EXISTENCE OF PERIODIC ORBITS FOR THE CR3BP UNDER A NEW FRAME

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Summary In this paper, a new coordinate system defined as "rotating-stretching frame" is firstly introduced. Unlike the previous literature devoted to the third-order approximation system of the circular restricted three-body problem (CR3BP), we primarily focus the attention on the periodic orbits to the non-approximation system of the CR3BP. Based on the equations of motion derived from the new frame, the existence of periodic orbits of the CR3BP is theoretically proven by using the method of functional analysis.

BACKGROUND AND A BRIEF OVERVIEW

The subject of periodic orbits of the circular restricted three-body problem (CR3BP) has received considerable attention in the past few decades. A detailed review, Farquhar and Kamel [1] dealt with an approximation for the periodic orbits, spurring a spate of investigation into the computation of periodic orbits. Richardson [2] analytically developed a local approximation for the periodic orbits of the CR3BP. Howison and Meyer [3] showed the existence of doubly symmetric periodic orbits of the spatial restricted three-body problem. Voyatzis and Kotoulas [4] studied a considerable number of resonant periodic orbits. They found that all the periodic orbits are symmetric and there is an indication of the existence of asymmetric ones only in a few cases. Yan [5] discussed the existence of a special symmetric periodic orbit in the planar three-body problem. Zhang et al. [6] investigated the existence of periodic orbits for the CR3BP with the third-order approximation in the inertial frame.
In spite of the large numbers of numerical and analytical work available, there are few papers on the existence of periodic orbits to the non-approximation system of the CR3BP. In present paper, we mainly focus the attention on the existence of periodic orbits to the non-approximation system of the CR3BP by using the method of functional analysis. The results achieved in the newly introduced rotating-stretching frame may throw more light than a thousand beautiful figures on the better understanding of the characteristics to the CR3BP.

ROTATING-STRETCHING FRAME

As is well known, the equations of motion for CR3BP in the inertial dimensionless frame can be expressed in the form:

$$\ddot{X} = -\widetilde{U}_X, \quad \ddot{Y} = -\widetilde{U}_Y, \quad \ddot{Z} = -\widetilde{U}_Z, \tag{1}$$

where the effective potential function $\widetilde{U}(X,Y,Z,t) = -\frac{1-\mu}{R_1} - \frac{\mu}{R_2}$, $R_1 = \left[(X+\mu\cos t)^2 + (Y+\mu\sin t)^2 + Z^2\right]^{1/2}$, $R_2 = \left[(X-(1-\mu)\cos t)^2 + (Y-(1-\mu)\sin t)^2 + Z^2\right]^{1/2}$, U_X, U_Y and U_Z are the partial derivatives of U with respect to X, Y, Z respectively. It is obvious that system (1) is time-dependent and has no simplified form. For this reason, we introduce a new frame with orbital plane x-y moves clockwise about the origin by an angle θ with unit angular velocity relative to the inertial frame with orbital plane X-Y, as shown in Fig.1. The z-axis completes the right-handed system with the scale stretched to the e^θ times of Z-axis, where e is known as the Euler's number. We will refer to this new frame throughout the paper as the *rotating-stretching frame*.

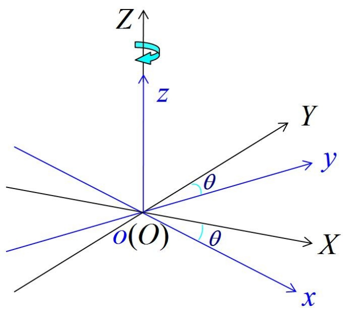

Fig.1. Schematic diagram of the inertial and rotating-stretching frames.

In normalized units, we have the following transformation of the third body's position between the two frames:

$$[X\ Y\ Z]^T = R(\theta)[x\ y\ z]^T, \tag{2}$$

where the coordinate transformation matrix $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & e^\theta \end{pmatrix}$.

