A NEW VARIATIONAL METHOD FOR FREE VIBRATION ANALYSIS OF CONICAL SHELLS WITH DISCONTINUITY IN THICKNESS

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Summary A new variational method is proposed to study the free vibration of stepped conical shells (SCSs). Multilevel partition hierarchy, viz., stepped SCS, shell segment and shell domain, is adopted to accommodate the computing requirements of high-order vibration modes. The continuity conditions between shell domains are enforced by means of a modified variational principle and least-squares weighted residual method. The present results show good agreement with those obtained from ANSYS, but with much less computational cost.

INTRODUCTION

Conical shells with discontinuities in thickness (i.e. the stepped conical shell, SCS) are efficient structural forms which have found widespread use in many branches of structural engineering. However, relatively few solutions available in the published literature address the vibration behaviour of these shells. The main objective of this paper is to develop an efficient variational method for vibration analysis of SCSs with classical and non-classical boundary conditions. The location of the step variations in a SCS can be completely arbitrary.

THEORETICAL FORMULATION

An elastic and isotropic SCS to be investigated is shown in Fig.1. The SCS consists of P steps with lengths $L_1, L_2, \cdots L_P$, step thicknesses $h_1, h_2, \cdots h_P$ and semi-vertex cone angle α_0. Let the co-ordinate system be chosen such that the origin O is at the vertex of the cone and on the middle surface of the shell; s axis lies on the curvilinear middle surface of the cone and θ is the circumferential co-ordinate. R_1 and R_2 are the radii of the SCS at its small and large edges. u, v and w are the displacement components in the s, θ and normal directions, receptively.

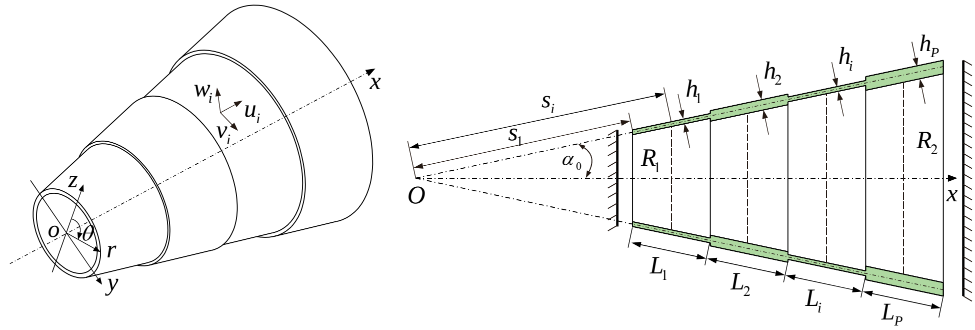

Fig.1 Geometry and coordinate system of a SCS with general boundary condition

The SCS is divided into P conical shell segments along the locations of the step variations and geometric boundaries, as depicted in Fig.1. In order to accommodate the computing requirements of high-order vibration modes, a shell segment with constant thickness may be further decomposed into N_c shell domains. The geometric boundaries herein are treated as special interfaces of those between shell domains. The continuity condition between adjacent conical shell domains (i) and (i+1) requires the continuity C^1, which indicates the displacement and slope at the interface must be continuous. A modified variational principle [1] is employed to incorporate the continuity conditions. This yields the functional as

$$\Pi_\pi = \int_{t_0}^{t_1} \sum_{i=1}^{P\times N_c} (T_i - U_i)dt + \int_{t_0}^{t_1} \sum_{i,i+1} \int_l \left[N_s(u_i - u_{i+1}) + \bar{N}_{s\theta}(v_i - v_{i+1}) + \bar{Q}_s(w_i - w_{i+1}) - M_s(\partial w_i/\partial s - \partial w_{i+1}/\partial s) \right] dl dt \quad (1)$$

where T_i and U_i are the maximum kinetic energy and potential energy of the i th shell domain, respectively. N_s, $\bar{N}_{s\theta}$, $\bar{Q}_s$ and M_s are the force and moment resultants defined on the interconnecting boundaries [2]. To ensure a numerically stable computation, we further modify the functional $\bar{\Pi}_\pi$ by appending a least-squares weighted residual term of the continuity equations scaled by weighted parameters. This yields a new variational functional $\tilde{\Pi}_\pi$, written by

$$\tilde{\Pi}_\pi = \Pi_\pi - \int_{t_0}^{t_1} \sum_{i,i+1} \frac{1}{2} \int_l \left[\kappa_u (u_i - u_{i+1})^2 + \kappa_v (v_i - v_{i+1})^2 + \kappa_w (w_i - w_{i+1})^2 + \kappa_r (\partial w_i/\partial s - \partial w_{i+1}/\partial s)^2 \right] dl dt \quad (2)$$

