UNIVERSAL DEFORMATIONS OF GROWING SOLIDS

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Summary A class of universal deformations of accreted hyperelastic incompressible bodies is studied. Accretion is realized by adding prestrained layers. The deformations correspond layerwise to the transformation of a parallelepiped to a hollow circular cylinder. Discrete and continuous accretion modes are considered and classified. Solutions of the boundary-value problems for the elastic Mooney-Rivlin potential are constructed. The solutions of the discrete accretion problems are shown to converge to solutions of the corresponding problems of continuous accretion as the number of layers increases and the layer thickness decreases.

The considered problems are related to the universal deformations of an incompressible medium:

$$\mathbf{x}(\mathbf{X}) = \sqrt{\frac{2(\mathbf{i}_1 \cdot \mathbf{X} + a)}{b}}[\mathbf{i}_1 \cos(b\mathbf{i}_2 \cdot \mathbf{X}) + \mathbf{i}_2 \sin(b\mathbf{i}_2 \cdot \mathbf{X})] + \mathbf{i}_3 \otimes \mathbf{i}_3 \cdot \mathbf{X}. \tag{1}$$

Here a and b are parameters of the deformation, which will further be assumed to be constant or differentiable functions of the coordinate r, $\mathbf{i}_1, \ldots, \mathbf{i}_3$ are the elements of cartesian basis. For deformations of the form (1), the position gradient $\mathbf{F}$ and left Cauchy–Green strain tensor $\mathbf{B}$ can be represented by the following decomposition ($\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ are the elements of local covariant basis, corresponding to the cylindrical co-ordinates):

$$\mathbf{F} = \frac{1}{br}\mathbf{e}_r \otimes \mathbf{i}^1 + b\mathbf{e}_\theta \otimes \mathbf{i}^2 + \mathbf{e}_z \otimes \mathbf{i}^3, \quad \mathbf{B} = \frac{1}{b^2r^2}\mathbf{e}_r \otimes \mathbf{e}_r + b^2\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_z \otimes \mathbf{e}_z. \tag{2}$$

The Cauchy stress (true stress) $\mathbf{T}$ can be determined up to the hydrostatic pressure p as follows: $\mathbf{T} = -p\mathbf{I} + J_1\mathbf{B} + J_{-1}\mathbf{B}^{-1}$, where $J_1 = 2\partial W/\partial I_1$ and $J_{-1} = -2\partial W/\partial I_2$ are reaction coefficients of the material. Since the tensor $\mathbf{B}$ is diagonal for the deformations under study, the stress tensor is also diagonal and its components have the following form:

$$\mathbf{T}=T^{rr}\mathbf{e}_r\otimes\mathbf{e}_r+T^{\theta\theta}\mathbf{e}_\theta\otimes\mathbf{e}_\theta+T^{zz}\mathbf{e}_z\otimes\mathbf{e}_z,\ T^{rr}=-p+\frac{J_1}{b^2r^2}+J_{-1}b^2r^2,\ T^{\theta\theta} = -\frac{p}{r^2}+J_1b^2+\frac{J_{-1}}{b^2r^4},\ T^{zz}=-p+J_1+J_{-1}.$$

The hydrostatic pressure p can be determined by equation $\boldsymbol{\nabla} \cdot \mathbf{T} = \mathbf{0}$. For the motion (1) it can be reduced to the form

$$T^{rr} = \int (J_1 - J_{-1}) \left(b^2r - \frac{1}{b^2r^3}\right) dr. \tag{3}$$

The above expressions for the stress remain true if the parameter b is assumed to be a function of the variable r. This fact will be used below. But if one assumes that $b = \mathrm{const}$ and takes the explicit expressions for the invariants $I_1 = I_2$ into account, then one can show that $T^{rr} = W + p_0$, where p_0 is the constant of integration.

Deformations of the form (1) belong to the class of controllable deformations, i.e., they can be realized by applying specially chosen surface forces to the boundary surfaces. Since the mathematical definition of deformations (1) contains the parameters a and b and the stress field is determined up to a constant p_0, it is possible to satisfy the prescribed boundary conditions pointwise on some boundary surfaces and then integrally on the other boundary surfaces, i.e., so that the corresponding net force and the net moment of the surface forces are equal to the prescribed values. The boundary surfaces of the first type are cylindrical surfaces π_0 and π_1 which can be specified analytically by the relations $r = r_0$ and $r = r_1$ with $r_0 = \sqrt{2a/b}, \quad r_1 = \sqrt{2(a + \Delta)/b}$. A constant pressure of respective intensity q_0 or q_1 can be prescribed on these surfaces, i.e., $\mathbf{T} \cdot \mathbf{e}^r\big|_{r=r_0} = q_0\mathbf{e}_r, \quad \mathbf{T} \cdot \mathbf{e}^r\big|_{r=r_1} = q_1\mathbf{e}_r$.

