# APPLICATION OF GJ-INTEGRAL TO SHAPE OPTIMIZATION PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS/SYSTEM WITH MIXED BOUNDARY CONDITIONS

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*Summary* GJ-integral is a generalization of J-integral in fracture mechanics proposed by the author in 1981 to express 3d crack extension, which is applicable to shape sensitivity analysis of partial differential equation/system. In this paper, a brief summary of GJ-integral is written; next, it is written how to apply GJ-ingegral to shape optimization for boundary value problems with Dirichlet-Neumann mix boundary condition; last, its numerical results are given.

## BRIEF SURVEY OF GJ-INTEGRAL

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ $(d \geq 2)$ and $L^p(\Omega, \mathbb{R}^m)$ Lebesgue space of all measurable functions $v : \Omega \to \mathbb{R}^m$ (a real number $1 < p < \infty$ and the integer $m \geq 1$) with the norm $\|v\|_{p,\Omega}$. Consider the problem $\mathrm{P}(f, V(\Omega, \Gamma_D))$: For a given function $f \in L^{p'}(\Omega, \mathbb{R}^m), p' = p/(p-1)$, find $u$ minimizing the following functional

$$\mathcal{E}(\boldsymbol{v}; f, \Omega) = \int_\Omega \left\{ \widehat{W}(x, v, \nabla v) - f \cdot v \right\} dx$$

over the space $V(\Omega, \Gamma_D) = \left\{ v \in W^{1,p}(\Omega, \mathbb{R}^m) : v = 0 \text{ on } \Gamma_D \right\}$, where $\Gamma_D$ stands for the part of $\partial\Omega$ and a scalar function $\widehat{W}(\xi, z, \zeta) : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times m} \to \mathbb{R}$ is in $C^1\left(\mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times m}\right)$ and $W^{1,p}(\Omega, \mathbb{R}^m)$ denote Sobolev space of functions $v \in L^p(\Omega, \mathbb{R}^m)$ with the norm $\|v\|_{1,p,\Omega}$. To simplify, we denote $\widehat{W}(x, v, \nabla v)$ by $\widehat{W}(v)$ whose derivatives are denoted as follows; $\nabla_\xi \widehat{W}(v) = \nabla_\xi \widehat{W}(\xi, z, \zeta)|_{(\xi,z,\zeta)=(x,u,\nabla u)}$, $\nabla_\zeta \widehat{W}(v) = \nabla_\zeta \widehat{W}(\xi, z, \zeta)|_{(\xi,z,\zeta)=(x,u,\nabla u)}$.

### Definition of GJ-integral and its fundamental property

For an open subset $\omega$ in $\mathbb{R}^d$ and $\rho \in W^{1,\infty}\left(\mathbb{R}^d, \mathbb{R}^d\right)$, GJ-integral [4, 5] $\mathcal{J}_\omega(u, \rho) = P_\omega(u, \rho) + R_\omega(u, \rho)$ is defined by

$$P_\omega(u, \rho) = \int_{\partial(\omega\cap\Omega)} \left\{ \widehat{W}(u)(\rho \cdot n) - \widehat{T}(u) \cdot (\nabla u \cdot \rho) \right\} ds$$

$$R_\omega(u, \rho) = -\int_{\omega\cap\Omega} \left\{ \nabla_\xi \widehat{W}(u) \cdot \rho + f \cdot (\nabla u \cdot \rho) - \left(\nabla_\zeta \widehat{W}(u)\right)^T \left(\nabla \rho^T\right) \nabla u + \widehat{W}(u) \operatorname{div} \rho \right\} dx$$

where $n = (n_1, \cdots, n_d)^T$ is the outward unit normal of $\partial(\omega \cap \Omega)$, $i$-th component of $\widehat{T}(u)$ is $n_j \nabla_{\zeta_{ij}} \widehat{W}(x, u, \nabla u)$ and $ds$ the surface(line) element of $\partial(\omega \cap \Omega)$.
If $u$ is regular inside $\Omega$, that is, if $u|_{\omega\cap\Omega} \in W^{2,p}(\omega \cap \Omega, \mathbb{R}^m)$ and the divergence formula

