MULTIPLE HOPF BIFURCATIONS FOR A NOVEL SMOOTH AND DISCONTINUOUS OSCILLATOR WITH IRRATIONAL NONLINEARITIES

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Summary In this paper, we investigate the Hopf bifurcations for a novel oscillator based upon the recently proposed smooth and discontinuous (SD) oscillator [1, 2]. The nonlinear property of this system is strong irrational nonlinearity due to the geometrical configuration. The Hopf bifurcation of the nonlinear damped system is examined near the catastrophe point C in parameter space. The Hopf bifurcation surfaces and the coexisted multiple limit cycles are obtained for multi-well dynamics of single, double and triple wells, respectively.

EQUATION OF MOTION OF NOVEL SD OSCILLATOR

The governing equation, seen in [3] for details, used in this paper is as the following:

$$\ddot{x} + F(x, \alpha, \beta) = 0. \tag{1}$$

where $F(x, \alpha, \beta) = (x + \alpha)\,(1 - 1/\sqrt{(x + \alpha)^2 + \beta^2}) + (x - \alpha)\,(1 - 1/\sqrt{(x - \alpha)^2 + \beta^2})$ is the nonlinear irrational force, it is smooth for $\beta > 0$ and discontinuous at $\beta = 0$, i.e. $F(x, \alpha, 0) = 2x - \mathrm{sign}(x + \alpha) - \mathrm{sign}(x - \alpha)$. The Taylor expansion of smooth $F(x, \alpha, \beta)$ can be obtained and written as

$$F(x, \alpha, \beta) = 2x - \sum_{n=0}^{\infty} P_n(0)[(x + \alpha)^{n+1} + (x - \alpha)^{n+1}]/\beta^{(n+1)}, \tag{2}$$

where $P_n(x)$ is the Legendre polynomial defined by $P_n(x) = \frac{1}{2^n n!}[\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2 - 1)^n], n = 0, 1, 2, 3, \cdots$. By truncating the Taylor series to the fifth order leads to the following equivalent system of fifth order, which is topological equivalent to the smooth phase of system (1), i.e.

$$\ddot{x} + C_1(\alpha, \beta)x + C_3(\alpha, \beta)x^3 + C_5(\alpha, \beta)x^5 = 0. \tag{3}$$

MULTIPLE HOPF BIFURCATION ANALYSIS

We assume that system (3) is perturbed by a Van der Pol [4] nonlinear damping $(\xi + \eta x^2)y$. This leads to the forced dissipative oscillator of the following form.

$$\ddot{x} + (\xi + \eta x^2)y + C_1 x + C_3 x^3 + C_5 x^5 = 0. \tag{4}$$

Re-scaling system (4) follows that

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\varepsilon_1 x - \varepsilon_2 x^3 + \mu y - x^5 - x^2 y. \end{cases} \tag{5}$$

In order to get the Hopf bifurcation surfaces for system (5), the following analysis are carried out.
Case 1. If $\varepsilon_1 \geq 0$, $\varepsilon_2 > -2\sqrt{\varepsilon_1}$, system (5) has a unique equilibrium (0,0), at which the Jacobian can be written as

$$J_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -\varepsilon_1 & -\mu \end{pmatrix},$$

with the characteristic equation

$$\lambda^2 + \mu\lambda + \varepsilon_1 = 0,$$

whose eigenvalues, $\lambda_\pm = -\mu/2 \pm \sqrt{\mu^2 - 4\varepsilon_1}/2$, imply that Hopf bifurcation occurs at $\varepsilon_1 > 0, \mu = 0$.
Case 2. If $\varepsilon_1 < 0$, system (5) have three equilibria, one saddle and a pair of symmetric centres, written as.

$$(x_s, y_s) = (0, 0), (\pm x_c, \pm y_c) = (\pm(-\varepsilon_2/2 + (\varepsilon_2^2 - 4\varepsilon_1)^{1/2}/2)^{1/2}, 0).$$

The Jacobian at centre (x_c, y_c) can be obtained as the following.

$$J_{(x_c, y_c)} = \begin{pmatrix} 0 & 1 \\ -\varepsilon_1 - 3x_c^2 - 5x_c^4 & -\mu - x_c^2 \end{pmatrix}.$$

