# THERMOELASTIC DAMPING OF THE AXISYMMETRIC VIBRATION OF BILAYERED CIRCULAR PLATE RESONATORS

Yuxin Sun [a)], Jialing Yang, Hua Liu & Yan Jiang
*The Solid Mechanics Research Centre, BUAA University, Beijing, 100191, China*

*Summary* In this paper, the governing equations of coupled thermoelastic problems are established for axisymmetric out-of plane vibration of bilayered circular plate. Then the analytical expression for thermoelastic damping is obtained. The importance of coating layers selection is highlighted.

## INTRODUCTION

For resonators, it is desired to design and construct systems with loss of energy as little as possible. There is an increasing use of sophisticated structural geometries and advanced materials, especially laminated composites of layered thin films [1], in the design and manufacture of microresonators. For example, bilayered structures are widely employed in MEMS- examples include gold-coated microscopy, and Al-coated SiC electrostatically actuated microreonators [2]. In addition, circular plates are common elements in many sensors and resonators [3]. This paper deals with thermoelastic damping effects on the out-of-plane vibration of circular plate resonators.

## FORMULATION AND SOLUTION OF BASIC EQUATIONS

Consider the axisymmetric out-of-plane flexural vibration of a bilayered thin circular plate with radius $a$. The thicknesses of the substrate and the coating are denoted by $z_1$ and $(z_2 - z_1)$, respectively. The cylindrical coordinate system ( $r,\theta,z$ ) is applied to study the vibration of the circular plate. In equilibrium, the plate is unstrained, unstressed, and keeps at environmental temperature $T_0$ everywhere. According to the Kirchhoff-Love plate theory, we can get the governing equations of the thermoelastic damping problem as

$$\begin{aligned}&(D_1 + D_2 - d_1 - d_2)\ \nabla^2\nabla^2 w + D_1(1+\upsilon_1)\alpha_{T1}\nabla^2 M_{T1} + D_2(1+\upsilon_2)\alpha_{T2}\nabla^2 M_{T2} + \rho_h \frac{\partial^2 w}{\partial t^2} = 0\\ &\rho_j C_{vj}\frac{\partial \phi_j}{\partial t} = k_j \frac{\partial^2 \phi_j}{\partial z^2} + \beta_j T_0 z \frac{\partial}{\partial t}(\nabla^2 w) \qquad (j=1,2)\end{aligned} \tag{1}$$

where $w$ is the axial deflection and $\phi_j$ is the temperature increment of the two layers, the subscript $j$ ranges from 1 to 2, with the convention that $j=1$ denotes the substrate and $j=2$ denotes the coating layer. $\rho_j$ is the density, $C_{vj}$ the specific heat at constant volume, $k_j$ the thermal conductivity, $\beta_j$ the thermal modulus, $\alpha_{Tj}$ the coefficient of thermal expansion, $\upsilon_j$ the Poisson's ratio. $\rho_h = \rho_1 z_1 + \rho_2 (z_2 - z_1)$ . $M_{T1} = \frac{3}{z_1^3}\int_0^{z_1}\phi_1 z dz$ and $M_{T2} = \frac{3}{z_2^3 - z_1^3}\int_{z_1}^{z_2}\phi_2 z dz$ are the thermal moments, and $D_1 = \frac{E_1 z_1^3}{3(1-\upsilon_1^2)}$ , $D_2 = \frac{E_2(z_2^3 - z_1^3)}{3(1-\upsilon_2^2)}$ , $d_1 = \frac{E_1 z_0 z_1^2}{2(1-\upsilon_1^2)}$ , $d_2 = \frac{E_2 z_0 (z_2^2 - z_1^2)}{2(1-\upsilon_2^2)}$ , $z_0 = \frac{z_2}{2}$ , $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$ .

To calculate the effect of thermoelastic coupling on the vibrations of a bilayered circular plate, we solve Eq. (1) for the case of harmonic vibrations. We set

$$w(r,t) = w_0(r)e^{i\Omega t}, \phi_1(r,z,t) = \phi_{10}(r,z)e^{i\Omega t},\ \ \phi_2(r,z,t) = \phi_{20}(r,z)e^{i\Omega t} \tag{2}$$

Substitute Eq. (2) into Eq. (1), and solve the obtained equations, and we can get the analytical expression of the thermoelastic damping as

$$Q^{-1} = 2\left|\frac{\mathrm{Im}(\Omega)}{\mathrm{Re}(\Omega)}\right| = 2\frac{\frac{T_0}{2D}\sum_{n=1}^{\infty}(H_1 A_{1n} + H_2 A_{2n})\mathrm{F_n}\ \frac{\gamma_n^2\Omega_0}{\gamma_n^4 + \Omega_0^2}}{1 + \frac{T_0}{2D}\sum_{n=1}^{\infty}(H_1 A_{1n} + H_2 A_{2n})\mathrm{F_n}\ \frac{\Omega_0^2}{\gamma_n^4 + \Omega_0^2}} \tag{3}$$

