A TOPOLOGY OPTIMIZATION PROCEDURE BASED ON A PHASE FIELD APPROACH AND AN ACTIVE SET STRATEGY.

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Summary We address the topology optimization using a phase-field approach. The presented formulation naturally includes a length scale which regularizes the problem and allows the perimeter of the optimal structure to be controlled. To solve the problem we consider both the mass-conserving Cahn-Hilliard approach as well as the Allen-Cahn model augmented with a mass constraint. The boundary value problems associated with both the mechanical balance and the phase field is solved using the finite element method. A double-well potential is used for penalizing intermediate densities. Moreover, we also consider a barrier augmented double-well function for strictly removing the possibility of non realistic densities $\rho \notin [0\ 1]$. The presentation is closed by numerical examples that illustrates features of the different approaches.

INTRODUCTION

The Simple Isotropic Material Penalization (SIMP) problem is well known to lack an intrinsic length scale which is manifested by a severe mesh-dependence when solved using the finite element method. Many suggestions to overcome this problem are available in the literature cf. e.g. the filtering procedure by Bruns and Tortorelli [1] and the perimeter control method by Peterson [2]. Another approach to introduce a length scale is via a phase-field approach where a cost for the perimeter of the structure, or gradient of the density function, is added, cf. e.g. Burger and Stainko [3]. In the present work we investigate the possibilities of using phase-field approaches in topology optimization. Phase field approaches have previously successfully been applied to design dependent loads (cf. Bourdin and Chambolle [4]) and to multimaterial (cf. Wang and Zhou [5]).

FORMULATION

We seek the optimal material distribution $\rho \in [0\ 1]$ that maximizes the stiffness within the domain Ω. For convenience, we use the equivalent formulation of minimizing the compliance instead of maximizing the stiffness. The amount of material available for the design is limited to M, i.e. $\int_\Omega \rho dV = M$ and we seek a distinct solution, i.e. a design $\rho \in \{0\ 1\}$ is sought. However, since the integer problem numerically is difficult we regularize the problem such that intermediate densities are allowed. Diffuse designs are penalized using a penalty function $F(\rho)$ which has the property $F(0) = F(1) = 0$ and $F > 0$ elsewhere. One possibility for the penalty function is the 'double-well' function $F(\rho) = F_0(\rho) = \rho^2(1-\rho^2)$. This choice will, however, not exclude the possibility of densities $\rho \notin [0\ 1]$ and therefore we also consider an alternative penalty function F that is augmented such that $F = F_0 + I$ where I is the indicator function with the properties $I(\rho) = 0$ for $\rho \in [0\ 1]$ and $I(\rho) = \infty$ for $\rho \notin [0\ 1]$.
Since the problem of minimizing the compliance, together with penalization of diffuse designs in combination with the finite element method leads to a mesh dependent solution, the problem is augumented such that a length scale is included in the formulation. The length-scale is introduced via adding a cost for gradients of the density distribution. It should be noted that the gradient can be related to the perimeter of the design. The functional, $\mathcal{L}$, that is subject to minimization is defined by

$$\mathcal{L}(\rho, \boldsymbol{u}) = \frac{1}{\epsilon}\int_\Omega F(\rho) dV + \frac{\epsilon}{2}\int_\Omega |\boldsymbol{\nabla}\rho|^2 dV + \eta C \tag{1}$$

where $\boldsymbol{u}$ represents the displacement field and enters the functional via the the compliance C which can be identified as two times the elastic stored energy, i.e.

$$C = 2\int_\Omega w dV, \quad w(\boldsymbol{u}, \rho) = \frac{1}{2}\boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{D}(\rho) : \boldsymbol{\epsilon}(\boldsymbol{u}) \tag{2}$$

where $\boldsymbol{\epsilon}$ denotes the strain defined as $\boldsymbol{\epsilon}[\cdot] = (\boldsymbol{\nabla}[\cdot] + \boldsymbol{\nabla}^s[\cdot])/2$. In contrast to the SIMP approach where the penalization of intermediate densities is performed via a non-linear scaling, $g(\rho)$, in the stiffness relation $\boldsymbol{D}(\rho) = g(\rho)\boldsymbol{D}_0$ we make use of a linear scaling and a threshold for small densities to avoid singular stiffness for zero density. The parameters ϵ and η present in (1) determine the influence of the compliance and the influence of the perimeter on the optimal design. For fixed η, decreasing ϵ results in decreasing cost for the perimeter, thus leading to a finer structure. Moreover, in addition to the effect of increased perimeter, decreasing ϵ also implies increased penalization of diffuse density $\rho \in]0\ 1[$ which leads to the width of the interface between material is being decreased. Evidently the mass constraint, equilibrium and boundary conditions must be respected when minimizing the functional $\mathcal{L}$. In the present approach we include these constraints via Lagrangian multipliers and it turns out that the Lagrangian multiplier enforcing equilibrium can be identified in terms of the displacement field.

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Two evolution laws for the density field will be considered. The first approach is based on the Cahn-Hilliard (CH) model and is given by

$$\dot{\rho} = \boldsymbol{\nabla} \cdot \boldsymbol{j}, \quad \boldsymbol{j} = -m\boldsymbol{\nabla}\mu$$

where μ is the functional derivative of $\mathcal{L}$ with respect to the density, i.e. $\mu = \frac{\delta \mathcal{L}}{\delta \rho}$. The numerical paramameter, m, is here taken as constant. The evolution law above along with proper boundary conditions has the advantage that it inherently preserves the total mass, i.e. the mass of the final, optimal, design is given by the initial guess. However, it should be noted that the method is computationally expensive since it is a fourth order PDE. Another approach to obtain a minimum to (1) is to adopt the Allen-Cahn (AC) approach. In the AC model the evolution of the density, ρ, is given by

$$\dot{\rho} = \xi\mu$$

where ξ is a numerical parameter. The AC model results in a second order PDE and it is therefore computationally more efficient than the Cahn-Hilliard model. However, in contrast to the CH model the AC model is not preserving the mass, and in the present approach the mass constraint is handled separately.

SOLUTION PROCEDURE

Several strategies for solving the problem stated above are available. The features of the volume preserving Cahn-Hilliard was previously demonstrated by e.g. Wallin et al.[6]. In the present study we consider both the Cahn-Hilliard approach and the Allen-Cahn approach. The Allen-Cahn model will be used together with the barrier function with excludes densities $\rho \notin [0\ 1]$. To solve this potentially more complex problem where barrier functions are included in the penalty for diffuse designs we will make use of the active set strategy proposed by Blank et al. [7]. In this approach the computational domain is split into regions defined as interface and regions which is either void or fully material. The strategy can easily be implemented and once the set of interface nodes is determined the evolution law (in each time step) is reduced to a boundary value problem only including the interface.

A Galerkin based finite element formulation is used for solving the elastic boundary value problem and the PDE associated with the phase field problem. The computational domain is discretisized using three node triangular element. The interface between void and full material needs to be resolved properly which typically implies that the interface between void and full material should be resolved by 4 elements. To avoid resolving the entire computational domain using the same element size, we use an adaptive strategy. The adaptivity is governed by an error estimate derived by Feng and Wu [8].

The proposed algorithms will be applied to boundary value problems where the properties of the solution procedures are evaluated. A fully implicit integration scheme as well as semi-implicit integration scheme will be considered. The efficiency of both integration schemes will be evaluated and compared. Moreover, in the numerical example the influence of the costs for creating surfaces will be investigated. It will be shown that the parameter, ϵ, can be used for controlling the total perimeter of the structure.

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