APPROXIMATIONS OF NONLINEAR DIFFERENTIAL EQUATION SOLUTIONS

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Summary A list of approximations of nonlinear functions of one and two arguments is done. The linearizations of nonlinear differential equations around stationary points correspond to equilibrium positions or relative equilibrium positions of mechanical system dynamics with trigger of coupled singularities are obtained. By using known analytical solutions of linearized nonlinear differential equations around stationary point, as the starting solutions, by application Krilov-Bogolyubov-Mitropolyski asymptotic methods and method of variation constants and averaging, different expressions of first approximations of nonlinear differential equation solution are obtained. First approximations of a nonlinear differential equation obtained by different methods and around different known analytical solutions were compared and corresponding conclusions are presented. As special examples are used nonlinear differential equations describing nonlinear dynamics of the mechanical system with coupled rotations in damping field.

APPROXIMATION OF NONLINEAR DIFFERENTIAL EQUATIONS

Let's consider the following system of differential equations:

$$\frac{d\varphi}{dt} = f_1(\varphi, v) \quad ; \quad \frac{dv}{dt} = f_2(\varphi, v) \tag{1}$$

describing nonlinear dynamics of mechanical system with series of the equilibrium positions or relative equilibrium positions (see refs. [1-8]). We accept that system posses series of the stationary points (φ_s, v_s), $s = 1,2,3,...$ containing a trigger of coupled singularities [2-5] and that is possible to made the approximation of the previous system of nonlinear differential equations (1) around stationary points (φ_s, v_s) in the form:

$$\begin{aligned}\frac{d\varphi}{dt} = {} & \left(\frac{\partial f_1(\varphi,v)}{\partial \varphi}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi + \left(\frac{\partial f_1(\varphi,v)}{\partial v}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} v + \frac{1}{2!}\left(\frac{\partial^2 f_1(\varphi,v)}{\partial \varphi^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi^2 + \frac{1}{2!}\left(\frac{\partial^2 f_1(\varphi,v)}{\partial v^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} v^2 + 2\frac{1}{2!}\left(\frac{\partial^2 f_1(\varphi,v)}{\partial v\partial \varphi}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi v + \\ & + \frac{1}{3!}\left[\left(\frac{\partial^3 f_1(\varphi,v)}{\partial \varphi^3}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi^3 + 3\left(\frac{\partial^3 f_1(\varphi,v)}{\partial \varphi^2\partial v}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi^2 v + 3\left(\frac{\partial^3 f_1(\varphi,v)}{\partial \varphi\partial v^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi v^2 + \left(\frac{\partial^3 f_1(\varphi,v)}{\partial v^3}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} v3\right] + ...\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{dv}{dt} = {} & \left(\frac{\partial f_2(\varphi,v)}{\partial \varphi}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi + \left(\frac{\partial f_2(\varphi,v)}{\partial v}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} v + \left(\frac{\partial^2 f_2(\varphi,v)}{\partial \varphi^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi^2 + \left(\frac{\partial^2 f_2(\varphi,v)}{\partial v^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} v^2 + 2\frac{1}{2!}\left(\frac{\partial^2 f_2(\varphi,v)}{\partial v\partial \varphi}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi v + \\ & + \frac{1}{3!}\left[\left(\frac{\partial^3 f_2(\varphi,v)}{\partial \varphi^3}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi^3 + 3\left(\frac{\partial^3 f_2(\varphi,v)}{\partial \varphi^2\partial v}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi^2 v + 3\left(\frac{\partial^3 f_2(\varphi,v)}{\partial \varphi\partial v^2}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}}\varphi v^2 + \left(\frac{\partial^3 f_2(\varphi,v)}{\partial v^3}\right)\Bigg|_{\substack{\varphi=\varphi_s\\ v=v_s}} v3\right] + ...\end{aligned} \tag{3}$$

in which after approximation we introduce that starts of the phase coordinate are in corresponding stationary point($\varphi - \varphi_s \xrightarrow{def} \varphi$, $v - v_s \xrightarrow{def} v$.

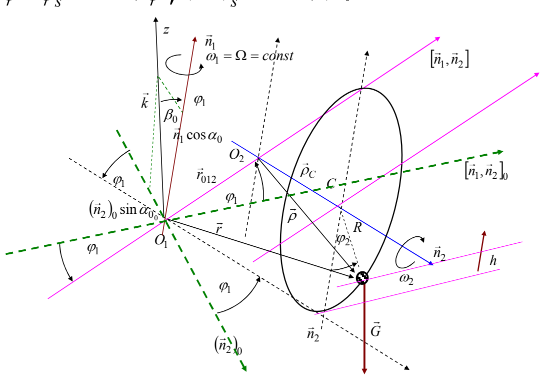

Figure 1. Coupled rotations about no intersecting axes

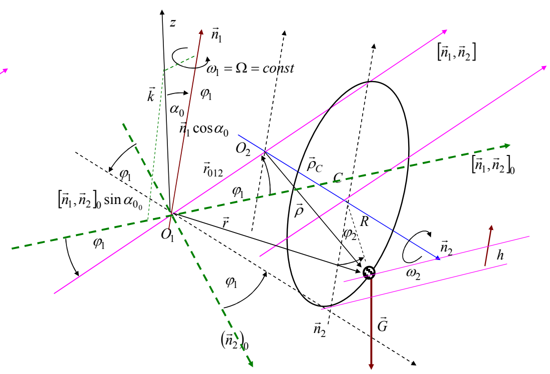

Figure 2. Coupled rotations about no intersecting axes

Nonlinear differential equations describing nonlinear dynamics of a heavy mass particle along rotating circle about axis inclined to the vertical direction with constant angular velocity are in the forms (see Figure 1 and 2):

