NONLINEAR VIBRATIONS OF HYPERELASTIC MEMBRANES RESTING ON A NONLINEAR ELASTIC FOUNDATION

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Summary The aim of the present work is to investigate the nonlinear vibration response of a pre-stretched circular hyperelastic membrane resting on a nonlinear elastic foundation. The membrane is composed of an isotropic, homogeneous and hyperelastic material, which is modeled as a neo-Hookean incompressible material. The elastic foundation is described by a Winkler type nonlinear model with cubic nonlinearity. First the exact solution of the membrane under a uniform radial stretch is obtained. Then the equations of motion of the pre-stretched membrane resting on the nonlinear foundation are derived. From the linearized equations, the natural frequencies and mode shapes of the membrane are obtained analytically. Then the natural modes are used to approximate the nonlinear deformation field and consistent reduced order models are derived using the Galerkin method. The results for free and forced oscillations compare well with the results evaluated for the same membrane using a nonlinear finite element formulation. The results show the strong influence of the initial stretching ratio and foundation parameters on the linear and nonlinear oscillations and stability of the membrane.

INTRODUCTION

Membranes have received considerable attention in recent years due to their use in numerous engineering areas, including space applications, actuators, sensors, robotics, bioengineering, biology, aero-space industry, acoustics and civil engineering structures. See references [1, 2] for related bibliography. In many cases the membrane is resting on a deformable foundation, for example, the human skin resting on a fat layer. Also in many circumstances the membrane is subjected to large deformations and, in such cases, geometric and material nonlinearities must be taken into account [3]. Although dozens of theoretical studies have been published on membrane vibrations, very little exists in the literature on the nonlinear vibrations of hyperelastic membranes, in particular, membranes resting on a deformable foundation [1, 2]. The aim of this paper is to present a detailed parametric analysis clarifying the influence of the membrane initial stretch and foundation parameters on the nonlinear vibrations and instabilities of the membrane.

FORMULATION

Consider a circular hyperelastic membrane of undeformed radius R_o, thickness H and mass density ρ. The membrane is first uniformly stretched in radial direction, reaching a final radius R_f, and then fixed along the outer edge. Assuming the complete recoverability after deformation, the strain energy density depends only on the final strains. Thus, given an undeformed reference state, the strain energy density is characterized by the principal stretches λ_1, λ_2 and λ_3 or, alternatively, by the strain invariants I_1, I_2 and I_3 [3]. The membrane material is assumed to be homogeneous, isotropic and incompressible ($I_3=1$). There are several constitutive laws in literature particularly adapted to the representation of hyperelastic materials [1]. Considering a neo-Hookean material, the energy density function can be described as $W = C_1(I_1 - 3)$, where C_1 an experimentally obtained material parameter and I_1 is the first strain invariant of the deformation field. Consider that the stretched membrane rests on a nonlinear Winkler type foundation [4]. The foundation reaction on the membrane is given by $F = K_1 w + K_3 w^3$, where w is the transversal membrane displacement, K_1 and K_3 are, respectively the linear and nonlinear foundation stiffness. First the nonlinear equilibrium equation of the membrane in polar coordinates under a uniform finite radial stretch is solved and the static displacement field is obtained. Then the equations of motion of the pre-stretched membrane resting on the nonlinear foundation are derived. From the linearized equations, the natural frequencies and mode shapes of the membrane are obtained analytically. Then the natural modes are used to approximate the nonlinear deformation field and the Galerkin method is used to derive a set of ordinary differential equations of motion, which are solved by numerical integration. Continuation techniques are used to obtain the nonlinear frequency-amplitude relation, resonance curves and bifurcation diagrams.

RESULTS AND DICUSSION

For the numerical analysis, a circular membrane with initial external radius Ro = 1 m, internal radius R_i = 0.20 m, thickness H = 0.001m, mass density Γ = 2200 Kg/m3 and material constant C_1 = 0.17 MPa is considered. Figure 1 shows the of the lowest natural frequency (axisymmetric mode) as a function of the linear foundation stiffness K_1 for two values of the stretching ratio $\delta = R_0 / R_f$. Figure 2 illustrates the influence of the initial stretching ratio on the nonlinear frequency-amplitude relation for $K_1 = 10^5$. The membrane shows a strong hardening behaviour for low values of the stretching ratio, but the nonlinearity decreases as δ increases and the response becomes practically linear for δ>2. Figures 3.a and 3.b show the influence of the nonlinear foundation parameter K_3 on the frequency-amplitude relation. The type and degree of nonlinearity depends on the stretching value and nonlinear foundation parameter. Finally, Figure 3.c shows the resonant responses for the three cases depicted in Fig. 3.b, where multiplicity of solutions can be observed in all cases due to saddle-node bifurcations. This membrane is also analysed using the finite element software Abaqus 6.5® [5].

