# PIECEWISE CONSTANT LEVEL SET APPROACH TO STRUCTURAL OPTIMIZATION OF CONTACT PROBLEMS

Andrzej Myśliński*
*Systems Research Institute, 01-447 Warsaw, Poland*

*Summary* The paper concerns the numerical structural optimization of the elastic contact problems using the level set approach. In standard PDE based level set method interfaces separating a given domain into subdomains are represented by the zero level set of the continuous level set functions. Here a piecewise constant level set approach is used to represent interfaces rather than the standard method. In the used approach the interface between subdomains is implicitly represented by the discontinuity of the set of the basis functions. In the paper the original structural optimization problem is approximated by a two-phase shape optimization problem. Different phases in a whole design domain are represented by a one piecewise constant level set function only. Using this approach the original problem is reformulated as equivalent constrained optimization problem with respect to this level set function. This problem is solved numerically using the projected Lagrangian method.

## STRUCTURAL OPTIMIZATION PROBLEM

Consider deformations of an elastic body occupying two – dimensional domain $\Omega$ with the smooth boundary $\Gamma$ [2, 3]. Assume $\Omega \subset D$ where $D$ is a bounded smooth hold – all subset of $R^2$. The body is subject to body forces $f(x) = (f_1(x), f_2(x))$, $x \in \Omega$. Moreover, surface tractions $p(x) = (p_1(x), p_2(x))$, $x \in \Gamma$, are applied to a portion $\Gamma_1$ of the boundary $\Gamma$. We assume, that the body is clamped along the portion $\Gamma_0$ of the boundary $\Gamma$, and that the contact conditions are prescribed on the portion $\Gamma_2$, where $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, $i, j = 0, 1, 2$, $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$. .We denote by $u = (u_1, u_2)$, $u = u(x)$, $x \in \Omega$, the displacement of the body and by $\sigma(x) = \{\sigma_{ij}(u(x))\}$, $i, j = 1, 2$, the stress field in the body. Consider elastic bodies obeying Hooke's law, i.e., for $x \in \Omega$ and $i, j, k, l = 1, 2$

$$\sigma_{ij}(u(x)) = a_{ijkl}(x)e_{kl}(u(x)) \;\text{ where }\; e_{kl}(u(x)) = \frac{1}{2}(u_{k,l}(x) + u_{l,k}(x)), \tag{1}$$

and $u_{k,l}(x) = \frac{\partial u_k(x)}{\partial x_l}$. We use here and throughout the paper the summation convention over repeated indices. The stress field $\sigma$ satisfies the system of equations

$$-\sigma_{ij}(x)_{,j} = f_i(x) \quad x \in \Omega, i, j = 1, 2, \tag{2}$$

where $\sigma_{ij}(x)_{,j} = \frac{\partial \sigma_{ij}(x)}{\partial x_j}$, $i, j = 1, 2$. The following boundary conditions are imposed

$$u_i(x) = 0 \;\text{ on }\; \Gamma_0, \;\; i = 1, 2, \;\text{ and }\; \sigma_{ij}(x)n_j = p_i \;\text{ on }\; \Gamma_1, \;\; i, j = 1, 2, \tag{3}$$

$$u_N \leq 0, \;\; \sigma_N \leq 0, \;\; u_N\sigma_N = 0 \;\text{ and }\; |\, \sigma_T \,| \leq 1, \;\; u_T\sigma_T + |\, u_T \,| = 0 \;\text{ on}\, \Gamma_2, \tag{4}$$

where $n = (n_1, n_2)$ is the unit outward versor to the boundary $\Gamma$. Let us formulate the contact problem (2) - (4) in the variational form. Denote by $V_{sp}$ and $K$ the space and set of kinematically admissible displacements:

$$V_{sp} = \{z \in [H^1(\Omega)]^2 \;:\; z_i = 0 \text{ on } \Gamma_0, \; i = 1, 2\}, \;\; K = \{z \in V_{sp} \;:\; z_N \leq 0 \;\text{ on }\; \Gamma_2\}. \tag{5}$$

$H^1(\Omega)$ denotes Sobolev space of square integrable functions and their first derivatives. $[H^1(\Omega)]^2 = H^1(\Omega) \times H^1(\Omega)$. Denote also by $\Lambda$ the set $\Lambda = \{\zeta \in L^2(\Gamma_2) \;: |\, \zeta \,| \leq 1\}$. Variational formulation of problem (2) - (5) has the form: *find a pair* $(u, \lambda) \in K \times \Lambda$ *satisfying for* $i, j, k, l = 1, 2$

$$\int_\Omega a_{ijkl}e_{ij}(u)e_{kl}(\varphi - u)dx - \int_\Omega f_i(\varphi_i - u_i)dx - \int_{\Gamma_1} p_i(\varphi_i - u_i)ds + \int_{\Gamma_2} \lambda(\varphi_T - u_T)ds \geq 0 \;\; \forall\varphi \in K, \tag{6}$$

$$\int_{\Gamma_2} (\zeta - \lambda)u_T ds \leq 0 \quad \forall\zeta \in \Lambda. \tag{7}$$

