TOPOLOGY OPTIMIZATION OF HYPERELASTIC BODIES BY MAXIMIZING THE EQUILIBRIUM POTENTIAL ENERGY

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Summary In this work topology optimization of hyperelastic bodies is performed by maximizing the equilibrium potential energy. Seven different hyperelastic potentials are considered and the results are compared to results obtained by optimizing the compliance, although these two objectives are no longer equivalent in this non-linear case, contrary to the case of linear elasticity. A loading consisting of both external forces and prescribed non-zero displacements are considered. Maximizing the equilibrium potential energy is a suitable objective when treating these two types of loads simultaneously, while, on the contrary, compliance optimization becomes a mixture of minimization and maximization. Furthermore, the sensitivity analysis of the equilibrium potential energy is straight-forward and no extra adjoint equation is needed. Numerical results show that, even in the case of moderate displacements, hyperelasticity and linear elasticity generate different optimal topologies.

STATE PROBLEM

Consider a hyperelastic body following [1]. Position vectors of material points in the reference configuration are denoted $\boldsymbol{X} = X_i \boldsymbol{e}_i$, where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is an orthonormal base. When the body deforms these points are mapped to spatial positions $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}) = x_i \boldsymbol{e}_i$. We use a SIMP-interpolation in order to achieve the design parametrization, and the total elastic energy is written as

$$V = V(\rho_e, E_{ij}) = \sum_{e=1}^{n_{\text{el}}} \rho_e^n \int_{\Omega_e} \Psi(E_{ij}) \, \mathrm{d}X_1 X_2 X_3, \tag{1}$$

where n_{el} is the number of finite elements, ρ_e is the design (density) parameter for each element e, n is the SIMP-factor, Ω_e is the reference configuration of element e and $\Psi = \Psi(E_{ij})$ is the hyperelastic strain energy function, which is a function of the Green-Lagrange strain tensor E_{ij}.
Seven different hyperelastic strain energies, all coinciding at infinitesimal strain, are studied. An example of such a function is

$$\Psi^6 = \lambda(J - \ln J - 1) + \frac{\mu}{2}(C_{ii} - 3) - \mu \ln J, \tag{2}$$

where $J = \det(F_{ij})$ is the determinant of the deformation gradient F_{ij}, $C_{ij} = 2E_{ij} + \delta_{ij}$ is the right Cauchy-Green deformation tensor, and λ and μ are Lame's material parameters. The Kirchhoff-St.Venant potential as well as five additional hyperelastic potentials and their corresponding second Piola-Kirchhoff stresses are investigated in [1].
For a fixed density distribution vector $\boldsymbol{\rho}$, our state problem is defined by minimizing the potential energy

$$\Pi = \Pi(\boldsymbol{d}, \boldsymbol{\rho}) = V\left(\boldsymbol{\rho}, E_{ij}(\boldsymbol{d})\right) - \boldsymbol{F}^T \boldsymbol{d} \tag{3}$$

with respect to the nodal displacement vector $\boldsymbol{d}$ under satisfaction of the kinematic constraint imposed by any prescribed displacement δ, i.e.

$$\boxed{\begin{cases} \min\limits_{\boldsymbol{d}} \Pi(\boldsymbol{d}, \boldsymbol{\rho}) \\ \text{s.t. } \delta - \boldsymbol{e}^T \boldsymbol{d} = 0 \end{cases}} \tag{4}$$

where $\boldsymbol{e}$ is a unit vector representing the direction of the prescribed displacement δ, and $\boldsymbol{F}$ represents the external forces. The corresponding Lagrangian function is

$$\mathcal{L} = \mathcal{L}(\boldsymbol{\rho}, \boldsymbol{d}, \lambda) = V\left(\boldsymbol{\rho}, E_{ij}(\boldsymbol{d})\right) - \boldsymbol{F}^T \boldsymbol{d} + \lambda(\delta - \boldsymbol{e}^T \boldsymbol{d}), \tag{5}$$

when $\boldsymbol{\rho}$ is taken to be constant. Here, λ is a Lagrange multiplier, which can be identified as the reaction force required to satisfy $\delta - \boldsymbol{e}^T \boldsymbol{d} = 0$.

