PERIODIC SOLUTIONS OF A SMOOTH AND DISCONTINUOUS (SD) OSCILLATOR

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Summary In this paper, periodic solutions of the perturbed SD oscillator are discussed by employing averaging method. Jacobian elliptic integrals are employed to obtain the primary responses directly. The stability of the periodic solutions are analysed by examining the averaged equation using Lyapunov method. All the primary resonances bifurcating from both hyperbolic saddle and non-hyperbolic centres are analysed and the distributions of the periodic orbits are presented. It is worth noticing that the results presented here are valid for both the smooth ($\alpha > 0$) and the discontinuous ($\alpha = 0$) cases. Numerical calculations are carried out to show a good agreement with the theoretical analysis and an excellent efficiency of the analysis for this particular system. This also suggests the analysis is applicable to strong nonlinear systems.

INTRODUCTION

The smooth and discontinuous (SD) oscillator [1, 2, 3] proposed and investigated recently is an archetypal oscillator with a strong irrational nonlinearity. This oscillator allows one to study the transition from smooth to discontinuous dynamics depending on the value of the smoothness parameter α. The system is smooth if $\alpha > 0$, while it is discontinuous when $\alpha = 0$. In this paper, we focus our attention on the periodic solutions of the system.
The dimensionless equation of SD oscillator can be written as the following:

$$\ddot{x} + \omega_0^2 x\left(1 - \frac{1}{\sqrt{x^2+\alpha^2}}\right) = 0, \tag{1}$$

which arises from the first mode approximation of a simple arch [1]. This system exhibits a large variety and complexity of responses and phenomena. The mechanism of transition from smooth to discontinuous dynamics was presented in [1, 3]. The codimension-two bifurcation [4], the Hopf bifurcations [5] and the resonance oscillations [6] have also been investigated, respectively. The dimensionless equation of the SD oscillator perturbed by a viscous damping and an external harmonic excitations is of the following form:

$$\ddot{x} + \delta\dot{x} + \omega_0^2 x\left(1 - \frac{1}{\sqrt{x^2+\alpha^2}}\right) = f\cos\omega t, \tag{2}$$

which admits multiple coexistences of the periodic solutions which are investigated for both smooth and discontinuous phases of all the equilibria of the system in the following sections.

PERIODIC SOLUTIONS OF THE SMOOTH SYSTEM

By using averaging method, we can get the response equation for the hyperbolic equilibrium, written as

$$\big(G(a,\alpha) + (\omega^2 - \omega_0^2)\big)^2 + \delta^2\omega^2 = \frac{f^2}{a^2}, \tag{3}$$

where $G(a,\alpha) = \frac{4\omega_0^2}{\pi a^2}\left[\sqrt{a^2+\alpha^2}E\left(\sqrt{\frac{a^2}{a^2+\alpha^2}}\right) - \frac{\alpha^2}{\sqrt{a^2+\alpha^2}}K\left(\sqrt{\frac{a^2}{a^2+\alpha^2}}\right)\right]$, $K(k)$ and $E(k)$ are, respectively, the complete elliptic integrals of the first kind and the second kind, with $0 < k < 1$.
The response equation for non-hyperbolic equilibrium is written by letting $x = a\cos(\tau+\theta) + c, \dot{x} = -a\sin(\tau+\theta) + \dot{c}$ as

$$\left[\omega^2 - \omega_0^2 + \frac{\omega_0^2}{a}Q_2(\tau,\alpha)\right]^2 + \delta^2\omega^2 = \frac{f^2}{a^2}, \tag{4}$$

where $Q_2(\tau,\alpha) = \frac{1}{\pi}\int_0^{2\pi}\frac{[a\cos(\tau+\theta)+c]\cos(\tau+\theta)}{\sqrt{(a\cos(\tau+\theta)+c)^2+\alpha^2}}d(\tau+\theta)$ and $c = \frac{1}{2\pi}\int_0^{2\pi}\frac{a\cos(\tau+\theta)+c}{\sqrt{(a\cos(\tau+\theta)+c)^2+\alpha^2}}d(\tau+\theta)$.

