PARAMETRIC INSTABILITY OF SLENDER BEAMS WITH UNILATERAL WINKLER SUPPORT: AN APPLICATION TO RISER DYNAMICS

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Summary The multiple scales method is applied to a reduced-order model of a slender beam with unilateral Winkler constraint to supply a closed-form solution of its non-linear dynamic response under parametric excitation caused by a harmonically-driven axial force. The subject is of utmost relevance to address current challenges within offshore technology, since it concerns the local dynamic analysis of the touch-down zone of steel catenary risers. A case study is presented for the sake of an illustration. A shear sensitivity of results with damping is found.

INTRODUCTION

The mathematical modelling of a *2D* slender beam unilaterally supported by a continuous Winkler medium with stiffness μ, under harmonically-driven axial force is studied here, as depicted in Figure 1.

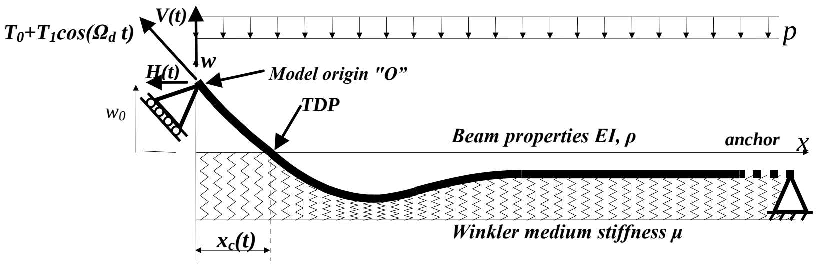

Figure 1: slender beam on unilateral Winkler support subjected to a harmonically-varying normal force

The motivation for this study is the local vibration analysis of the Touch-Down Zone (TDZ) of offshore catenary risers [1]. The riser global dynamics, from the hang-off to the Touch-Down Point (TDP), is usually studied with a cable model, therefore neglecting bending. Imposed motion at the hang-off, due to wave loading, or vortex-induced vibrations, due to sea currents, cause a harmonic variation of the cable normal force. Yet, non-negligible bending effects start at a distance $x_c(t) \cong 4\sqrt{EI/T_{TDP}}$ from the TDP, where EI is the bending stiffness and T_{TDP} is the static normal force at the TDP [2]. Bending-dominated vibration may cause fatigue and dynamic buckling. Parametric instability, although less likely, can also happen and will be addressed in this paper.

EQUATION OF MOTION OF A SLENDER BEAM ON UNILATERAL WINKLER SUPPORT

The equation of motion of a semi-infinite slender beam on unilateral Winkler support can be written as:

$$EIw^{IV} + H(x)\mu w + \rho\ddot{w} - Tw'' - T'w' + p = 0 \tag{1}$$

where $T(x,t)$, ρ and p stand for the beam normal force; mass and submerged weight per unit length, respectively. $H(x)$ is the Heaviside function, which is *zero* for $x < x_c(t)$ and *one* for $x \geq x_c(t)$. This is a problem with moving boundary conditions, since the TDP position changes with time. The unilateral Winkler constraint is a strong source of non-linearity. A variable transformation is introduced to transform this problem into one with fixed boundary conditions:

$$z = \frac{\alpha x}{c(\tau)} - 1,\quad \alpha = \sqrt[4]{\frac{\mu}{4EI}},\quad c(\tau) = \alpha x_c(t),\quad u = \frac{\mu}{p}w,\quad \tau = \beta t,\quad \beta = \sqrt{\frac{\mu}{\rho}},\quad \gamma = \frac{T}{2\sqrt{\mu EI}}. \tag{2}$$

One arrives at:

$$\frac{1}{4}u^{IV} - 2(1+z)\dot{u}'\dot{c}c^3 + \left[(1+z)^2 c^2\dot{c}^2 - \gamma c^2\right]u'' + u'\left[(1+z)\left(2\dot{c}^2 - c\ddot{c}\right) - \gamma'\right]c^2 + c^4\ddot{u} + \left[1 + H(z)u\right]c^4 = 0\,. \tag{3}$$