This form is an extension of the concept of Nitsche's method [3] for multiple domains. κ_u, κ_v, κ_w and κ_r are pre-assigned weighted parameters. Non-classical boundary conditions can be easily incorporated in the present model by removing the second term from Eq.(1). Double mixed series, i.e. the Fourier series and Chebyshev orthogonal polynomials, are adopted as admissible functions for each conical shell domain. Substituting the admissible functions into Eq. (2) and setting the variation of the preceding functional $\tilde{\Pi}_\pi$ to zero, namely $\delta\tilde{\Pi}_\pi = 0$, the equations of motion of a SCS shell can be obtained as

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$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \left[\boldsymbol{K} - \bar{\boldsymbol{K}}_{\lambda} + \bar{\boldsymbol{K}}_{\kappa}\right]\boldsymbol{q} = \boldsymbol{0} \tag{3}$$

where $\boldsymbol{q}$ the is global generalized coordinate vector. $\boldsymbol{M}$ and $\boldsymbol{K}$ are, respectively, the disjoint generalized mass and stiffness matrices. $\bar{\boldsymbol{K}}_{\lambda}$ and $\bar{\boldsymbol{K}}_{\kappa}$ are the interface stiffness matrices introduced by the modified variational and least-squares weighted residual terms, respectively. By assuming a harmonic motion, $\boldsymbol{q} = \bar{\boldsymbol{q}}e^{i\omega t}$, the governing equations of motion can be written in the standard form: $[-\omega^2\boldsymbol{M} + (\boldsymbol{K} - \bar{\boldsymbol{K}}_{\lambda} + \bar{\boldsymbol{K}}_{\kappa})]\bar{\boldsymbol{q}} = 0$. A non-trivial solution is obtained by setting the determinant of the coefficient matrix equal to zero. Roots of the determinant are the square of the eigenfrequency ω.

NUMERICAL EXAMPLES AND DISCUSSIONS

A typical SCS with four step variations in thickness is examined. The geometrical dimensions of the SCS used for the following analysis are: $R_1 = 0.5$ m, $R_2 = 1$ m, $L_1 : L_2 : L_3 : L_4 = 1:1:1:1$, $h_1 : h_2 : h_3 : h_4 = 1:2:3:4$, $h_4 = 0.04$ m, $\alpha_0 = \pi/10$. The SCS is of Young's modulus $E = 2.11\times10^{11}$ Nm^{-2}, Poisson's ratio $\mu = 0.3$ and density $\rho = 7800$ kgm^{-3}. Reissner-Naghdi-Berry's thin shell theory [2] is adopted to formulate the theoretical model. In Fig.2, the frequency convergence of the present method is examined for a free-clamped SCS with its large end clamped. The frequency error is defined as follows: error $= (\omega_{nm,\text{Present}} - \omega_{nm,\text{ANSYS}})/\omega_{nm,\text{ANSYS}} \times 100\%$. Here, n represents the circumferential wave number and m is the longitudinal mode number. Fig.3 presents the frequency convergence of a free-elastic supported SCS, which is free at the small end and supported by an elastic foundation at the large edge. The foundation is considered to be circumferentially elastic only (i.e., $v \neq 0$, $u = w = \partial w/\partial s = 0$), with stiffness constant per unit length $\kappa_v = 2.5\times10^7$ N/m in circumferential direction. For simplicity, all weighted parameters are defined with the same value $\kappa = 1\times10^{14}$, and eight terms of the Chebyshev polynomials are used for each shell domain. Some selected mode shapes and frequencies obtained by means of the present method for the free-elastic supported SCS are given in Fig.4.

