# ON PRINCIPLE OF STATIONARY ACTION FOR ELECTROMAGNETIC ELASTIC SOLIDS

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*Summary* The possible consequences of the principle of stationary action formulated for electromagnetic elastic bodies are shown. The material structure with internal state variables for those bodies is considered and the appropriate action functional is proposed. The possible variations of fields of the dependent state variables are introduced together with a non-commutative rule between operations of taking variations of the field and their partial time or spatial derivatives. The balance of linear momentum in differential form is received together with the remaining Maxwell's law and the boundary conditions.

## INTRODUCTION

Historically the classical Lagrange and Hamilton's formalisms were formulated for the point mechanics problems. Accordingly, if a dynamical system is described by the vector-valued coordinate $\boldsymbol{q}$ and the Lagrangian $L = T - V$, where $T$ and $V$ are, respectively, the kinetic and potential energy, then one formulates the variational principle of the dynamical system by requiring that between all curves $\boldsymbol{q} = \boldsymbol{q}(t)$ in a configuration space $\mathcal{V}$ the actual path (i.e. the solution of the system) is that which makes the action integral

$$I = \int_{t_0}^{t_1} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\, dt \tag{1}$$

stationary. Taking the first variation $\delta\boldsymbol{q}$ subject to the conditions $\delta\boldsymbol{q}(t_0) = \delta\boldsymbol{q}(t_1) = 0$ the stationarity of the action requires $\delta I = 0$, which is equivalent to the Euler-Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} = 0\,. \tag{2}$$

It is well known that the general equations of continuum mechanics and dissipative phenomena employed at the present time cannot be derived from Hamilton's variational principle. The case of bodies described by hyper-elastic material structure is exceptional. On the other hand there are several attempts to formulate the stationary action principle for dissipative systems, and the general review of the different attempts one can find e.g. in [1], [2].

In order to derive the governing equations of irreversible phenomena by the variational technique some artificial restrictions must be made, concerning the basic rules of variational calculus.

The interesting procedure was proposed by Vujanović in [3] and applied to governing equation of the generalized (hyperbolic) theory of heat conduction with finite wave speed, in which the new Lagrangian was proposed with an explicit dependence on time through the exponential term $\exp t/\tau$ appearing as a factor. This term has the power $t/\tau$, where $\tau$ is the thermal relaxation time. The corresponding transition to the classical case, i.e. with infinite speed of thermal disturbance (parabolic case) is performed by setting the relaxation time equal to zero.

## VARIATIONAL PRINCIPLE

The proposed here the new method for deriving the class of equations appearing in some physical irreversible processes is based on the variational principle which has a Hamiltonian structure, and in most cases its form does not differ from that known for conservative systems. However, the crucial assumption of the proposed method is in non-commutative rule between operations of taking variations of the field and their partial time and/or spatial derivatives.

For dissipative systems the lost of energy is a crucial effect which is called irreversibility. In the case of mechanical systems, especially for deformable bodies, dissipation of mechanical energy is described by the second law of thermodynamics, which restricts all mechanical (or coupled field) processes to those which satisfy the internal dissipation inequality. The irreversibility means that it is impossible to reverse the process, in which dissipation energy occurs without changing the environment. It also means that the variation of a quantity (e.g. a variable, which is crucial in description of dissipative phenomena) and the variation of its time derivative are related by the dissipative mechanism governing the process considered, not the time differentiation. If denote such a variable by $\boldsymbol{\xi}$ then we claim that the time derivative $\frac{\partial}{\partial t}\delta\boldsymbol{\xi}$ must be different from the variation of the time derivative of the quantity, i.e. $\delta\frac{\partial}{\partial t}\boldsymbol{\xi}$. From the other side this variation is the dynamic quantity and it should depend on the irreversible forces (nonconservative forces, according to [7])) acting upon the system. For the first time these observations were made by Vujonović in [6], [7].