The expressions obtained above for the radial component of the stress tensor (3) permit us to formulate conditions as a system of two nonlinear equations for a and p_0:

$$W\big|_{r=\sqrt{2a/b}} = q_0 - p_0, \quad W\big|_{r=\sqrt{2(a+\Delta)/b}} = q_1 - p_0. \tag{4}$$

If the reaction coefficients J_1 and J_{-1} are functions of the invariants I_1 and I_2, then system (4) must be solved numerically. But if the reaction coefficients are constant, as in the case of the Mooney–Rivlin material $W = C_1(I_1 - 3) + C_2(I_2 - 3)$, $C_1 + C_2 = \mu/2$, $C_1 - C_2 = \mu\beta/2$, then it is possible to obtain solutions of system (4) in closed form:

$$a = \frac{\Delta}{2}\left(\sqrt{1 + \frac{1}{\Delta b(\Delta b - \delta)}} - 1\right), \quad p_0 = q_0 - \mu\left(ba + \frac{1}{4ba}\right).$$

On the plane boundaries ω_0 and ω_1, the form of the stress distribution is prescribed by the chosen deformation (1), but the net forces on these surface depend on the parameter b and can be chosen so that the conditions on the surfaces ω_0 and ω_1 can be satisfied integrally. The corresponding net force $\boldsymbol{F}$ and net moment $\boldsymbol{M}$ can be obtained by the formulas

$$\boldsymbol{F} = \mathbf{e}_{\langle\theta\rangle}h\left\{p_0(r_1-r_0)+\frac{\mu}{2}\left[b^2(r_1^3-r_0^3)+\frac{1}{b^2r_1}-\frac{1}{b^2r_0}\right]\right\},\ \mathbf{M} = \mathbf{e}_z\frac{\mu h}{2}\left[\frac{r_1^4 - r_0^4}{4}b^2 + \frac{r_1^2 - r_0^2}{2}\left(\frac{1}{b^2r_0r_1} - b^2r_0r_1\right) - \frac{1}{b^2}\ln\frac{r_1}{r_0}\right].$$

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Discrete growth. Consider finite sets Ω of bodies that, in the unstressed reference configuration, are parallelepipeds $\iota^k \subset \mathrm{E}$, $k = 1, 2, \ldots, n$, deformed by the transformation (1) into annual domains $\phi^k \subset \mathrm{E}$, i.e., they are hollow circular cylinders whose bases lie in the planes Π_0 and Π_1. Assume that the sides of the bases Δ^k and h^k of the parallelepipeds ι^k are the same, i.e., $\Delta^k = \Delta$ and $h^k = h$, and their heights H^k are generally distinct. The fact that the results of the deformation are annular domains rather than sectors is expressed by the relation $b^k = 2\pi/H^k$. Moreover, also assume that the images ϕ^k of the actual configurations of elements of Ω do not intersect pairwise and their union $\bigcup_{k=1}^{n} \phi^k$ is an annular domain in E. From the mechanical viewpoint, this condition determines a geometrically ideal contact of elements of Ω along the cylindrical boundary surfaces. The obtained deformed body can be treated as the result of discrete accretion if one assumes that, after such an assembly, there arise constraints on the contacting surfaces, and these constraints prevent elements of Ω from independent deformations. The following types of accretion will be distinguished:

Type 1. Accretion with a given actual geometry. The values of the parameters b^k and a^k are unknown. The radius of the outer boundary surface of the extreme layer is given. Consider a sequence of assemblies of one, two, and more elements of the set Ω, i.e., of the domain $\hat{\phi}^1 = \phi^1$, $\quad \hat{\phi}^2 = \phi^1 \cup \phi^1$, $\quad \ldots$, $\quad \hat{\phi}^m = \phi^1 \cup \phi^1 \cup \cdots \cup \phi^m$. Each m-assembly is associated with a system of equations for $m + 1$ unknown variables $\{b^m, \delta^1, \delta^2, \ldots, \delta^m\}$:

$$\frac{\Delta b^k - \delta^k + \sqrt{(\Delta b^k - \delta^k)^2 + 1 - \delta^k/\Delta b^k}}{(b^k)^2 - b^k\delta^k/\Delta} = \frac{\Delta b^{k+1} - \delta^{k+1} + \sqrt{(\Delta b^{k+1} - \delta^{k+1})^2 + 1 - \delta^{k+1}/\Delta b^{k+1}}}{(b^{k+1})^2 - b^{k+1}\delta^{k+1}/\Delta},$$

$$k = 1, \ldots, m-1, \quad \sum_{k=1}^{n} \delta^n = \delta, \quad R^2 = \frac{\Delta b^m - \delta^m + \sqrt{(\Delta b^m - \delta^m)^2 + 1 - \delta^m/\Delta b^m}}{(b^m)^2 - b^m\delta^m/\Delta}.$$

Let us solve the system successively as follows: first, we solve the system of two equations for the 1-assembly $\hat{\phi}^1$ and, thus, determine the elements b^1 and δ^1; then assuming that b^1 is known, we solve the system for the 2-assembly $\hat{\phi}^2$ for the unknowns b^2, δ^1, δ^2, etc. It should be noted that the parameters δ^k do not have to be recalculated anew for each assembly.

Type 2. Accretion with a given tension. The values of the parameters b^k and a^k are unknown. The tension F (or the bending moment M) in the extreme layer is given. Consider the sequence of assemblies $\hat{\phi}^m$. Each m-assembly is associated with a system of equations for $m + 1$ unknowns $\{b^m, \delta^1, \delta^2, \ldots, \delta^m\}$:

$$\frac{\Delta b^k - \delta^k + \sqrt{(\Delta b^k - \delta^k)^2 + 1 - \delta^k/\Delta b^k}}{(b^k)^2 - b^k\delta^k/\Delta} = \frac{\Delta b^{k+1} - \delta^{k+1} + \sqrt{(\Delta b^{k+1} - \delta^{k+1})^2 + 1 - \delta^{k+1}/\Delta b^{k+1}}}{(b^{k+1})^2 - b^{k+1}\delta^{k+1}/\Delta},$$

$$k = 1, \ldots, m-1, \quad \sum_{k=1}^{n} \delta^n = \delta, \quad \mathbf{e}_{\langle\theta\rangle} F = \mathbf{F}(b^m, \delta^m).$$

Continuous growth. The parameter b is replaced by a continuous function $b = b(t)$, which will be called the distortion parameter. This allows us to describe continuous accretion realized as a continuous process of adding material surfaces. The tensor field $\mathbf{F} = 1/(b(r)r)\mathbf{e}_r \otimes \mathbf{i}^1 + b(r)\mathbf{e}_\theta \otimes \mathbf{i}^2 + \mathbf{e}_z \otimes \mathbf{i}^3$ cannot now be the gradient of any deformation.

Type 1. Accretion with a given position of the growth surface. The volume of the accreted material is given as a function of a time-like parameter $\tau \in (0, t)$, i.e., $V = V(\tau)$. The position of the growth boundary $r_1(\tau)$ is given. Just as in the preceding case, the pressures $q_0(\tau)$ and $q_1(\tau)$ on the boundary surfaces π_0 and π_1 are given. It is required to determine the distortion function $b(r)$ and the functions $r_0(\tau)$ and $p_0(\tau)$. They can be found by solving the integral equation

$$\mu \int_{\sqrt{r_1(\tau)^2 - V/(\pi h)}}^{r_1(\tau)} \left[b^2(\zeta)\zeta - \frac{1}{b^2(\zeta)\zeta^3} \right] d\zeta + q_0(\tau) = q_1(\tau).$$

Type 2. Accretion with a given tension. Just as above, the volume of the accreted material $V = V(\tau)$, the pressures $q_0(\tau)$ and $q_1(\tau)$ on the boundary surfaces π_0 and π_1, and the surface stresses (tension) $g(\tau)$ on the growth boundary are given. It is required to find the distortion function $b(r)$ and $r_0(\tau)$, $r_1(\tau)$, $p_0(\tau)$. They are determined by the equations

$$q_1 + \mu \left(b^2 r^2 - \frac{1}{b^2 r^2} \right) = g(\tau), \quad \mu \int_{\sqrt{r_1(\tau)^2 - V/(\pi h)}}^{r_1(\tau)} \left[b^2(\zeta)\zeta - \frac{1}{b^2(\zeta)\zeta^3} \right] d\zeta + q_0(\tau) = q_1(\tau).$$

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