OPTIMIZATION PROBLEMS

Two different objectives are studied in this work. Letting $\boldsymbol{d}$=$\boldsymbol{d}(\boldsymbol{\rho})$ and λ=$\lambda(\boldsymbol{\rho})$ be defined implicitly by (4), we maximize the equilibrium potential energy, i.e., $\mathcal{L}(\boldsymbol{\rho}, \boldsymbol{d}(\boldsymbol{\rho}), \lambda(\boldsymbol{\rho})) = \Pi(\boldsymbol{d}(\boldsymbol{\rho}), \boldsymbol{\rho})$, and we minimize/maximize the compliance of the external forces and the prescribed displacement. Thus, our two objectives read

$$c^1 = c^1(\boldsymbol{\rho}) = -\mathcal{L}(\boldsymbol{\rho}, \boldsymbol{d}(\boldsymbol{\rho}), \lambda(\boldsymbol{\rho})), \quad c^2 = c^2(\boldsymbol{\rho}) = \frac{1}{2}\boldsymbol{F}^T \boldsymbol{d}(\boldsymbol{\rho}) - \frac{1}{2}\lambda(\boldsymbol{\rho})\delta. \tag{6}$$

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In the case of linear elasticity these two objectives are equivalent. However, this is not true when (hyperelastic) non-linear elasticity is considered.
In summary, we consider the following nested optimization problems:

$$\boxed{\begin{cases} \min\limits_{\boldsymbol{\rho}} c^i(\boldsymbol{\rho}) \\ \text{s.t.} \begin{cases} V_{\text{vol}}(\boldsymbol{\rho}) \leq \hat{V}_{\text{vol}} \\ \boldsymbol{\epsilon} \leq \boldsymbol{\rho} \leq \mathbf{1} \end{cases} \end{cases}} \tag{7}$$

Here, we have introduced

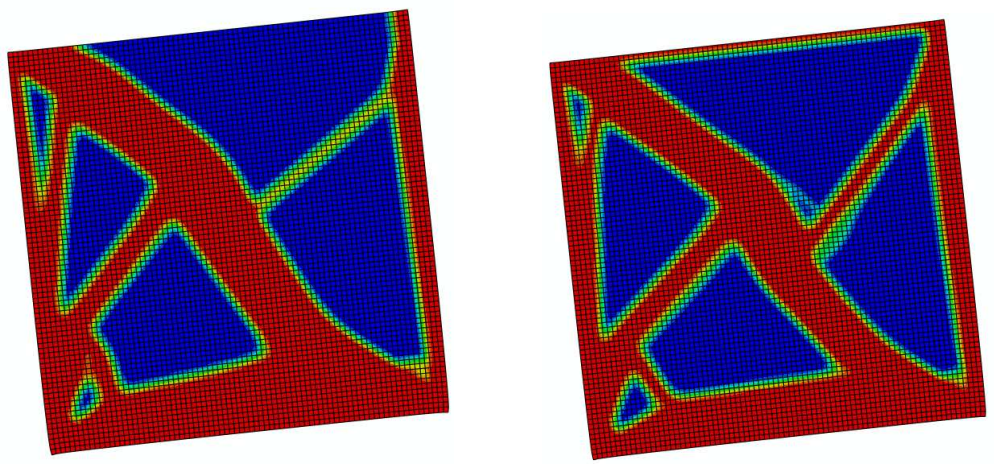
$$V_{\text{vol}}(\boldsymbol{\rho}) = \sum_{e=1}^{n_{\text{el}}} V_e \rho_e, \qquad V_e = \int_{\Omega_e} \mathrm{d}X_1 \mathrm{d}X_2 \mathrm{d}X_3, \tag{8}$$

which we constrain by $\hat{V}_{\text{vol}}$. Furthermore, singular design domains are prevented by representing zero element densities by $\boldsymbol{\epsilon}=\{\epsilon,\ldots,\epsilon\}^T$, where $\epsilon > 0$ is a small number, and $\mathbf{1}=\{1,\ldots,1\}^T$.
A nice feature of the first objective in (6) is that the sensitivities

$$s_e^1 = \frac{\partial c^1}{\partial \rho_e} = \frac{\partial \mathcal{L}}{\partial \rho_e} + \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{d}}\right)^T \frac{\partial \boldsymbol{d}}{\partial \rho_e} + \frac{\partial \mathcal{L}}{\partial \lambda}\frac{\partial \lambda}{\partial \rho_e} \tag{9}$$

are easily evaluated without introducing any extra adjoint equation. The optimality conditions of (4) imply that the two latter terms of (9) disappear and we obtain

$$s_e^1 = \frac{\partial \mathcal{L}}{\partial \rho_e} = \frac{\partial V}{\partial \rho_e}. \tag{10}$$



(a) Linear elasticity (b) Hyperelasticity

Figure 1. *An example with moderate displacements generating different optimal topologies for linear elasticity and hyperelasticity, respectively. Results from [1].*

CONCLUSIONS

Topology optimization of hyperelastic bodies is performed by maximizing the equilibrium potential energy. Numerical results show that even for moderate displacements linear elasticity and hyperelasticity might generate different optimal topologies, see Figure 1. It is also demonstrated that maximizing the equilibrium potential energy might generate a different solution than obtained by optimizing the compliance. Finally, one concludes that the Kirchhoff-St.Venant model generally has a significantly worse performance in topology optimization than models based on the other six hyperelastic potentials considered in this work.

References

[1] Klarbring A., Strömberg N., Topology Optimization of Hyperelastic Bodies including Non-Zero Prescribed Displacements, in review, 2012.