In the case of parametric excitation due to a harmonically-driven normal force, a reduced-order model (ROM) can be proposed, after projecting (3) onto the excited non-linear normal mode [3]. The modal oscillator equation is obtained:

$$\ddot{U} + \hat{a}_1\dot{U} + \left(\hat{a}_{2,0} + \hat{a}_{2,1}\cos\Omega\tau\right)U = \hat{b}_{0,1}\cos\Omega\tau + \left(\hat{b}_{1,0} + \hat{b}_{1,1}\cos\Omega\tau\right)U^2 + \hat{b}_{2,0}\dot{U}^2 + \hat{b}_{3,0}U\ddot{U} + \left(\hat{b}_{4,0} + \hat{b}_{4,1}\cos\Omega\tau\right)U^3$$
$$+ \hat{b}_{5,0}U\dot{U}^2 + \hat{b}_{6,0}U^2\ddot{U} + \hat{b}_{7,0}U^4 + \hat{b}_{8,0}U^2\dot{U}^2 + \hat{b}_{9,0}U^3\ddot{U} + \hat{b}_{10,0}U^5 + \hat{b}_{11,0}U^3\dot{U}^2 + \hat{b}_{12,0}U^4\ddot{U} \tag{4}$$

where the modal co-ordinate U is the TDP horizontal displacement. Introducing scaling by means of a dummy parameter $0 < \varepsilon << 1$, and keeping only the most relevant terms, a simplified MOR equation comes out:

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$$\ddot{U} + 2\varepsilon^2\xi\omega_0\dot{U} + \left(\omega_0^2 + \varepsilon^2\eta\cos\Omega\tau\right)U = \varepsilon\lambda\cos\Omega\tau + \varepsilon\zeta\dot{U}^2 + \varepsilon\vartheta U\ddot{U} + \nu U\dot{U}^2 + \kappa U^2\ddot{U} \qquad (5)$$

where $\varepsilon^2\xi$ is the equivalent viscous damping ratio, ω_0 is the modal frequency, $\varepsilon^2\eta$ and $\varepsilon\lambda$ are caused by the normal force harmonic variation, and $\varepsilon\zeta$, $\varepsilon\vartheta$, ν and κ are coefficients of non-linear terms.

PARAMETRIC INSTABILITY

The harmonic variation of the normal force can cause parametric instability, for small detuning $\sigma \cong 0$, with $\Omega = 2\omega_0 + \varepsilon^2\sigma$. In such a case, among other possible scenarios, stable periodic attractors may appear, with important implication on the structural performance. The method of multiple scales is used to obtain the approximate solution:

$$U = const + \varepsilon a\cos\frac{\Omega}{2}\tau + \ldots \qquad (6)$$

$$\varepsilon a = 2\sqrt{\frac{1}{r_3}\left[\left(\varepsilon^2 r_1 - \varepsilon^2\sigma\omega_0\right) \pm \sqrt{\left(\varepsilon^2 r_2\right)^2 - \left(2\varepsilon^2\xi\omega_0^2\right)^2}\right]} \qquad (7)$$

$$\varepsilon^2 r_1 = \left[\frac{\kappa\left(2\Omega^2 + \omega_0^2\right) + \nu\Omega^2}{2\left(\Omega^2 - \omega_0^2\right)^2}\right](\varepsilon\lambda)^2; \quad \varepsilon^2 r_2 = \left[\frac{\left(\Omega^2 + \omega_0^2\right)(\varepsilon\vartheta) - 2\Omega\omega_0(\varepsilon\zeta)}{2\left(\Omega^2 - \omega_0^2\right)}\right](\varepsilon\lambda) - \frac{\left(\varepsilon^2\eta\right)}{2}; \quad r_3 = (\nu - 3\kappa)\omega_0^2 \qquad (8)$$

