DYNAMIC MODELLING FOR THERMOELASTIC COSSERAT ROD

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Summary A thermoelastic Cosserat rod with a heat flux along the length is modeled based on simple Cosserat rod theory. Extended Kirchhoff constitutive relations included thermal effects are obtained using the first law of thermodynamic by defining a free energy. The thermal dissipation power is estimated by Clausius-Duhem inequality. Nonlinear dynamic equations of thermoelastic Cosserat rod with a heat conduction equation are obtained by assembling the extended Kirchhoff constitutive relations into the simple Cosserat rod model.

INTRODUCTION

Antman [1] has established a simply Cosserat theory for dynamic modelling of slender structures. The simple Cosserat rod theory has been extensively studied and found to have wide applicability [2,3]. Burton and Tucker [4] provide a detailed review of the history and applications of this theory. A special feature of the Cosserat rod model is the transparency of its formulations and the ease by which different approximation models can be exploited within its general framework. When materials endured heat to some extent their thermomechanical characteristics will affect their dynamic response to external loadings. This paper aims to explore a nonlinear dynamic model which can be used to describe the dynamic behaviors of one dimensional slender structure subjected to external loadings and time-varying temperature. We attempt to model a thermoelastic Cosserat rod with heat conduction along the length by generalizing the classical Kirchhoff constitutive relations used in the simple Cosserat rod model. Through this paper, repeated indices follow the Einstein summation convention.

REVIEW OF SIMPLE COSSERAT ROD THEORY

The simple Cosserat rod theory is fundamentally formulated in the Lagrangian picture in which material elements of a rod are labeled by s. The dynamic equations of a Cosserat rod with mass density, $s \in [0,\ell] \mapsto \rho(s)$, and cross-sectional area, $s \in [0,\ell] \mapsto A(s)$, are governed by Newton's dynamic laws

$$\begin{cases} \rho(s)A(s)\ddot{\mathbf{r}}(s,t) = \mathbf{n}'(s,t) + \mathbf{f}(s,t) \\ \partial_t\left(\mathbf{I}(s,t)\cdot\mathbf{w}(s,t)\right) = \mathbf{m}'(s,t) + \mathbf{r}'(s,t)\times\mathbf{n}(s,t) + \mathbf{l}(s,t) \end{cases} \tag{1}$$

which are applied to a triad of orthonormal vectors

$$s \in [0,\ell] \mapsto \{\mathbf{d}_1(s,t), \mathbf{d}_2(s,t), \mathbf{d}_3(s,t)\} \tag{2}$$

over the space-curve line of centroids of the cross-sections $s \in [0,\ell] \mapsto \mathbf{r}(s,t)$ at time t, where $\mathbf{n}' = \partial_s \mathbf{n}$ and $\dot{\mathbf{r}} = \partial_t \mathbf{r}$, $\mathbf{f}$ and $\mathbf{l}$ denote external force and torque densities respectively and $s \in [0,\ell] \mapsto \mathbf{I}(s,t)$ is moment of inertia tensor. In these field equations the contact forces $\mathbf{n}$ and contact torques $\mathbf{m}$ are determined by constitutive relations involving the strains $\mathbf{u}(s,t)$ and $\mathbf{v}(s,t)$. One can specify that $\mathbf{d}_1$ is vertical to its corresponding cross-section, $\mathbf{d}_2$ and $\mathbf{d}_3$ are on the plane of their corresponding cross-section, and $\mathbf{d}_1 \times \mathbf{d}_2 = \mathbf{d}_3$. The strains $\mathbf{v}$ and $\mathbf{u}$, and the angular velocity of the cross-section $\mathbf{w}$ are defined in terms of the configuration variables $\mathbf{r}$ and $\mathbf{d}_k$ ($k = 1,2,3$) by the relations:

$$\mathbf{v} = \mathbf{r}', \quad \mathbf{u} = \frac{1}{2}\mathbf{d}_k \times \mathbf{d}'_k, \quad \mathbf{w} = \frac{1}{2}\mathbf{d}_k \times \dot{\mathbf{d}}_k \tag{3}$$

then the Kirchhoff constitutive relations for simple Cosserat rod are

$$\mathbf{n} = \mathbf{K}\cdot(\mathbf{v} - \mathbf{v}_0), \quad \mathbf{m} = \mathbf{J}\cdot(\mathbf{u} - \mathbf{u}_0) \tag{4}$$

where $\mathbf{K}$ and $\mathbf{J}$ are second-order tensors whose definitions can be found in references [1-4], and $\mathbf{v}_0$ and $\mathbf{u}_0$ are reference strains.

DYNAMIC EQUATIONS FOR THERMOELASTIC COSSERAT ROD

Thermodynamics in Cosserat Rod

Assume further that the structure possesses a Kelvin temperature $\vartheta(s,t)$, internal energy $e(s,t)$ and entropy density $\eta(s,t)$ per unit reference mass, and a heat flux $q(s,t)$ directed along the direction $\mathbf{d}_1(s,t)$. In addition there may be an applied heat power $h(s,t)$ per unit reference mass along the structure. In this case, the material parameters of the