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We are not going to tamper with the usual notation of the variation of $\delta\boldsymbol{\xi}$ and the velocity of variation $\frac{\partial}{\partial t}\delta\boldsymbol{\xi}$. These two vectors are regarded as purely kinematic in nature. The vector $\delta\boldsymbol{\xi}$ means that we consider the *infinitesimal transformation* (i.e. first variation) replacing $\boldsymbol{\xi}(t,x)$ by $\boldsymbol{\xi}(t,x)+s\boldsymbol{h}(t,x)$, where $\boldsymbol{h}(t,x)$ is an arbitrary differentiable function of $t$ and $x$, and $s$ is a small parameter passing through zero. Then, from the definition

$$\delta\boldsymbol{\xi} = \frac{\partial}{\partial s}(\boldsymbol{\xi}(t,x)+s\boldsymbol{h}(t,x))\delta s = \boldsymbol{h}(t,x)\delta s\,. \tag{3}$$

In this notation we have

$$\frac{\partial}{\partial t}\delta\boldsymbol{\xi} = \frac{\partial}{\partial t}\boldsymbol{h}(t,x)\delta s\,. \tag{4}$$

Since the vector $\delta\frac{\partial}{\partial t}\boldsymbol{\xi}$ has a purely dynamic character and its form depends on the nature of dissipative (nonconservative) phenomena (forces) acting on the body the infinitesimal transformation replaces $\frac{\partial}{\partial t}\boldsymbol{\xi}(t,x)$ by $\frac{\partial}{\partial t}\boldsymbol{\xi}(t,x)+s\boldsymbol{k}(t,x)$, where $\boldsymbol{k}(t,x)$ is not arbitrary differentiable function of $t$ and $x$. This function, however, may differ from $\frac{\partial}{\partial t}\boldsymbol{h}(t,x)$ by a part that is related to the function $\boldsymbol{h}(t,x)$ via some relationship depending on the irreversible phenomena of the system. Hence we can write

$$\delta\frac{\partial}{\partial t}\boldsymbol{\xi} = \frac{\partial}{\partial s}\{\frac{\partial}{\partial t}\boldsymbol{\xi}(t,x)+s\boldsymbol{k}(t,x)\}\delta s = \boldsymbol{k}(t,x)\delta s\,, \tag{5}$$

together with

$$\boldsymbol{k}(t,x) = \frac{\partial}{\partial t}\boldsymbol{h}(t,x) + \mathcal{F}(\boldsymbol{\xi}(t,x),\frac{\partial}{\partial t}\boldsymbol{\xi}(t,x),t,x)\boldsymbol{h}(t,x)\,. \tag{6}$$

Notice that dependence of the function $\mathcal{F}$ on time $t$ and space variable $x$ may be through some additional state variable, not written explicitly, here.
Comparing (3) and (5) with (6) we end up with the following noncommutative rule

$$\left[\delta,\frac{\partial}{\partial t}\right]\boldsymbol{\xi} = \mathcal{F}(\boldsymbol{\xi}(t,x),\frac{\partial}{\partial t}\boldsymbol{\xi}(t,x),t,x)\delta\boldsymbol{\xi}\,. \tag{7}$$

We can see that the case when the function $\mathcal{F}=0$ corresponds to a commutative rule. This noncommutative rule will be crucial in developing new variation principle for electromagnetic elastic body.

## CONCLUSIONS

In this paper starting with the action functional in which kinetic energy, free energy and energy related to the both coupled fields: electric and magnetic, appear, as well as the energy related to applied generalized forces, the principle of its stationarity is formulated. Making the first variation of the functional compatible with the assumed initial and boundary conditions and postulating the appropriate non-commutative laws between the variation and partial differentiation operators the consequence in the form of Euler-Lagrange equations is obtained. The equations are: momentum balance law together with Gauss electric law and Ampere law.
In the previous papers of the first author [4, 5] the same variational technique was developed and applied to long-line (telegraph) equation, hyperbolic and parabolic model of heat conduction as well as for dissipative solid body described by internal state variable approach. In the next paper we will generalize the present derivation to the case of thermodynamics.

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