FREE MATERIAL DESIGN WITH MULTIPLE LOAD CASES

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Summary Multiple load cases and the consideration of strength is a reality that most structural designs are exposed to. Improved possibility to produce specific materials, say by fiber lay-up, put focus on research on free material optimization. A formulation for such design problems together with a practical recursive design procedure is presented and illustrated with examples. The presented finite element analysis involve many elements as well as many load cases. Separating the local amount of material from a description with unit trace for the local anisotropy, gives the free materials formulation a more physical interpretation of the material constraint.

Optimal design of continua may include design of anisotropic material and free material is the ultimate case with total freedom. From a practical point of view strength consideration under multiple load cases should be accounted for, and in general pure compliance design is not enough. The simple analytical solution proved in [1] is in the present work applied directly in a practical approach that is closely related to "fully stressed iterations". Separating the local amount of material from a description with unit trace for the local anisotropy, gives free material optimization (FMO) a more physical resource interpretation.

The total approach is described shortly as follows: A given design (to be redesigned), subjected to a number of design independent load cases, is with FE analyzed and gives for each load case the strain state and the state of elastic energy densities. For each element, the maximum energy density is determined with the relative value for the individual load cases. An approach to redesign towards uniform maximum energy density by changing the non-dimensional volume densities ρ_e is applied. This approach has earlier been applied with confidence to rather different problems. The non-dimensional anisotropy of the element material constitutive matrices is also redesigned, but this redesign is obtained directly from the analytical solution in addition to the redesigned material densities ρ_e. The chosen formulation for this part of the material design is given below. The approach is also practical for given material without FMO and examples of strength design for multiple loads with isotropic material are shown, but here only FMO results are exemplified. The suggested approach and formulation is found practical and reliable, although it is heuristic in general. Design histories in terms of maximum elastic energy density (and for isotropic cases maximum von Mises stress) are presented together with total elastic energy (compliance), but not shown in this note.

REDESIGN OF ELEMENT CONSTITUTIVE MATRICES

In free material optimization (FMO) for a 2D problem, the non-dimensional optimal anisotropic constitutive matrix $[\alpha]$ is derived and presented in [1], and by an assumed strain state with principal strains ϵ_1, ϵ_2 given by

$$[\alpha]_{\epsilon_1,\epsilon_2} = \frac{1}{\epsilon_1^2+\epsilon_2^2}\begin{bmatrix} \epsilon_1^2 & \epsilon_1\epsilon_2 & 0 \\ \epsilon_1\epsilon_2 & \epsilon_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1}$$

The size of this $[\alpha]$ matrix is measured either by the trace norm $T = ||[\alpha]||_T$ or by the Frobenius norm $F = ||[\alpha]||_F$, both equal to one $T = 1$ and $F = 1$. However, the matrix (1) describes the anisotropy in the principal strain directions of ϵ_1, ϵ_2. To apply the matrix in a global x, y coordinate system, a rotational transformation based on an angle θ is needed. This angle is here defined from the direction of ϵ_1 to the x-direction. The result of such a transformation follow from classical laminate theory, and in a notation with invariant trace and Frobenius norms it is denoted

$$[\alpha]_{\epsilon_1,\epsilon_2} \quad\Rightarrow\quad [\alpha]_{x,y} = \begin{bmatrix} \alpha_{1111} & \alpha_{1122} & \sqrt{2}\alpha_{1112} \\ \alpha_{1122} & \alpha_{2222} & \sqrt{2}\alpha_{2212} \\ \sqrt{2}\alpha_{1112} & \sqrt{2}\alpha_{2212} & 2\alpha_{1212} \end{bmatrix} \tag{2}$$

When two ratios ζ_1, ζ_2 are defined, $\zeta_1 = (\epsilon_1 - \epsilon_2)^2/(\epsilon_1^2 + \epsilon_2^2)$ and $\zeta_2 = (\epsilon_1^2 - \epsilon_2^2/(\epsilon_1^2 + \epsilon_2^2)$, then the constitutive transformation in a simplified notation is

$$\begin{aligned} \alpha_{1111} &= \tfrac{1}{2} + \tfrac{1}{2}\zeta_2\cos 2\theta - \tfrac{1}{8}\zeta_1(1-\cos 4\theta), & \alpha_{1122} &= \tfrac{1}{2}(1-\zeta_1) + \tfrac{1}{8}\zeta_1(1-\cos 4\theta) \\ \alpha_{2222} &= \tfrac{1}{2} - \tfrac{1}{2}\zeta_2\cos 2\theta - \tfrac{1}{8}\zeta_1(1-\cos 4\theta), & \alpha_{1112} &= \tfrac{1}{4}\zeta_2\sin 2\theta + \tfrac{1}{8}\zeta_1\sin 4\theta \\ \alpha_{1212} &= \tfrac{1}{8}\zeta_1(1-\cos 4\theta), & \alpha_{2212} &= \tfrac{1}{4}\zeta_2\sin 2\theta - \tfrac{1}{8}\zeta_1\sin 4\theta \end{aligned} \tag{3}$$

The trace T is seen to be independent of the rotation θ and $T = \alpha_{1111} + \alpha_{2222} + 2\alpha_{1212} = 1$. The Frobenius norm is also independent of the rotation and $F^2 = \alpha_{1111}^2 + \alpha_{2222}^2 + 4\alpha_{1212}^2 + 2\alpha_{1122}^2 + 4\alpha_{1112}^2 + 4\alpha_{2212}^2 = 1$.