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rod ρ, E, and G, and the area of cross section A are dependent on temperature ϑ, denote them as $\tilde{\rho}(\vartheta)$, $\tilde{E}(\vartheta)$, $\tilde{G}(\vartheta)$ and $\tilde{A}(\vartheta)$, respectively. Then, one can denote $\mathbf{K}$, $\mathbf{J}$, $\mathbf{I}$, $\mathbf{n}$, and $\mathbf{m}$ as $\tilde{\mathbf{K}}$, $\tilde{\mathbf{J}}$, $\tilde{\mathbf{I}}$, $\tilde{\mathbf{n}}$ and $\tilde{\mathbf{m}}$ respectively, for distinction. Applying the first law of thermodynamic and Clausius-Duhem inequality, one can obtain

$$\tilde{\rho}\tilde{A}\dot{e} = \tilde{n}_i\dot{v}_i + \tilde{m}_i\dot{u}_i - \varepsilon_{ijk}v_i\tilde{n}_j w_k + \tilde{\rho}\tilde{A}h - \tilde{A}\partial_s q \tag{5}$$

and

$$\tilde{n}_i\dot{v}_i + \tilde{m}_i\dot{u}_i - \varepsilon_{ijk}v_i\tilde{n}_j w_k - \frac{\tilde{A}q}{\vartheta}\partial_s\vartheta + q\partial_s\tilde{A} - \tilde{\rho}\tilde{A}(\dot{\phi} + \dot{\vartheta}\eta) \geq 0 \tag{6}$$

where $\varepsilon_{ijk} = (\mathbf{d}_i \times \mathbf{d}_j)\cdot\mathbf{d}_k$, and $\phi \equiv e - \vartheta\eta$ which is called as free energy. Thus one can see that in addition to six independent dynamical fields encoded in the fields $\mathbf{r}$ and $\mathbf{d}_i$, one has four additional dynamical thermal fields η, q, ϑ and ϕ (or e) over the Cosserat rod.

Extended Kirchhoff Constitutive Relations

Analyzing on Eq. (5), one can come to a conclusion that the free energy is a function with $\mathbf{u}$, $\mathbf{v}$ and ϑ. At this course one can also obtain the heat conduction equation in the rod as

$$\tilde{\rho}\tilde{A}\vartheta\dot{\eta} - \tilde{\rho}\tilde{A}h + \tilde{A}\partial_s q + \varepsilon_{ijk}v_i\tilde{n}_j w_k = 0 \tag{7}$$

The free energy of the rod with unit length can be expanded into Taylor series around the reference state, in which coefficients can be determined through a series of methods: the sake of unity between the simple Cosserat rod model and the thermoelastic Cosserat rod model, the common expanding and contracting behaviors of materials suffering changes of temperature, Fourier's law for heat conduction, and the necessary unity between the heat conduction equation and the balance of heat energy in a non-strain state of the rod. Then at last one can obtain an extended Kirchhoff constitutive relations in the thermoelastic Cosserat rod as

$$\tilde{\mathbf{n}} = \tilde{\mathbf{K}}\cdot(\mathbf{v} - \mathbf{v}_0) - \alpha(\vartheta)\tilde{K}_{11}(\vartheta - \vartheta_0)\mathbf{d}_1, \quad \tilde{\mathbf{m}} = \tilde{\mathbf{J}}\cdot(\mathbf{u} - \mathbf{u}_0) \tag{8}$$

and the heat conduction equation in terms of temperature, the applied heat power, strains, stresses, and motions of rod as

$$\tilde{\rho}\tilde{A}C_p\dot{\vartheta} = \alpha\tilde{K}_{11}\tilde{\rho}\tilde{A}\vartheta\dot{v}_1 + \tilde{\rho}\tilde{A}h + k\tilde{A}\vartheta'' - \varepsilon_{ijk}v_i\tilde{n}_j w_k \tag{9}$$

where ϑ_0 is the reference temperature, α is thermal expansion coefficient varying with temperature, C_p is thermal capacity, k is thermal conductivity.

Dynamic Equations for Thermoelastic Cosserat Rod

Finally, dynamic equations for thermoelastic Cosserat rod can be obtained by assembling Eq. (8) into Eq. (1). Combining them with Eq. (9) as

$$\begin{cases} \tilde{\rho}\tilde{A}\ddot{\mathbf{r}} = \left[\tilde{\mathbf{K}}\cdot(\mathbf{v} - \mathbf{v}_0) - \alpha\tilde{K}_{11}(\vartheta - \vartheta_0)\mathbf{d}_1\right]' + \mathbf{f} \\ \dfrac{\partial}{\partial t}(\tilde{\mathbf{I}}\cdot\mathbf{w}) = \left(\tilde{\mathbf{J}}\cdot(\mathbf{u} - \mathbf{u}_0)\right)' + \mathbf{v}\times\left[\tilde{\mathbf{K}}\cdot(\mathbf{v} - \mathbf{v}_0) - \alpha\tilde{K}_{11}(\vartheta - \vartheta_0)\mathbf{d}_1\right] + \mathbf{l} \\ \tilde{\rho}\tilde{A}C_p\dot{\vartheta} = \alpha\tilde{K}_{11}\tilde{\rho}\tilde{A}\vartheta\dot{v}_1 + \tilde{\rho}\tilde{A}h + k\tilde{A}\vartheta'' - \varepsilon_{ijk}v_i\tilde{n}_j w_k \end{cases} \tag{10}$$

The thermal dissipation power of thermoelastic Cosserat rod can be obtained by using Clausium-Duhem inequality as

$$\dot{D}^{\mathrm{T}}(s,t) = -\vartheta\varepsilon_{ijk}v_i\tilde{n}_j w_k - \tilde{A}q\vartheta_s + q\vartheta\partial_s\tilde{A} \tag{11}$$

CONCLUSIONS

The governing equations for a heat conducting Cosserat rod have been derived based on the tenets of simple Cosserat model. Using the first law of thermodynamic, thermomechanical modeling of a rod is developed in terms of a free energy that extends the classic Kirchhoff constitutive relations to include thermal effects. The thermal dissipative behavior of the model is explored by the Clausius-Duhem inequality. The thermoelastic Cosserat rod model can be applied to simulate the nonlinear dynamics of slender structures working under conditions of large extent temperature, such as drill-string, structures in the MEMS, and so on.

References

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