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In the same way, the response equation for saddle-like singularity[3] can be obtained and written as

$$\left[\frac{4\omega_0^2}{\pi a} + (\omega^2 - \omega_0^2)\right]^2 + \delta^2\omega^2 = \frac{f^2}{a^2}, \tag{5}$$

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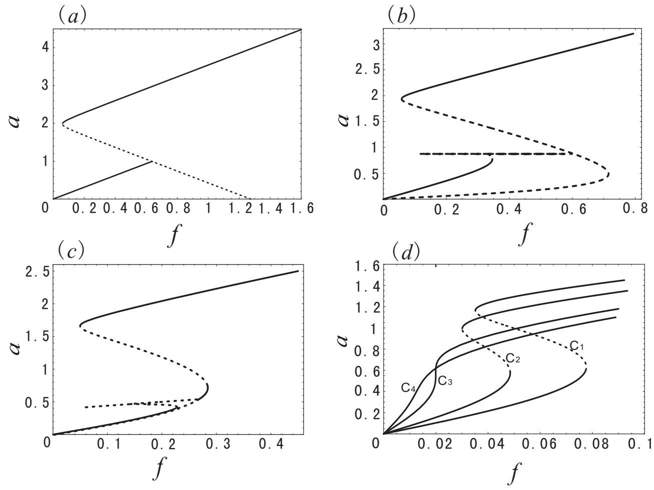

Figure 1. Response curves (a) for $\alpha = 0$, (b) for $\alpha = 0.25$, (c) for $\alpha = 0.75$ and (d) c_1-c_4 for $\alpha = 1.2, 1.3, 1.438$ and 1.5 respectively.

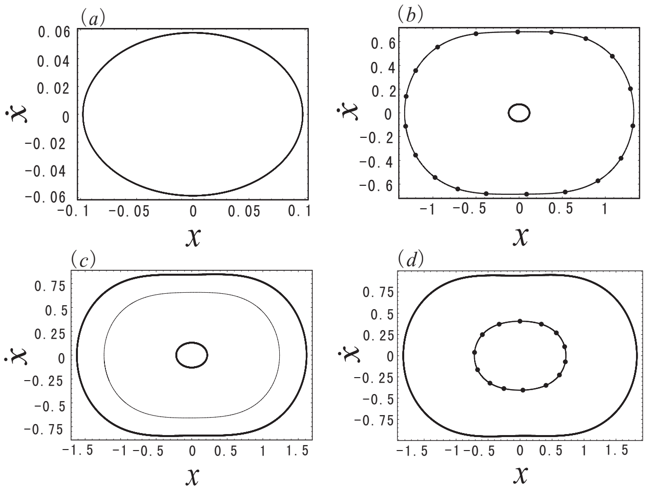

Figure 2. Distributions of the periodic solutions of the system for $\alpha = 1$, $(a)f = 0.02$, $(b)f = 0.043$, $(c)f = 0.075$, $(d)f = 0.154$.

and for non-hyperbolic equilibrium also be given by

$$(\omega^2 - \omega_0^2)^2 + \delta^2\omega^2 = \frac{f^2}{a^2}. \tag{6}$$

Here (5) and (6) can also be obtained directly from (3) and (4) by letting $\alpha \to 0$ respectively. This approach is valid for both the smooth and the discontinuous stages. The response curves for both the trivial solution, the saddle-like singularity, and the centres at the discontinuous stage for $\delta = 0.05$, $\omega = 0.6$ and $\omega_0 = 1$ are presented in Fig.1(a), and the response curves for both hyperbolic and non-hyperbolic equilibria at the smooth stage are plotted in Fig.1(b), (c) and (d), respectively. The solid curves represent the stable and the dashed ones mark the unstable branches. Fig.2 plots the distributions of the periodic solutions of the system when $\alpha = 1$, $\delta = 0.05$, $\omega = 0.6$ and $\omega_0 = 1$, in which, the thick, dotted and thin curves mark the stable, semi-stable, and the unstable periodic solutions, respectively.

CONCLUSIONS

In this paper, periodic solutions of the smooth-and-discontinuous (SD) oscillator with strong irrational nonlinearity have been studied. By introducing the complete Jacobian integrals of the first and second kinds, the primary responses for both smooth and discontinuous phases of all the equilibria of the system have been obtained. The stability analysis of the periodic solutions have also been carried out by employing liapunov's criteria. Numerical investigations show the excellent agreement with the theoretical results obtained herein. This provides a way to overcome the barrier of conventional methods in solving strong irrational nonlinear systems.

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