ADVANCES IN MECHANICS OF GROWING SOLIDS

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Summary Deformation processes in a solid whose composition, mass, or volume varies in a piecewise continuous manner due to the accretion of new material to the outer surface of a body are the subject of this study. Basic fundamentals of the mathematical theory of growing solids are under consideration. The constitutive equations are formulated with respect to complete distortion tensor which may be representing as the composition of initial distortion and compatible deformation gradient. The initial distortion induces the linear connection on the material manifold which becomes a flat space of affine connectivity with nontrivial Cartan torsion.

GROWING BODIES

We will employ the concept of a *body* $\mathfrak{B}$ which is connected subset of an abstract topological space such that its image in the physical space is an region with a regular boundary. Its elements $\mathfrak{p} \in \mathfrak{B}$ are called particles. We assume that the particles of the body are *simple particles*. The set $\mathfrak{B}$ endowed with a structure defined by the class $\mathfrak{C}$ of *configuration* $\varkappa : \mathfrak{B} \to \mathcal{E}$. The image of set $\mathfrak{B}$, otherwise the collection of places of all material points, is called *shape*. Since each shape is embedded in affine space with Euclidian structure it has natural parametrization, for example, by means of Cartesian coordinates. Hence the class $\mathfrak{C}$ endow subset $\mathfrak{B}$ with the structure of manifold. Smoothness of this manifold depends on differential properties of functions belongs to class $\mathfrak{C}$.

In classical mechanics of solids bodies are treated as invariant sets. Mechanics of growing solids admit the *evolution* of the set $\mathfrak{B}$ that represents body and treat it as *variable* set. In following we will refer to the bodies of invariable composition as to *permanent bodies*. If a permanent body has more then one configuration (up to rigid motion), i.e. undergoes deformation in physical space, we call it *permanent solid*.

We introduce material manifold $\mathfrak{M}$ as a fibre bundle of a smooth three-dimensional manifold. As one knows from differential geometry, a fibre bundle is defined by base $\mathfrak{N}$ which is a smooth manifold itself, structure group G, and projection mapping. We assume that growing body is bounded, i.e. all its shapes may be placed into sufficiently large ball with fixed finite diameter, and the process of growing is implemented as continuously adhering of material surfaces to the boundary of the body. Under this assumptions the dimension of the base $\mathfrak{N}$ is equal to one and material manifold $\mathfrak{M}$ is represented as bundle over interval.

One may define a permanent body $\mathfrak{B}$ as an open subset of a material manifold $\mathfrak{M}$ bounded by non-identical fibers. In this case the body $\mathfrak{B}$ can be represented as a union of fibers $\mathfrak{M}_{\mathfrak{X}^3}$ whose indices are in the open interval (α, β), i.e.

$$\mathfrak{B} = \mathfrak{B}(\alpha, \beta) = \bigcup_{\mathfrak{X}^3 \in (\alpha, \beta)} \mathfrak{M}_{\mathfrak{X}^3}.$$

A growing body can be defined as a parametric family of such sets

$$\mathfrak{C} = \left\{ \mathfrak{B}_\gamma = \mathfrak{B}(\alpha, \gamma) \middle| \gamma \in (\alpha, \beta) \right\},$$

where γ is a parameter of the family. While $\alpha \to \beta$ the body degenerates into an infinitely thin ply or a point. Obviously this definition may be generalize to the next one

$$\mathfrak{C} = \left\{ \mathfrak{B}_\gamma = \mathfrak{B}(\alpha_\gamma, \beta_\gamma) \middle| (\alpha_\gamma, \beta_\gamma) \subset (\alpha, \beta) \right\},$$

where $(\alpha_\gamma, \beta_\gamma)$ are nested intervals.

According to the above definitions the boundary of growing solid should be topologically equivalent to the typical fiber which is smooth manifold itself and hence the growing boundary should be geometrically closed surface. If the growing boundary is topologically equivalent to the *manifold with edge* then a growing body can be defined as follows

$$\mathfrak{C} = \left\{ \mathfrak{B}_\gamma = \mathfrak{B}_0 \cap \mathfrak{B}(\alpha_\gamma, \beta_\gamma) \middle| (\alpha_\gamma, \beta_\gamma) \subset (\alpha, \beta) \right\}.$$

Here $\mathfrak{B}_0$ is fixed subset of $\mathfrak{M}$ with smooth boundary.

CONFIGURATIONS

In common case the natural shape of growing solids may not exists. This lead to so-called theory of *inhomogeneous bodies*. So, one can find the configuration that relax neighborhood of fixed point locally, but there is no configuration that carries the whole body to natural state. More precisely there is no such configuration with image imbedded in Euclidian

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space. At the same time there is possibility to find the configuration that maps the whole body to stress free shape if it permissible to embed this shape into Cartan space with nontrivial torsion.

Let the reference shape $\varkappa_R(\mathfrak{B})$ is the image of body $\mathfrak{B}$ under the mapping $\varkappa_R$. It is stressed. One can associate with each material point the deformation that carries the neighborhood of the point to stress free state. However this deformations for different points in general corresponds to different configurations. One may construct the collection of such local deformations associated with all material points and obtain the tensor field $\boldsymbol{K}^{-1}$ over set $\mathfrak{B}$ that is not in general a gradient of any vector field. The corresponding tensor field of local configurations we denote by $\mathcal{K}$. It's clear that $\mathcal{K} = \boldsymbol{K}^{-1} \circ \varkappa_R$.