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EQUATIONS OF MOTION

By using the lagrangian approach, the equations of motion in the new frame are given by

$$\ddot{x} - 2\dot{y} = x - U_x, \tag{3a}$$

$$\ddot{y} + 2\dot{x} = y - U_y, \tag{3b}$$

$$\ddot{z} = z - U_z, \tag{3c}$$

where $U(x,y,z) = -\frac{1-\mu}{r_1} - \frac{\mu}{r_2}$, $r_1 = \left[(x+\mu)^2 + y^2 + z^2\right]^{1/2}$, $r_2 = \left[(x-1+\mu)^2 + y^2 + z^2\right]^{1/2}$.
System (3) can also be reduced to the following form:

$$\ddot{\mathbf{r}} + M\dot{\mathbf{r}} - grad\Omega = 0, \tag{4}$$

where $\mathbf{r} = (x,y,z)^T$, $M = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and the augmented potential $\Omega = \frac{1}{2}\left(x^2 + y^2 + z^2\right) - U(x,y,z)$.

EXISTENCE OF PERIODIC ORBITS

For simplicity, we denote $|\cdot|$ as the absolute value and the Euclidean norm on $\mathbb{R}^3$, $I = [0,P]$, $C^0 = C^0\left(I, \mathbb{R}^3\right)$, $C^1 = C^1\left(I, \mathbb{R}^3\right)$, $C_P = \left\{u \in C^0 \mid u(0) = u(P)\right\}$, $C_P^1 = \left\{u \in C^1 \mid u(0) = u(T), u'(0) = u'(T)\right\}$. In addition, the norm in C_P will be denoted by $\|\cdot\|_\infty$, where $\|\cdot\|_\infty = \max_{t\in I}|\cdot|$. We now state and prove the following theorem.

Theorem. *System (4) has at least one P-periodic orbit.*

Proof. We embed system (4) into the one parameter family of equations

$$\ddot{\mathbf{r}} + \epsilon M\dot{\mathbf{r}} - \epsilon grad\Omega = 0, \quad \epsilon \in]0,1]. \tag{5}$$

Assume that $\mathbf{r} \in C_P$. Then, integrating both sides of Eq.(5) on the interval I, we have

$$\int_0^P grad\Omega \, \mathrm{d}t = 0. \tag{6}$$

On the other hand, there exists a positive constant c_1 such that $\int_0^P grad\Omega \, \mathrm{d}t \neq 0$ for $\|\mathbf{r}\| > c_1$, which contradicts Eq.(6). Therefore, $\|\mathbf{r}\| \leqslant c_1$.

Multiplying both sides of Eq.(5) by $\mathbf{r}$ and integrating it from 0 to P, we obtain $\int_0^P |\dot{\mathbf{r}}|^2 \, \mathrm{d}t \leqslant c_1 P$.

From $\mathbf{r}(0) = \mathbf{r}(P)$, it is easy to see that there is a $t_0 \in I$ such that $\mathbf{r}(t_0) = 0$. Then, we have

$$\|\dot{\mathbf{r}}\|_\infty = \max_{t\in I}\left|\int_{t_0}^t \ddot{\mathbf{r}}(s)\,\mathrm{d}s\right| \leqslant \int_0^P |M\dot{\mathbf{r}}(s)|\,\mathrm{d}s + \int_0^P |grad\Omega|\,\mathrm{d}s.$$

Then, it can be followed from $\|\mathbf{r}\| \leqslant c_1$ and $\int_0^P |\dot{\mathbf{r}}|^2 \, \mathrm{d}t \leqslant c_1 P$ that there is a positive constant c_2 such that $\|\dot{\mathbf{r}}\|_\infty \leqslant c_2$.

Let B be the open ball in C_P^1 with center zero and radius $\rho > \max\{c_1, c_2\}$, then Eq.(5) has no solution on ∂B for $\forall\, \epsilon \in\ ]0,1]$, and

$$F(-\rho)F(\rho) < 0, \quad \text{where } F(a) := \int_0^P grad\Omega \, \mathrm{d}t.$$

By using the method of functional analysis, we can conclude that system (4) has at least one P-periodic solution on $\overline{B}$ with $\|\mathbf{r}\| \leqslant c_1$. This completes the proof of theorem.

CONCLUSIONS

The main conclusion drawn from this paper is that unlike the most previous literature dealt with the third-order approximation system of the CR3BP numerically or analytically. Based on the method of functional analysis, this paper mainly investigate the existence of periodic orbits to the non-approximation system of the CR3BP in the rotating-stretching frame, which is firstly introduced. Moreover, one can see that the problem becomes much easier to treat in the new coordinate system. The results may give new sights on this classical three-body problem of celestial mechanics.

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