PROBABILITY DENSITY EVOLUTION ANALYSIS OF MULTIDIMENSIONAL NONLINEAR STOCHASTIC DYNAMICAL SYSTEMS

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Summary A new family of random-frequency stochastic harmonic function representation of stochastic processes is incorporated into the probability density evolution method to implement stochastic dynamic analysis of multidimensional nonlinear systems. The stochastic response of a complicated nonlinear structure subjected to random ground motion is investigated as a numerical example for validation. The studies indicate that the proposed method is of both fair accuracy and efficiency for stochastic response analysis of multidimensional nonlinear dynamical systems.

INTRODUCTION

Since the1950s, the random vibration of different types of structures attracted a variety of researchers in different engineering fields. Great endeavours have been doted to random vibrations of linear and/or nonlinear systems with either deterministic or random parameters. The methods to capture the probabilistic information of linear systems are readily available now. Nevertheless, huge difficulty still arises in dynamic analysis of multidimensional nonlinear systems subjected to non-stationary stochastic excitations, such as strong ground motion. The probability density evolution method (PDEM), which reveals the intrinsic relationship between a stochastic system and the deterministic counterpart, can capture the instantaneous probability density functions (PDFs) of general stochastic system and thus provides a new insight into such problems (Li and Chen, 2009). To further reduce the computational efforts without loss of accuracy, the stochastic harmonic function (SHF) representation of stochastic processes is incorporated into PDEM to carry out stochastic dynamic analysis of multidimensional nonlinear systems.

STOCHASTIC HARMONIC FUNCTION REPRESENTATION OF STOCHASTIC PROCESSES

Mathematically speaking, a stochastic process can be specified completely by the probabilistic information such as the family of finite-dimensional joint probability distribution functions. However, this is almost impossible for most stochastic processes due to the unavailability of huge data and the so-called curse of dimensions. As an alternative, the second-order statistics, e.g. the power spectral density (PSD) or the covariance function, are mostly adopted in practice. Though the PSD is adequate for stochastic dynamic analysis of linear structures, time histories of the stochastic process is a key prerequisite to implement stochastic analysis of nonlinear systems. Thus the task is changed to be generating time histories from the known probabilistic information, say PSD function. Several methods have been developed. Nonetheless, the truncation of infinite series and large number of random variables involved in these methods may lead to the predicament of the unbalance between the accuracy and efficiency of stochastic dynamic analysis of multidimensional nonlinear systems. Recently, two kinds of stochastic harmonic function (SHF) representations of stochastic processes have been put forward respectively (Chen & Li, 2011; Sun et al, 2011). For instance, the stochastic harmonic function of the first kind (SHF-I) for the ground motion acceleration takes the form

$$\ddot{X}_{\mathrm{g}}(t) = \sum_{i=1}^{N} A(\Omega_i)\cos(\Omega_i t + \phi_i) \tag{1}$$

where Ω_i 's are independent random frequencies, of which the probability densities are consistent with the target PSD function in the sub-interval, ϕ_i 's are mutually independent random variables uniformly distributed over $[0, 2\pi]$ (Chen & Li, 2011). It is proved that the PSD function of SHF-I is identical to the target PSD and the one-dimensional PDF is quite close to normal distribution when the number of harmonic components is over seven.

GENERALIZED DENSITY EVOLUTION EQUATION

The equation of motion of a general multidimensional system subjected to random ground motion reads

$$\mathbf{M}(\boldsymbol{\eta})\ddot{\mathbf{Y}} + \mathbf{C}(\boldsymbol{\eta})\dot{\mathbf{Y}} + \mathbf{f}(\boldsymbol{\eta}, \mathbf{Y}) = -\mathbf{M}\mathbf{I}\ddot{X}_{\mathrm{g}}(t) \tag{2}$$

where $\mathbf{M}$ and $\mathbf{C}$ are, respectively, the mass and damping matrices, $\mathbf{f}(\boldsymbol{\eta}, \mathbf{Y})$ is the n-dimensional linear or nonlinear restoring force vector, $\mathbf{I} = \{1, 1, \cdots, 1\}^{\mathrm{T}}$, $\boldsymbol{\eta}$ is the random vector involved in structural parameters. $\ddot{X}_{\mathrm{g}}(t)$ is the random ground motion described as Eq.(1). Denote the basic random vector $\boldsymbol{\Theta} = (\eta_1, \eta_2, \cdots, \eta_s, \Omega_1, \cdots, \Omega_N, \phi_1, \cdots, \phi_N)$ $= (\Theta_1, \Theta_2, \cdots, \Theta_s)$ of which the joint PDF $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is known accordingly. Then Eq. (2) turns to be

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$$\mathbf{M}(\mathbf{\Theta})\ddot{\mathbf{Y}}+\mathbf{C}(\mathbf{\Theta})\dot{\mathbf{Y}}+\mathbf{f}(\mathbf{\Theta},\mathbf{Y})=-\mathbf{MI}\ddot{X}_{\mathrm{g}}(\mathbf{\Theta},t) \tag{3}$$