With multiple load cases, several principle strain cases need to be accounted for in each element (point). A suggested model for an element stiffness matrix $[\widehat{C}_e]$ in the global x,y coordinate system is

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$$[\widetilde{C}_e] = E_{iso}[\alpha]_{iso} + E_{ani}(\rho_e \sum_l (\eta_l [\alpha]_l)_e) \tag{4}$$

with all $([\alpha]_l)_e$ matrices referring to the global x, y coordinate system. The isotropic constitutive part may model a basic isotropic material (say epoxy), assumed to be given and satisfying $E_{iso} << E_{ani}$ for the two modulus of elasticity. The isotropic small term may also be artificial and similar to a minimum thickness (density) in size optimization (topology optimization). Thus, E_{iso} may describe an involved basic material or just a practical manner to avoid holes or materials with zero eigenvalues (semi-definite). The factor ρ_e is the non-dimensional material density and the weight factors η_l satisfy $\sum_l (\eta_l)_e = 1$. The change in the the constitutive matrices is limited by a non-dimensional parameter $0 \leq \beta \leq 1$, i.e.,$[C_e]_{new} = \beta[\widetilde{C}_e] + (1 - \beta)[C_e]_{old}$. The value of β applied in the examples is $\beta = 0.5$ (in one case $\beta = 0.1$ is used in 5 additional iterations). With $[\widetilde{C}_e]$ determined as in (4) and $[\alpha]_{iso}$ being positive definite all constitutive matrices are known to have positive eigenvalues, even with only one load case.

EXAMPLE OF STRENGTH DESIGN FOR MULTIPLE LOADS WITH FREE MATERIAL

In Figure 1 a 2D example with 8 load cases illustrate the optimal design to obtain uniform maximum energy density. Different total material and different stiffness interpolations are investigated. Only specific final designs are shown here with focus on density distributions and on directions of largest material stiffness, including information on the level of anisotropy. At the loaded and supported domains, Figure 2 shows relative low level of anisotropy, while in most other domains the level of anisotropy is larger and rather uniform.

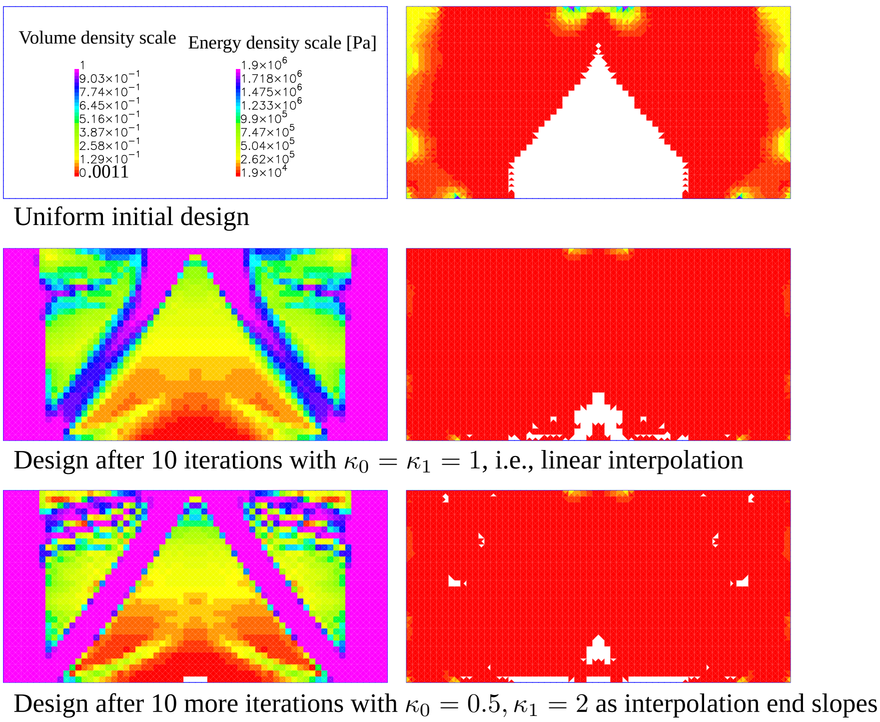

Figure 1. Initial and optimized design distributions in left column and their response in terms of distributions of elastic energy densities in right column. Mean volume (material) density for this example including free material design (FMO) is $\rho_{mean} = 0.6$.

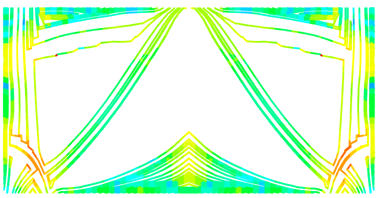

Figure 2. Directions of largest longitudinal stiffness, added color scale to illustrate the level of anisotropy.

References

[1] Bendsøe, M. P., Guedes, J. M., Haber, R. B., Pedersen, P., and Taylor, J. E.: An analytical model to predict optimal material properties in the context of optimal structural design. *J. Applied Mechanics 61*:930–937,1994