MINIMUM-TIME DAMPING OF A PHYSICAL PENDULUM

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Summary We study the minimum-time damping of a physical pendulum by means of a bounded control. In the similar problem for a linear oscillator each optimal trajectory possesses a finite number of control switchings from the maximal to the minimal value. If one considers simultaneously all optimal trajectories with any initial state, the number of switchings can be arbitrary large. We show that for the physical pendulum there is a uniform bound for the switching number for all optimal trajectories. We find asymptotics for this bound as the control amplitude goes to zero.

INTRODUCTION

The problem of minimum-time damping of a pendulum is a classical problem of control theory. In the linear case, described by the equation $\ddot{x} + x = u,\ |u| \leq 1$, its solution is stated in [1]. The optimal control is of bang-bang type, i.e. it takes values $u = \pm 1$, and the switching curve which separates the domain of the phase plane, where $u = -1$ from the domain $u = +1$ consists of unit semicircles centered at points of the form $(2k + 1, 0)$, where k is an integer. The real physical pendulum controlled by a torque in the joint is governed by the equation $\ddot{x} + \sin x = \varepsilon u,\ |u| \leq 1$, where x is the vertical angle, and ε is the maximal amplitude of the control torque. The parameter ε is arbitrary: it might be large, small, of order 1. We are interested most in the case of a small ε. The maximum principle says that the optimal control has the form $u = \operatorname{sign}\psi$, where the "adjoint" variable satisfies the equation $\ddot{\psi} + (\cos x)\psi = 0$. Thus, the control is still of the bang-bang type, but the time instants of switchings are roots of a rather nontrivial function, a solution of the general Sturm–Liouville/Schrödinger equation. The complexity of a control is characterized mainly by the number of switchings. In the linear case this number for a trajectory connecting the initial point $(x, \dot{x})$ with $(0, 0)$ is $\frac{T}{\pi} + O(1)$, where T is the duration of the motion. In its turn, $T = \pi\sqrt{\frac{E}{2}} + O(1)$, where $E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2$ is the energy. Thus, each trajectory possesses a finite number of switches, but if the initial energy is large this number is $\sqrt{\frac{E}{2}} + O(1)$ and is also large.

In the nonlinear case the switching number behaves quite differently. The best result, known to the author, is due to Reshmin [2]. It says that if the parameter ε is large enough, all optimal trajectories possess no more than a single switch.

RESULTS

We show that for any ε the switching number for all optimal trajectories possesses a common bound.

Theorem 1 *Suppose $N_\varepsilon(x, \dot{x})$ is the number of zeroes of the adjoint variable ψ along an optimal trajectory connecting $(x, \dot{x})$ with (0,0). Then the quantity $N_\varepsilon = \sup N_\varepsilon(x, \dot{x})$, where $\sup$ is taken over the entire phase space is finite.*

Another result is an upper and lower bound for N_ε which is sharp with respect to the order of magnitude.

Theorem 2 *There exist positive constants $c_1,\ c_2$, such that*

$$\frac{c_1}{\varepsilon} \leq N_\varepsilon \leq \frac{c_2}{\varepsilon}$$

as the parameter ε is small enough.

Our main result is a promotion of inequalities of theorem 2 to an asymptotic equality:

Theorem 3 *As $\varepsilon \to 0$ there is the asymptotic equivalence*

$$N_\varepsilon \sim \frac{D}{\varepsilon},\ \textit{where}\ D = \frac{1}{2}\mathrm{Si}\,(\pi) = \int_0^\pi \frac{\sin x}{2x}\,dx = 0.925968526\ldots \tag{1}$$

The theorem can be regarded, like the "Feigenbaum universality", as an asymptotic formula $\varepsilon_n \sim D/n$ for bifurcation values of the parameter ε. Here, the bifurcation is the increment by 1 of the maximal number of the control switches. Unlike the period-doubling bifurcation, studied by Feigenbaum, the relevant constant D can be expressed via standard mathematical constructions, and its computation with any accuracy is not a problem.

The paper is based on a lemma saying that in the large speed area the optimal trajectory possesses no more than a single switch. We use heavily the Sturm theory of zero loci for solutions of a Sturm–Liouville equation. It allows us to relate N_ε to the optimal time of motion from points with energy of order 1 to the point (0,0). The lower bound in Theorem 2 is based on energy considerations, which allows us to estimate this time. The upper bound is more complicated and follows from a computation of the elapsed time in a motion under a quasioptimal control. The asymptotic equivalence (1) stems from the idea of the Poincaré map control, coupled with a special nonlinear Sturm-like theorem.

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Bounds for the damping time

Let K be a compact in the phase space $S^1 \times \mathbf{R}$, and $T_\varepsilon = T_\varepsilon(K)$ the maximum of damping times over all initial conditions $(x, y) \in K$. Assume that K is not the singleton (0,0). The next estimate for the time T_ε makes a ground for the finiteness Theorems 1 and 2:

Theorem 4 *There exist positive constants $C_i = C_i(K),\ i = 1, 2$ such that*

$$\frac{C_1}{\varepsilon} \leq T_\varepsilon \leq \frac{C_2}{\varepsilon}$$

as the (*positive*) ε *is small enough.*

Theorems 1 and 2 follows from the above bounds relatively easily.

Optimal time theorem

In fact, we can describe the limit behavior of the damping time T_ε with a high precision. Suppose that the compact K consists of a single point $p = (x, y)$, and $E = E(p) = \frac{1}{2}y^2 + (1 - \cos x)$ is the energy of p.
For $E \geq 2$ put

$$\tau_+(E) = \frac{1}{2\sqrt{2}\pi} \int\limits_2^E dE \int\limits_0^{2\pi} (E - 1 - \cos\phi)^{-1/2} d\phi,$$

and for $E \leq 2$, define $\Phi = \Phi(E) \in [0, \pi]$ is the solution of $1 - \cos\Phi = E$, and

$$\tau_-(E) = \frac{1}{\sqrt{2}} \int\limits_0^\Phi \frac{\sin x}{x}\, dx \int\limits_0^x \frac{d\phi}{\sqrt{\cos\phi - \cos x}}.$$

Theorem 5 *Let $T_\varepsilon(p)$ be the minimum time required to bring the point p to the lower equilibrium point $(0, 0)$. Then*

$$T_\varepsilon(p) \sim \frac{\tau(E)}{\varepsilon},\ \textit{as}\ \varepsilon \to 0,\ \textit{where}\ \tau(E) = \begin{cases} \tau_+(E) + \tau_-(2),\ \textit{if}\ E \geq 2 \\ \tau_-(E),\ \textit{if}\ E \leq 2 \end{cases}.$$

CONCLUSIONS

The paper reveals a sharp difference between behavior of naturally related linear and nonlinear control systems. We regard the finiteness phenomenon for the switching number deserving a further study in a more general setup.

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