The solution of Eq.(3) could be assumed as

$$\mathbf{Y}(t)=\mathbf{H}(\mathbf{\Theta},t) \tag{4}$$

Usually, we are concerned not only with the displacement of a stochastic system, but also with many other related physical quantities such as the accelerations, internal forces or stresses and strains, etc.. Denote the physical quantities of interest by $\mathbf{Z}=(Z_1,Z_2,\cdots,Z_m)^{\mathrm{T}}$, which is of course related to $\mathbf{Y}(t)$ .The augmented system $(\mathbf{Z},\mathbf{\Theta})$ is a probability preserved system. According to the principle of preservation of probability (Li and Chen, 2008), we have

$$\frac{D}{Dt}\int_{\Omega_t\times\Omega_\theta} p_{\mathbf{Z\Theta}}(\mathbf{z},\boldsymbol{\theta},t)dtd\boldsymbol{\theta}=0 \tag{5}$$

A partial differential equation, called the generalized density evolution equation (GDEE), can be derived after a series of mathematical manipulations (Li and Chen, 2009).

$$\frac{\partial p_{\mathbf{Z\Theta}}(\mathbf{z},\boldsymbol{\theta},t)}{\partial t}+\sum_{j=1}^{m}\dot{Z}_j(\boldsymbol{\theta},t)\frac{\partial p_{\mathbf{Z\Theta}}(\mathbf{z},\boldsymbol{\theta},t)}{\partial z_j}=0 \tag{6}$$

where $p_{\mathbf{Z\Theta}}(\mathbf{z},\boldsymbol{\theta},t)$ denotes the joint PDF of $(\mathbf{Z},\mathbf{\Theta})$ in Eqs.(5) and (6), $\dot{Z}_j(\boldsymbol{\theta},t)=\partial Z_j(\boldsymbol{\theta},t)\big/\partial t$ is the rate of the physical quantity $Z_j(\boldsymbol{\theta},t)$. In the case $m=1$, the equation reduces to a one-dimensional partial differential equation, which is mostly used in practice.

Under the initial condition $p_{\mathbf{Z\Theta}}(\mathbf{z},\boldsymbol{\theta},t)\,|_{t=t_0}=\delta(\mathbf{z}-\mathbf{z}_0)p_{\mathbf{\Theta}}(\boldsymbol{\theta})$, the PDF could be evaluated via

$$p_{\mathbf{Z}}(\mathbf{z},t)=\int_{\Omega_{\mathbf{\Theta}}} p_{\mathbf{Z\Theta}}(\mathbf{z},\boldsymbol{\theta},t)d\boldsymbol{\theta} \tag{7}$$

ILLUSTRATIVE EXAMPLE

A 9-story shear frame subjected to random ground motion is studied to illustrate the methodology. In the situation of strong nonlinearity, the Bouc-Wen model is employed to characterize the nonlinear restoring force v.s. inter-story drift. Then the time histories of random ground motion are generated by SHF-I, where the Clough-Penzein spectrum is adopted as the target PSD. Obviously, excellent accordance can be noticed between the reproduced PSD and the original PSD (Figure 1). Further, incorporating the time histories into PDEM will obtain the probabilistic information of the multidimensional stochastic dynamic system. Again, the second-order statistics of the response accord very well with that by Monte Carlo Simulation (MCS) (Figure 2). However, the computational cost of PDEM via SHF-I is far more less. It is concluded that the proposed method could achieve tradeoffs between the accuracy and efficiency in the stochastic dynamic analysis of multidimensional nonlinear systems. In addition, it is clear that the PDFs exhibit irregularities, which is quite different from general imaginations (Figure 3). It should be noted that it is the instantaneous PDF, rather than the second-order statistics, that is the output of GDEE. To quantitatively characterize the difference between the instantaneous PDFs and the normal distributions with the same mean and standard deviation, the relative entropy is adopted. Again, the result shows that the evolutionary PDFs are far away from the corresponding normal distributions (Figure 4).

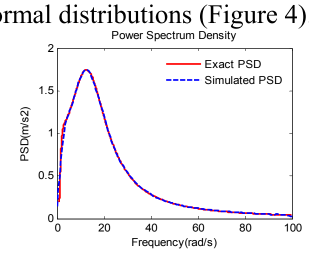

Figure 1. Target and reproduced PSD

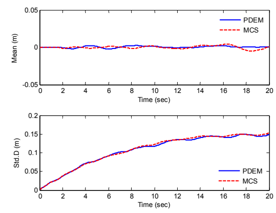

Figure 2. The mean and standard deviation

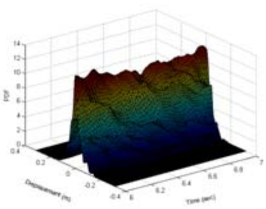

Figure 3. PDF surface

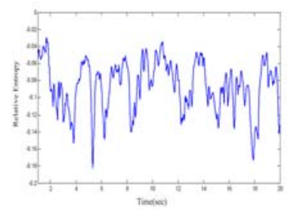

Figure 4. Relative entropy

CONCLUDING REMARKS

The stochastic harmonic function representation can greatly ease the efforts of stochastic dynamic analysis of multidimensional nonlinear systems with fair accuracy when incorporated into PDEM. The proposed approach is numerically validated.

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