Let us introduce the set $U_{ad}$ of admissible domains. This set has the form: $U_{ad} = \{\Omega \;:\; E \subset \Omega \subset D \subset R^2 \;:\; \Omega \text{ is suitable regular}, \;\; Vol(\Omega) - Vol^{giv} \leq 0, \;\; P_D(\Omega) \leq const_1\}$ where $Vol(\Omega) = \int_\Omega dx$ and $P_D(\Omega) = \int_\Gamma dx$. The subsets $E$ and $D$ as well as constants $Vol^{giv}, const_1 > 0$ are given. The set $U_{ad}$ is assumed to be nonempty. The cost functional approximating the normal contact stress on the contact boundary has the form

$$J_\eta(u(\Omega)) = \int_{\Gamma_2} \sigma_N(u)\eta_N(x)ds. \tag{8}$$

The auxiliary given bounded function $\eta(x) \in M^{st} = \{\eta = (\eta_1, \eta_2) \in [H^1(D)]^2 : \eta_i \leq 0 \;\text{ on}\, D, \; i = 1, 2, \;\; \| \eta \|_{[H^1(D)]^2} \leq \; 1\}$. Consider the following structural optimization problem: *for a given function* $\eta \in M^{st}$*, find a domain* $\Omega^\star \in U_{ad}$ *such that*

$$J_\eta(u(\Omega^\star)) = \min_{\Omega \in U_{ad}} J_\eta(u(\Omega)). \tag{9}$$

a) Corresponding author. E-mail: myslinsk@ibspan.waw.pl

## PIECEWISE CONSTANT LEVEL SET APPROACH

Assume $D$ is partitioned into $N$ subdomains $\{\Omega_i\}_{i=1}^N$ such that $D = \bigcup_{i=1}^N (\Omega_i \cup \partial\Omega_i)$ where $N$ is a given integer and $\partial\Omega_i$ denotes the boundary of the subdomain $\Omega_i$ [1, 4, 5]. Define function $\phi : D \to R$ such that $\phi = i$ in $\Omega_i$, $\quad i = 1, 2, ..., N$. This function is used to identify all the phases in $D$. In order to guarantee that there is no vacuum or overlap between different subdomains assume function $\phi$ satisfies the constraint $K(\phi) = 0$ where $K(\phi) \stackrel{def}{=} (\phi - 1)(\phi - 2)...(\phi - N) = \prod_{i=1}^N (\phi - i)$. Using this approach the characteristic function $\chi_i$, $i = 1, 2, ..., N$, of the subdomain $\Omega_i$ is represented as $\chi_i = \frac{1}{\alpha_i} \prod_{j=1, j\neq i}^N (\phi - j)$ and $\alpha_i = \prod_{k=1, k\neq i}^N (i - k)$. Any piecewise constant density function $\rho : D \to R^2$ defined as $\rho(x) = \epsilon$ if $x \in D \setminus \bar{\Omega}$ and $\rho(x) = 1$ if $x \in \Omega$ where $\epsilon > 0$ is a small constant, can be constructed as a weighted sum of the characteristic functions $\chi_i$. Denoting by $\{\rho_i\}_{i=1}^N$ a set of real scalars, we can represent a piecewise constant function $\rho$ taking these $N$ distinct constant values by $\rho(x) = \sum_{i=1}^N \rho_i \chi_i(\phi(x))$. We consider two - phase model problem, i.e., $N = 2$. Therefore $\chi_1(x) = 2 - \phi(x)$ and $\chi_2(x) = \phi(x) - 1$, $\rho(x) = \rho_1\chi_1(x) + \rho_2\chi_2(x) = (1 - \epsilon)\phi(x) + 2\epsilon - 1$ [5]. Moreover function $K(\phi)$ takes the form $K(\phi) = (\phi - 1)(\phi - 2)$. The structural optimization problem (9) takes the form: find $\phi \in U_{ad}^\phi$ such that:

$$\min_{\phi \in U_{ad}^\phi} J_\eta(\phi) = \int_{\Gamma_2} \rho(\phi)\sigma_N(u_\epsilon)\eta_N ds, \tag{10}$$

where the admissible set $U_{ad}^\phi = \{\phi : Vol(\phi) = \int_\Omega \rho(\phi)dx - Vol^{giv} \leq 0, \quad K(\phi) = (\phi - 1)(\phi - 2) = 0, \quad P_D(\phi) = \int_\Omega \mid \nabla\phi \mid dx \leq const_1\}$ as well as $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$ satisfies the state system in the domain $D$:

$$\int_D \rho(\phi) a_{ijkl} e_{ij}(u_\epsilon) e_{kl}(\varphi - u_\epsilon) dx - \int_D \rho(\phi) f_i(\varphi_i - u_{\epsilon i}) dx -$$

$$\int_{\Gamma_1} p_i(\varphi_i - u_{\epsilon i}) ds + \int_{\Gamma_2} \lambda_\epsilon(\varphi_T - u_{\epsilon T}) ds \geq 0 \quad \forall \varphi \in K, \quad i, j, k, l = 1, 2, \tag{11}$$

$$\int_{\Gamma_2} (\zeta - \lambda_\epsilon) u_{\epsilon T} ds \leq 0 \quad \forall \zeta \in \Lambda. \tag{12}$$

## NUMERICAL RESULTS

The necessary optimality condition for the problem (10) is formulated in [3]. Augmented Lagrangian algorithm with projection is used to solve numerically the discretized optimization problem (10). The obtained optimal domain is displayed in Fig. 1. The areas with low values of density function appear in the central part of the body and near the fixed edges. The obtained normal contact stress is almost constant along the optimal shape boundary and has been significantly reduced comparing to the initial one.

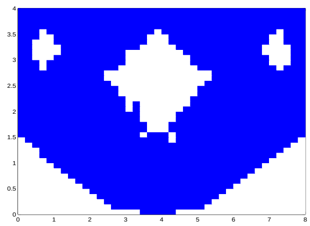

**Figure 1.** Optimal domain $\Omega^\star$.

## CONCLUSIONS

The structural optimization problem for elastic contact problem with the prescribed friction is solved numerically in the paper. The obtained results indicate that the proposed numerical algorithm allows for significant improvements of the structure from one iteration to the next and is more efficient than the algorithms based on standard level set approach.

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