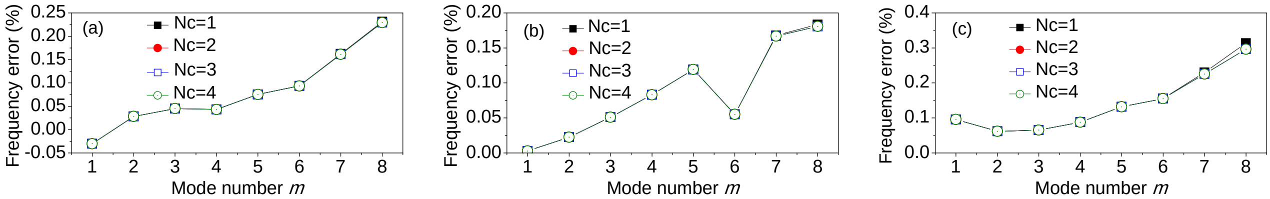

Fig.2. Discrepancies between the present results and those of ANSYS for the free-clamped SCS: (a) n=0; (b) n=1; (c) n=2

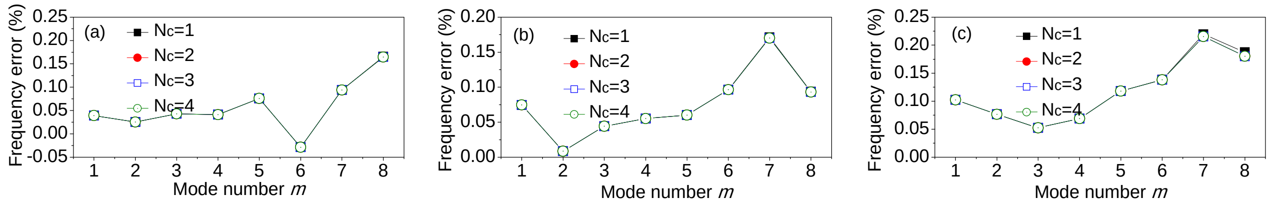

Fig.3. Discrepancies between the present results and those of ANSYS for the free-elastic supported SCS: (a) n=0; (b) n=1; (c) n=2

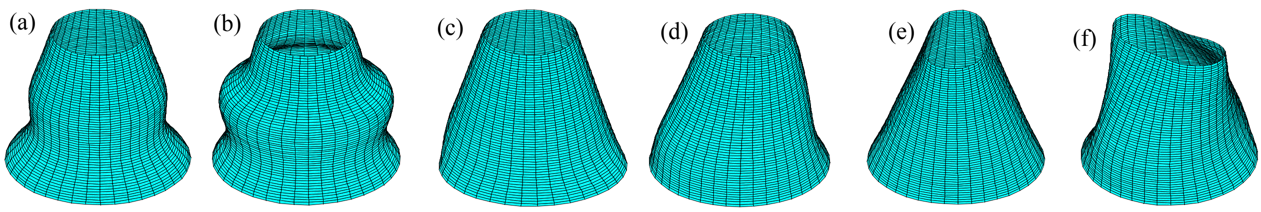

Fig.4. Mode shapes and frequencies for the free-elastic supported SCS: (a) ω_{02}=943.762Hz; (b) ω_{03}=1054.829Hz; (c) ω_{11}=299.173Hz; (d) ω_{12}=725.363Hz; (e) ω_{21}=272.512Hz; (f) ω_{22}=471.985Hz

The results converge very rapidly. The natural frequencies obtained from the present solution ($N_c = 1$) agree closely with those from ANSYS with a maximum difference of 0.31% for a total of 48 modes (see in Fig.2 and Fig.3). Using the present method, only 96 dofs ($N_c = 1$) are needed for each circumferential wave number. Furthermore, for different boundary conditions, there is no need to reformulate system matrix in the present variational approach, and only the line integrals along the geometric boundaries need to be reevaluated. This leads to reduced computational efforts.

CONCLUSIONS

A new variational approach is proposed to study the free vibration of the SCSs with classical and non-classical boundary conditions. Reissner-Naghdi-Berry's thin shell theory is adopted to formulate the theoretical model. The present results converge very rapidly, and show good agreement with those obtained from ANSYS.

References

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