# EXISTENCE OF TRAVELING WAVE SOLUTIONS FOR AN INITIAL VALUE PROBLEM OF PIEZO-VISCOELASTICITY

O. P. Layeni* & A. P. Akinola**
*Center for Research in Computational and Applied Mechanics, University of Cape Town, Rondebosch 7701, Cape Town, South Africa
**Department of Mathematics, Obafemi Awolowo University, Ile-Ife 220005, Nigeria

*Summary* We study the existence of travelling wave (TW) solutions arising from the deformation of an electrically sensitive, generalized viscoelastic material (of inclusion type), under the quasistatic approximation. This is posed as a doubly nonlinear initial value evolution inclusion problem. The approach taken consists in the application of a TW transform, elements of set-valued analysis and calculus, together with a qualitative study of the resulting nonlinear evolution inclusions, while sieving out pertinent wave propagation properties.

Keywords: Traveling waves, evolution inclusion, Clarke's subdifferential
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## INTRODUCTION

In this article, we establish the existence of traveling waves (TWs) for an initial value problem governing the evolution of an electrically sensitive nonlinear viscoelastic material. Suppose that the material considered occupies an open bounded subset $\Omega \subset \mathbb{R}^3$ in time $\mathbb{I} \subset \mathbb{R}_0^+$. Specifically, we study the existence of traveling waves resultant of the system driven by the constitutive coupling

$$\begin{cases} \Sigma(\vec{x},t) & \in \mathfrak{A}\mathsf{E}(\dot{\vec{u}}) + \mathfrak{G}\mathsf{E}(\vec{u}) + \pi^T \nabla\varphi \\ \mathsf{D} & = \pi \mathsf{E}(\dot{\vec{u}}) + \mathsf{B}\nabla\varphi \end{cases} \tag{1}$$

and the dynamics

$$\begin{cases} \nabla \cdot \Sigma + \vec{f} = \rho \ddot{\vec{u}} & \text{in} \quad \Omega \times \mathbb{I}, \\ \vec{u}|_{t=0} = \vec{u}_0 \ \text{ and } \ \dot{\vec{u}}|_{t=0} = \vec{u}_1. \end{cases} \tag{2}$$

In equations $(1)$ and $(2)$ above, $\mathsf{E}$ is assumed the infinitesimal strain tensor map defined, via the displacement $\vec{u} = \vec{u}(\vec{x},t)$, as

$$\mathsf{E}\left(\vec{u}\right) = \frac{1}{2}\left(\nabla\vec{u} + \nabla\vec{u}^T\right), \tag{3}$$

$\Sigma$ the Cauchy stress tensor map, $\vec{f}$ a volume force and $\rho$ the material density. The nonlinear fourth rank maps $\mathfrak{A}$ and $\mathfrak{G}$, respectively, determine the viscoelastic and elastic response in the purely mechanical model, while the electrical sensitivity is determined by the piezo-electric map $\pi$, electric potential map $\varphi$, electric displacement map $\mathsf{D}$ and the permittivity map $\mathsf{B}$. Stacked dots denote derivative with respect to the temporal variable $t$, $\vec{x}$ the spatial variable vector, and superscripted $T$ the transpose operation.

It is worth noting that the model here contains for instance, as a special case, the generalized viscoelastic models of Blanchard and Le Tallec [1] and Visintin [2] as well as intersects, under certain restrictions on the defining maps, the electro-mechanical models of Selmani [3] and Kraynyukova and Alber [4].

## FORMULATION AND EXISTENCE RESULTS

The contribution made in this paper is two-fold: reformulation and the proof of existence of traveling waves. The one consists of re-framing the problem as a system of evolution inclusions in a finite dimensional space through the application of a pertinent traveling wave transform. The other regards placing appropriate and flexible constraints on the defining maps, thereby showing existence of solutions to the inclusion problem, making pertinent recourse to elements of multivalued differential equations, convex and functional analysis.

In the first instance, we have the following TW representation of equations $(1)$ and $(2)_1$.