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Similarly, it follows that the Hopf bifurcation happens when $-\varepsilon_1 - 3x_c^2 - 5x_c^4 > 0, \mu - x_c^2 = 0$.
Case 3. If $\varepsilon_1 \geq 0$, $\varepsilon_2 \leq -2\sqrt{\varepsilon_1}$, the system (5) have five equilibria: three centres and a pair of symmetric saddles, written as

$$(x_{c1}, y_{c1}) = (0,0),\ (\pm x_{c2}, \pm y_{c2}) = (\pm(-\varepsilon_2/2 + (\varepsilon_2^2 - 4\varepsilon_1)^{1/2}/2)^{1/2}, 0),$$
$$(\pm x_s, \pm y_s) = (\pm(-\varepsilon_2/2 - (\varepsilon_2^2 - 4\varepsilon_1)^{1/2}/2)^{1/2}, 0).$$

The Jacobian at centres can be written as

$$J_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -\varepsilon_1 & -\mu \end{pmatrix}, J_{(\pm x_{c2}, \pm y_{c2})} = \begin{pmatrix} 0 & 1 \\ -\varepsilon_1 - 3x_{c2}^2 - 5x_{c2}^4 & -\mu - x_{c2}^2 \end{pmatrix},$$

which lead to the Hopf bifurcation conditions, $\varepsilon_1 > 0, \mu = 0$ and $-\varepsilon_1 - 3x_{c2}^2 - 5x_{c2}^4 > 0, \mu - x_{c2}^2 = 0$, for centres $(0,0)$ and $(\pm x_{c2}, \pm y_{c2})$, respectively.
In summary, three Hopf bifurcation surfaces, marked with **HI**, **HII**, and **HIII**, as shown in Figure 1(a), present the corresponding single, double and triple Hopf bifurcation surfaces of system (5) in $\varepsilon_1 - \varepsilon_2 - \mu$ space, which are listed as:

1. **HI** $= \{\varepsilon_1 > 0, \mu = 0\}$;
2. **HII** $= \{\varepsilon_1 < 0, \mu - \varepsilon_2/2 - (\varepsilon_2^2 + 4\varepsilon_1)^{1/2}/2) = 0\}$;
3. **HIII** $= \{\varepsilon_1 > 0, \mu = 0\} \cup \{\varepsilon_2 \leq -2\sqrt{\varepsilon_1}, \mu - \varepsilon_2/2 - (\varepsilon_2^2 + 4\varepsilon_1)^{1/2}/2) = 0\}$.

We define the Hopf bifurcation surface **H** as **H** $=$ **HI** $\cup$ **HII** $\cup$ **HIII**, in which the Hopf bifurcations occur [5]. While **H** divides the three-dimensional parameter space $\varepsilon_1 - \varepsilon_2 - \mu$ into seven persistence regions **I**$-$**VII**, in each of which the system is structurally stable. The coexisted multiple limit cycles are given in Figures 1(b), (c) and (d).

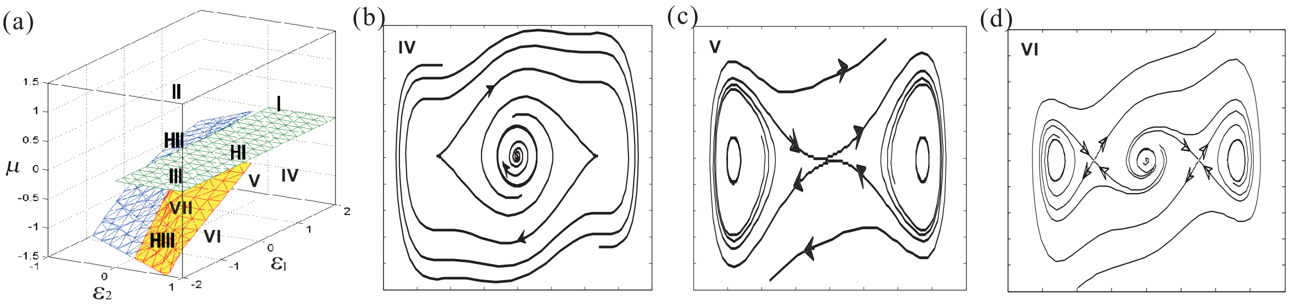

Figure 1. (a) Hopf bifurcation surfaces, marked **HI** (in green), **HII** (in blue) and **HII** (in yellow), and seven persistence regions marked **I**$-$**VII**, respectively; (b-d) the coexisted single (**IV**), double (**V**) and triple (**VI**) limit cycles for system (5) respectivrly.

CONCLUSIONS

We have investigated multiple Hopf bifurcations for a novel oscillator with irrational nonlinearities and nonlinear damping, this system typically posses multiple limit cycles. Three Hopf bifurcation surfaces have been obtained, in which multiple Hopf bifurcations occurs and the coexisted single, double and triple limit cycles surrounding equilibria point emerge from the equilibria.

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