We can determine the connection coefficients $\Gamma^{\beta}_{\gamma\alpha}$ on material manifold $\mathfrak{M}$: $\Gamma^{\beta}_{\gamma\alpha} = \partial_\alpha \left(\boldsymbol{K}^{-1}\right)^{\rho}_{\cdot\gamma} K^{\beta}_{\cdot\rho}$. The corresponding connection can be nontrivial, i.e. with nonzero torsion. This geometrization of the manifold $\mathfrak{M}$ transforms it into a configuration that is not embedded in three-dimensional Euclidean space.

Note that material manifold is a bundle and each its fiber $\mathfrak{M}_{\mathfrak{X}^3}$ is associated with a material surface strained before its adhesion to a body. From the mechanical point of view it means that each fiber was deformed from natural state before adhesion. So each individual fiber has a a global two-dimensional natural reference configuration.

Taking into account this, we introduce the tensor field $\boldsymbol{K}$ as a local deformation associated with individual material surface $\mathfrak{M}_{\mathfrak{X}^3}$. It can be quantify as pre-deformation of a surface from natural reference configuration onto configuration that match its position in reference shape. Thus, at each point of the material manifold tensor field $\boldsymbol{K}(\boldsymbol{\mathfrak{X}})$ is defined. This field is continuously differentiable due to the the continuity of accretion process. Moreover the field $\boldsymbol{K}^* \cdot \boldsymbol{K}$ specifies a metric on a surface embedded in three-dimensional Euclidean space on each fiber $\mathfrak{M}_{\mathfrak{X}^3}$. The complete distortion has the form $\boldsymbol{H} = \boldsymbol{K} \cdot \boldsymbol{F}$. Stresses may be represent as smooth tensor field $\boldsymbol{T}(\boldsymbol{H}) = \boldsymbol{T}(\boldsymbol{K}, \boldsymbol{F})$.

The boundary value problem for an accreted solid is determined by the equations of equilibrium in $V(\gamma)$ with boundary $\Omega(\gamma)$ parametrically dependent on γ, i.e.

$$\nabla \cdot \boldsymbol{T} + \boldsymbol{b} = \boldsymbol{0}, \quad \boldsymbol{n} \cdot \boldsymbol{T}\Big|_{\Omega(\gamma)} = \boldsymbol{p}, \quad \boldsymbol{P} \cdot \boldsymbol{T} \cdot \boldsymbol{P}\Big|_{\Omega(\gamma)} = \boldsymbol{\mathcal{T}}.$$

Here $\boldsymbol{P} = (\boldsymbol{E} - \boldsymbol{n} \otimes \boldsymbol{n})$ is a projector onto the tangent plane to $\Omega(t)$. This equation expresses the fact that the fibers align with the specified tension determined by the surface tensor $\boldsymbol{\mathcal{T}}$, i.e., two-dimensional tensor of second rank defined in the tangent space of the adhering material surface.

DISTORTION

The equation for distortion tensor $\boldsymbol{K}$ provided that the increase is the result of continuous adherence of prestressed surfaces can be obtained from the relations of the theory of material surfaces. Effects of material surface adhering leads to an infinitesimal change of the stress–strain state of an accreted solid but as an elementary act of adhesion occurs during an infinitely small time interval the rate of stress state is finite. This rate can be found from the equations of contact interaction between the spatial body and adhering the material surface. The equation of physical boundary equilibrium (which from the geometrical point of view is a bounding surface of the body in its actual state and from mechanical point of view is a thin film undergoing a membrane state of stress) can be written as

$$\nabla_s \cdot \boldsymbol{\mathcal{T}} + \boldsymbol{b}_s = \boldsymbol{n}_\varkappa \cdot \boldsymbol{T}_\varkappa\Big|_{\Omega(t)},$$

where ∇_s is the surface nabla operator, $\boldsymbol{b}_s$ is the surface density of external forces acting on $\Omega(t)$. To complete the boundary value problem formulation the condition on the curvilinear boundary $\partial\Omega(t)$ of the surface $\Omega(t)$ must be specified. This conditions may be of the form $\widetilde{\boldsymbol{n}} \cdot \boldsymbol{\mathcal{T}}\Big|_{\partial\Omega(t)} = \widetilde{\boldsymbol{f}}$. Here $\widetilde{\boldsymbol{n}}$ is an external unit normal to the curve $\partial\Omega(t)$ in the tangent plane and $\widetilde{\boldsymbol{f}}$ is a linear density of forces distributed on the curve $\partial\Omega(t)$.

Defined distortion field allows to specify the geometry on the material manifold. The resulting coefficients of affine connection do not possess the symmetry and define a nonzero torsion tensor

$$\mathcal{T}^{\alpha}_{\gamma\beta} = \partial_\gamma \left(\boldsymbol{K}^{-1}\right)^{\rho\cdot}_{\cdot\beta} K^{\alpha\cdot}_{\cdot\rho} - \partial_\beta \left(\boldsymbol{K}^{-1}\right)^{\rho\cdot}_{\cdot\gamma} K^{\alpha\cdot}_{\cdot\rho}.$$

The non-Euclidean features of reference natural configuration, quantitatively defined by the torsion tensor which characterizes the incompatibility of fibers stress-strain states joined in the process of accretion.

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