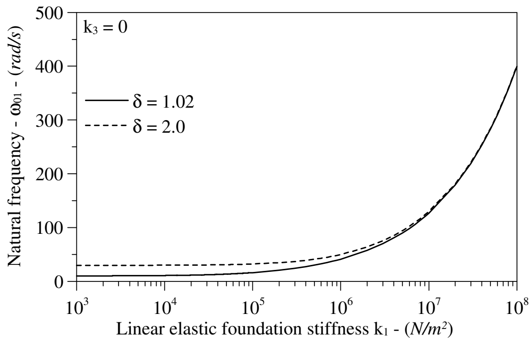

Figure 1 – Influence of the radial stretching ratio δ and linear foundation parameter on the fundamental frequency.

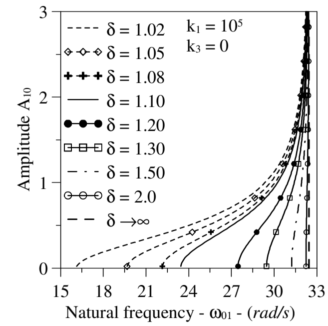

Figure 2 – Influence of the radial stretching ratio on the frequency-amplitude relation.

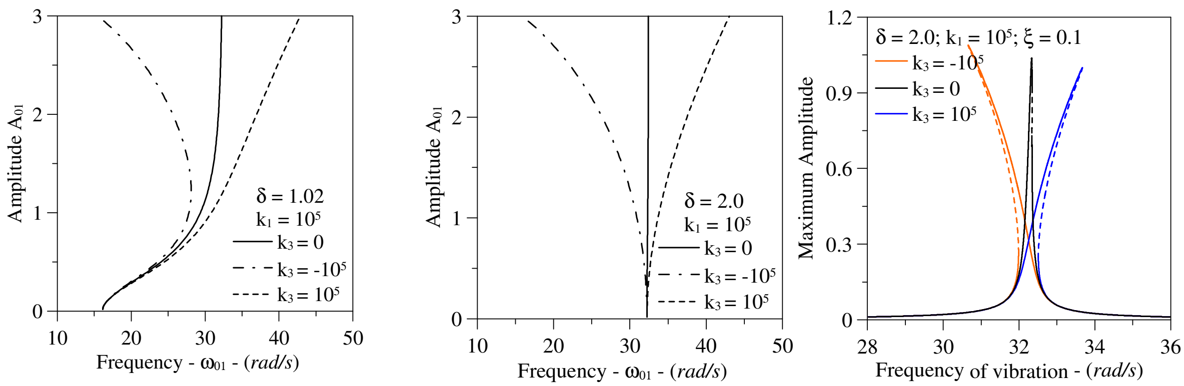

Figure 3 – Influence of the nonlinear foundation parameter K_3 on (a, b) the nonlinear frequency-amplitude relation and (c) bifurcation diagram in the resonant region.

CONCLUSIONS

The results highlight the influence of the stretching ratio and nonlinear foundation parameters on the natural frequencies, the nonlinear frequency-amplitude relation and bifurcation diagrams. It is shown that a lightly stretched membrane resting on a linear Winkler type foundation displays a highly nonlinear hardening response, that the nonlinearity decreases as the stretching ratio increases, and the response becomes practically linear for a deformed radius of twice the initial value or higher. Both the frequency-amplitude relation and the corresponding linear frequency converge to the same frequency upper bound as the radial stretch and/or vibration amplitude increases. For a highly stretched membrane, depending on the value of the nonlinear foundation stiffness, the membrane may exhibit either a hardening or softening behaviour. In the main resonance region multiplicity of solutions is observed due to both material and geometric nonlinearities. This may lead to sudden jumps in vibration amplitude and complex transients which are a function of the load control parameters and damping. The present results may lead to a better understanding of the dynamics of this class of structures and to new technological applications of thin membranes.

References

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