**Lemma 1.** *Suppose a TW transform $\chi$ is given such that*

$$\begin{cases} \chi : \Omega \times \mathbb{R} \to \mathbb{R}, \ \ \chi\left(\vec{x},t\right) = \xi; \\ \xi = \vec{w}\cdot\vec{x} + \lambda t, \ \ \lambda \neq 0\,, \quad |\vec{w}| = 1, \end{cases} \tag{4}$$

*and $\vec{u}(\vec{x},t) = \vec{U}(\xi)$ with $\vec{U} : \mathbb{R} \longrightarrow \mathbb{R}^3$. Then equations $(1)$ and $(2)_1$ recast as*

$$\begin{cases} \mathsf{z} = \vec{w} \otimes \vec{U} \\ \begin{cases} \left(\mathfrak{A} - \alpha\lambda\rho\mathbb{1}\mathbb{1}\right)(\lambda\mathsf{z})^{''} + \mathfrak{G}\mathsf{z}^{'} - \pi^T(\vec{w}\varphi^{'}) + \vec{w}\otimes\vec{f}(\xi) \ni \tilde{0} \\ \pi\mathsf{z}^{'} - \mathsf{B}\vec{w}\varphi^{'} = q, \ \ \alpha \in \{0,1\}\,, \end{cases} \end{cases} \tag{5}$$

[a)] Corresponding author. E-mail: olawanle.layeni@uct.ac.za

*where the prime denotes derivative with respect to $\xi$, $\mathbb{1}\mathbb{1}$ is the fourth rank identity map, $\vec{w}$ a constant vector whose magnitude is the wavenumber, $\lambda$ the frequency of the wave, while $\alpha$ takes values $0$ or $1$ according as the deformation process is quasistatics or dynamics.*

A decoupling of the system of differential inclusion in equation $(5)$ is clearly possible when $\mathsf{B}$ is invertible. In such a case it leads to the subsequent, for which many schemes hold depending on the regularity of the maps therein.

**Lemma 2.** *Suppose $\mathsf{B}$ is invertible with $\mathsf{B}^{-1} = \mathsf{C}$. Then, finding solution(s) to the system of differential inclusion in equation $(5)$ is equivalent to seeking $\Phi$, such that*

$$\left\{ \begin{array}{l} \Phi = \Big[\mathsf{z}, \mathsf{a}\Big]^T \text{ , with } \mathsf{a} = \mathsf{z}^{'}; \\ \left\{ \begin{array}{l} \mathsf{H}\left(\Phi(\xi), \xi\right) = \begin{bmatrix} \mathsf{a} \\ \mathfrak{A}_+^{-1}\left(-\mathfrak{G}\mathsf{a} + \pi^T \mathsf{C}\left(\pi\mathsf{a} - \mathbb{1}\, q\right) - \vec{w} \otimes \vec{f}(\xi)\right) \end{bmatrix}, \\ \Phi^{'} \in \mathsf{H}\left(\Phi(\xi), \xi\right) . \end{array} \right. \end{array} \right. \tag{6}$$

It appeals readily to consider, as a point of departure, a selection or approximate selection of the multivalued operator $\mathsf{H}$. Another approach is that of the consideration of $\mathsf{H}$ measurable and hemicontinuous monotone together with necessary estimates [5]. Even at that, certain difficulties arise due to the presence of composition operators and the difference of operators. Kraynyukova and Alber [4] employed a regularization procedure in their approach to show existence of solution to a certain different problem (of piezoelectricity) involving composition operators.

In this paper, we exploit pertinent results from the concept of extended sum and composition of monotone operators, Fitzpatrick functions [6] and selection theory in our study of the existence of solutions to equation $(6)$ (which leads to that of TW solutions for the piezo-viscoelastic initial value problem). We show the existence TW solutions, amidst various scenarios, for the following cases: $1)$ The purely mechanical problem, under necessary assumptions on $\mathfrak{A}$ and $\mathfrak{G}$; $2)$ $\mathfrak{A}_+^{-1} = \partial$, a Clarke subdifferential of locally Lipschitz and not necessarily convex arguments. Furthermore, through estimates taken in passage to our proofs of existence, we furnish measures of the frequencies of these traveling waves.

## References


[1] Blanchard, D., Le Tallec, P.: Numerical analysis of the equations of small strains quasistatic elastoviscoplasticity. *Numerische Mathematik* **50 (2)**: 147–169, 1986.

[2] Visintin A.: Homogenization of the nonlinear Kelvin-Voigt model of viscoelasticity and of the Prager model of plasticity *Continuum Mech. Thermodyn.* **18**: 223–252, 2006.

[3] Selmani A., Selmani L.: A frictional contact problem with wear and damage for electro-viscoelastic materials *Applications of Mathematics* **55**: 89–109, 2010.

[4] Kraynyukova N., Alber H.-D.: A doubly nonlinear problem associated with a mathematical model for piezoelectric material behaviour *Z. Angew. Math. Mech.* **DOI 10.1002/zamm.201000129**: 1–19, 2011.

[5] Hu S., Papageorgiou N.: Handbook of Multivalued analysis Vol. II: Applications. Kluwer, Dordrecht (2000).

[6] Garcia Y., Lassonde M., Revalski J.P.: Extended sums and extended compositions of monotone operators. *Journal of Convex Analysis* **13 (3/4)**: 721–738, 2006.