Figure 2 displays results of the numerical integration of (5), for a case study with $EI = 9.443\times10^6\ Nm^2$, $D = 0.2032m$, $\mu = 2032Nm^{-2}$, $\alpha = \sqrt[4]{\mu/4EI} = 0.08564m^{-1}$, $\rho = 141kgm^{-1}$, $p = 727Nm^{-1}$, $T_0 = 200kN$, $T_1 = 30kN$ and zero detuning. For the multiple scales solution, the following non-dimensional parameters should be used accordingly: $\omega_0 = 0{,}2036$, $\Omega = 0.4072$, $\varepsilon^2\sigma = 0$, $\varepsilon^2\xi = 0.2954$, $\varepsilon^2\eta = -0.00197$, $\varepsilon\lambda = 0.04304$, $\varepsilon\zeta = -0.01169$, $\varepsilon\vartheta = -0.71995$, $\kappa = -0.19377$, $\nu = 0.02385$.

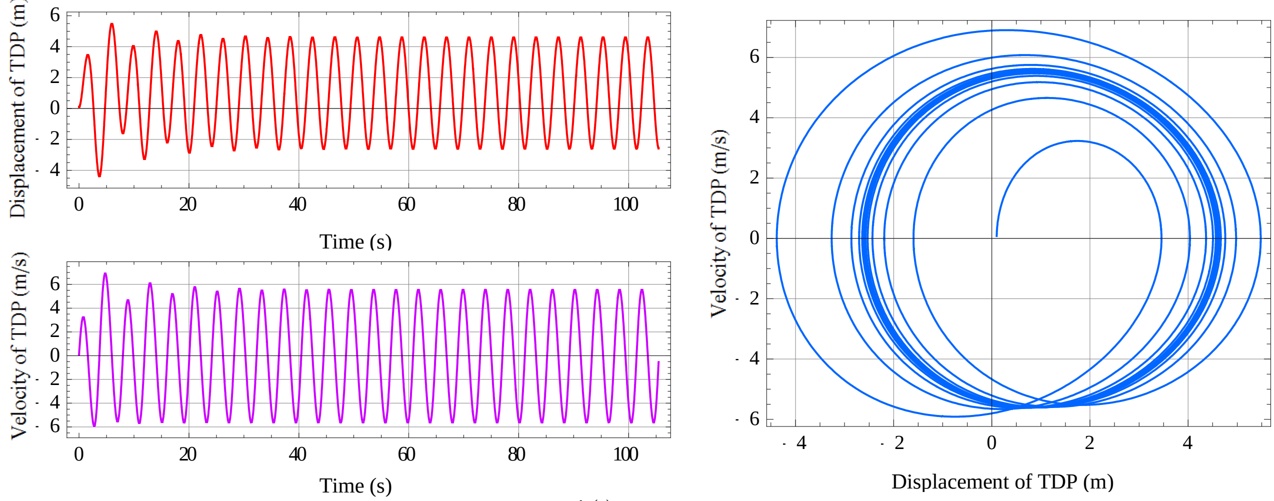

Figure 2: left top, $U(t)$; left bottom, $V = \dot{U}(t)$; right, phase plane; extracted from case P9 of [3]

From (7), the amplitude of the TDP excursion is found to be $\varepsilon a/\alpha = 3.96m$, which should be compared with the average of maximum and minimum displacements shown in Figure 2 ($\cong 3.55m$), with a good agreement. It should be emphasised the shear sensitivity of both the numerical integration and the multiple-scales solutions with respect to the damping ratio.

CONCLUSIONS

The use of reduced-order modelling together with the multiple-scale method has been demonstrated for the analysis of a slender beam with unilateral Winkler support, subjected to a harmonically-varying normal force, under parametric resonance. This approach is expected to be of great help in the analysis of offshore catenary risers.

References

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