[a)] Corresponding author. Email: yxsun@buaa.edu.cn.

where $\Omega_0 = \frac{q_n}{a^2}\sqrt{\frac{D}{\rho h}}$ is the frequency without considering the thermoelastic coupling effect.

$$D = D_1 + D_2 - d_1 - d_2,\quad F_n = \frac{\beta_1 A_{1n} + \beta_2 A_{2n}}{N_n},\quad H_1 = \frac{E_1\alpha_1}{1-\upsilon_1},\quad H_2 = \frac{E_2\alpha_2}{1-\upsilon_2},\quad A_{1n} = \int_0^{z_1} z\phi_{1n}dz, A_{2n} = \int_{z_1}^{z_2} z\phi_{2n}dz,$$

$$N_n = \rho_1 C_{v1}\int_0^{z_1}\phi_{1n}^2 dz + \rho_2 C_{v2}\int_{z_1}^{z_2}\phi_{2n}^2 dz,\ n = 1,2,\ldots,\ \phi_{1n} = \cos(\gamma_n a_1 z),$$

$$\phi_{2n} = \frac{b\sin(\gamma_n a_1 z_1)\cos(\gamma_n a_2 z_2)}{\sin[\gamma_n a_2 z_2(\frac{z_1}{z_2}-1)]}\cos(\gamma_n a_2 z) + \frac{b\sin(\gamma_n a_1 z_1)\sin(\gamma_n a_2 z_2)}{\sin[\gamma_n a_2 z_2(\frac{z_1}{z_2}-1)]}\sin(\gamma_n a_2 z)$$

$$a_1 = \sqrt{\frac{\rho_1 C_1}{k_1}}, a_2 = \sqrt{\frac{\rho_2 C_2}{k_2}}, b = \sqrt{\frac{k_1\rho_1 C_1}{k_2\rho_2 C_2}},$$

Here, $\gamma_n$ is defined by the following equation:

$$\det\begin{vmatrix} -\cos(\gamma_n a_1 z_1) & -\cos(\gamma_n a_2 z_1) & \sin(\gamma_n a_2 z_1) \\ -b\sin(\gamma_n a_1 z_1) & \sin(\gamma_n a_2 z_1) & -\cos(\gamma_n a_2 z_1) \\ 0 & -\sin(\gamma_n a_2 z_2) & \cos(\gamma_n a_2 z_2) \end{vmatrix} = 0 \tag{4}$$

## 3. Numerical results

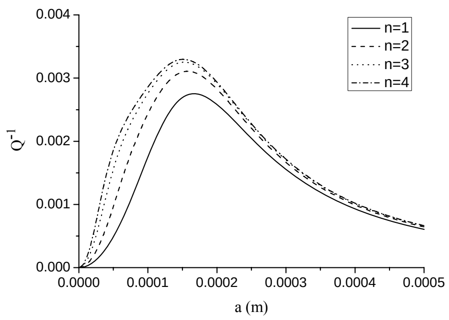


Fig.1 The convergence of $Q^{-1}$

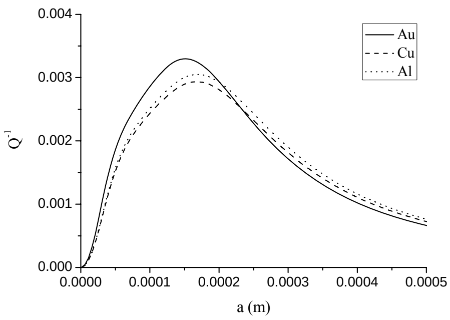


Fig. 2 Variation of $Q^{-1}$ with radius $a$ for coating layers of Au, Cu and Al.

Consider the substrate as Si, and the coating layers as Au, Cu, Al, respectively. The thicknesses of the substrate and coating layers are 10 and $1.11\ \mu m$, respectively. Figure 1 verifies the convergence of Eq. (3). The coating layer is Au. It is shown that the expression converges quickly and is essentially unchanged for $n \geq 3$. Figure 2 shows the variation of $Q^{-1}$ with radius $a$ for coating layers of Au, Cu and Al. It is shown that the metallic coating layers lead to a considerable difference in the damping. The analysis highlights the importance of coating layers selection: Au leads to higher damping than Au and Al. In addition, as the radius increases, $Q^{-1}$ increases first and then decreases, and there is a critical radius, denoted as $a_c$, with the maximum value of $Q^{-1}$.

## References

[1] Srikar V.T., Spearing S.M.: Materials selection in micromechanical design: an application of the Ashby approach. *J. Microelectromech. Syst.* **12**:3-10, 2003.

[2] Huang X.M.H., Zorman C.A., Mehregany M., et al.: Nanodevice motion at microwave frequencies. *Nature.* **421**:496, 2003.

[3] Bao G., Jiang W.: A heat transfer analysis for quartz microresonator IR sensors. *Int. J. Solids. Struct*. **35**:3635-3653, 1998.