\sigma_{12} + \partial_{22}w\Delta\sigma_{22}\right)$$

$$+ 2\left(\overline{\sigma}_{11,\alpha}\partial_{11}w_{,\alpha} + 2\overline{\sigma}_{12,\alpha}\partial_{12}w_{,\alpha} + \overline{\sigma}_{22,\alpha}\partial_{22}w_{,\alpha}\right).$$

The calculation and analysis by these expressions of a symbolical determinant show that the characteristic form of the system (4) and (5) may be strongly elliptical, hyperbolical and parabolical or mixed type. Other than that, in statical cases the system may have shock waves solutions.

References

[1] Fletcher C.: Computitional Teqniqus for Fluid Dynamics, 2. Springer Verlag, 1988.
[2] Ciarlet P.: Mathematical Elasticity Vol. II: Theory of Plates. North-Holland, Elsevier, 1997.
[3] Vashakmadze T.: The Theory of Anisotropic Elastic Plates. Kluwer Acad. Publ. & Springer, 2010.