$$\ddot{\varphi}_2 + 2\delta\dot{\varphi} + \Omega^2(\lambda\cos\beta_0 - \cos\varphi_2)\sin\varphi_2 - \Omega^2\frac{r_{012}}{R}\cos\varphi_2 = \Omega^2\lambda\cos\varphi_2\sin\beta_0\sin\Omega t \tag{4}$$

$$\ddot{\varphi}_2 + 2\delta\dot{\varphi} + \Omega^2(\lambda\cos\beta_0 - \cos\varphi_2)\sin\varphi_2 - \Omega^2\frac{r_{012}}{R}\cos\varphi_2 = \Omega^2\lambda\cos\varphi_2\sin\beta_0\sin\Omega t \qquad (5)$$

Previous differential equations (4) or (5), also, represent the analogous differential equations of the self rotation heavy rigid body, skew and eccentrically positioned to the axis of self rotation, with coupled rotations about two no intersecting orthogonal axes. To previous differential equations we add classical nonlinear differential equation in the form [1]:

$$\ddot{\varphi}_2 + 2\delta\dot{\varphi} + \Omega^2(\lambda - \cos\varphi_2)\sin\varphi = 0 \qquad (6)$$

Taking into account possible approximation of nonlinear differential equations (4)-(5)-(6) in the form: (2)-(3) around stationary points, we obtain series of the approximations and for this time we separate the three following types: $\ddot{\varphi} + \Omega^2(\lambda - 1)\varphi \approx \Omega^2\lambda ctg\alpha\cos\Omega t$; $\ddot{\varphi} + 2\delta\dot{\varphi} + \omega_{0,lin}^2\varphi = \kappa_2\varphi^2 + \kappa_3\varphi^3$ and $\ddot{\varphi}_2 + 2\delta\dot{\varphi} + \varphi_2[\lambda + \gamma\sin\varphi_{s2}\cos\Omega t] + f = h_s\cos\Omega t$.

CONCLUDING REMARKS

Let we made a general review of the obtained results for approximately solving of the nonlinear differential equation with small cubic nonlinearity and linear damping in the form:

$$\ddot{x}_1(t) + 2\delta_1\dot{x}_1(t) + \omega_1^2 x_1(t) = \mp\tilde{\omega}_{N1}^2 x_1^3(t), \qquad (7)$$

in which hard or soft, refers to $\mp$ sign, $\delta_1 = \varepsilon_1\tilde{\delta}_1$ and $\tilde{\omega}_{N1}^2 = \varepsilon\tilde{\tilde{\omega}}_{N1}^2$, and ε and ε_1 are small parameters. By use two methods [9] and [8] starting known analytical solutions $x_1(t) = R_1(t)e^{-\delta_1 t}\cos(p_1 t + \phi(t))$, $p_1 = \sqrt{\omega_1^2 - \delta_1^2}$ and $x(t) = a(t)\cos[\omega_1 t + \phi(t)]$, and we obtained same first approximation of the solution in the following forms [9]:

$$x_1(t) = R_{01}e^{-\delta_1 t}\cos\left[p_1 t \mp \frac{3}{16\delta_1 p_1}\omega_{N1}^2 a_o^2\left(e^{-2\delta_1 t} - 1\right) + \Phi_o\right]. \text{ for } \delta_1 \neq 0,\ \varepsilon \neq 0,\ \omega_1^2 > \delta_1^2,\quad p_1 = \sqrt{\omega_1^2 - \delta_1^2} \qquad (8)$$

$$x_1(t) = a_o e^{-\delta_1 t}\cos\left[\omega_1 t \mp \frac{3}{16\delta_1\omega_1}\omega_{N1}^2 a_o^2\left(e^{-2\delta_1 t} - 1\right) + \Phi_o\right]. \text{ for } \delta_1 \neq 0,\ \varepsilon \neq 0,\ \omega_1^2 > \delta_1^2 \qquad (9)$$

For the case that damping coefficient tends to zero, from both first approximations (8) and (9), we obtain same analytical approximation of the solution for conservative nonlinear system dynamics. For the case that coefficient of the cubic nonlinearity tends to zero, from first approximation (8), we obtain known analytical solution of the linear no conservative system dynamics in the following form: $x_1(t) = R_{01}e^{-\delta_1 t}\cos(p_1 t + \alpha_{01})$, for $\delta_1 \neq 0$, $\varepsilon = 0$, $\omega_1^2 > \delta_1^2$ $\tilde{\omega}_{N1}^2 = o$, but from the second form (9) obtained solution $x_1(t) = a_0 e^{-\delta_1 t}\cos(\omega_1 t + \alpha_{01})$ is not correct. Because is not solution of the differential equation: $\ddot{x}_1(t) + 2\delta_1\dot{x}_1(t) + \omega_1^2 x_1(t) = 0$. Then we can conclude that, starting different known analytical solutions, for obtaining first approximations are acceptable, but limited by corresponding conditions. Approximation of the solution of nonlinear differential George Duffing differential equations (7) in the form (8) is better them (9) known from numerous literatures. Presentation of full original results is limited by length of the paper.

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