$$\int_{\omega\cap\Omega} \nabla \widehat{W}(u) \cdot \rho dx = \int_{\partial(\omega\cap\Omega)} \widehat{W}(u)(\rho \cdot n) ds - \int_{\omega\cap\Omega} \widehat{W}(u) \operatorname{div} \rho dx \tag{1}$$

holds, then

$$\mathcal{J}_\omega(u, \rho) = 0 \ \text{ for all } \rho \in W^{1,\infty}\left(\Omega, \mathbb{R}^d\right) \tag{2}$$

Consider the perturbation $\Omega(t), 0 \leq t < \epsilon$ of $\Omega$ and the problems $\mathrm{P}(f, V(\Omega(t), \Gamma_D(t)))$ that are given by $\varphi_t$ such as $\Omega(t) = \varphi_t(\Omega)$ and $\Gamma_D(t) = \varphi_t(\Gamma_D)$.

[M1] For each $t \in [0, \epsilon)$, $\varphi_t$ is 1-1 mapping and has the inverse $\varphi_t^{-1}$, where $\varphi_0(x) = x$.

[M2] $[t \mapsto \varphi_t] \in C^1\left([0, \epsilon), W^{1,\infty}\left(\Omega_0, \mathbb{R}^d\right)\right)$.

Assume that $\mathrm{P}(f, V(\Omega(t), \Gamma_D(t)))$ satisfy the conditions; the unique solvability of $\mathrm{P}(f, V(\Omega(t), \Gamma_D(t)))$, the uniform coercivity of $\mathcal{E}(v, f, \Omega; \varphi_t)$ with respect to $\varphi_t$ and $\mathrm{P}(f, V(\Omega, \Gamma_D))$ satisfy $(S_+)$ condition. Here

$$\mathcal{E}(v, f, \Omega; \varphi_t) = \int_\Omega \left\{ \widehat{W}(\varphi_t(x), v(x), [A(\varphi_t)(x)]\nabla v(x)) - f \circ \varphi_t(x) v(x) \right\} \kappa(\varphi_t)(x) dx \tag{3}$$

where $A(\varphi) = \left(\nabla \varphi^T\right)^{-1} \in L^\infty(\Omega_0, \mathbb{R}^{d\times d})$, $\quad \kappa(\varphi) = \det \nabla \varphi^T \in L^\infty(\Omega_0, \mathbb{R})$..

**Theorem 1** *For a family of mappings $\varphi_t$ satisfying [M1] and [M2], if $f \in W^{1,p'}(\Omega, \mathbb{R}^m)$ and $P(\varphi_t(\Omega), f, V(\varphi_t(\Omega), \varphi_t(\Gamma_D)))$ satisfy the conditions just above, then the following holds*

$$\left.\frac{d}{dt} \mathcal{E}(u(t); f, \Omega(t))\right|_{t=0} = -R_\Omega(u, \mu_\varphi) - \int_{\partial\Omega} f \cdot u(\mu_\varphi \cdot n) ds \tag{4}$$

*where $\mu_\varphi = d\varphi_t/dt|_{t=0}$.*

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See [6] for the proof. Theorem 1 is fundamental results on shape sensitivity analysis, because in linear case, (4) express the differentiability of bilinear form which derive the differentiability of static response and eigenvalues with respect to shape.

## APPLICATION TO SHAPE OPTIMIZATION

In Theorem 1, we get the shape sensitivity analysis of the potential energy $\Omega \mapsto \mathcal{E}(u; f, \Omega)$. We introduce Azegami's method[1, 3] to find optimum shape $\Omega^o$ assuming that the cost function is the energy, that is, find $u^o$ and $\Omega^o$ such that

$$\mathcal{E}(f; u^o, \Omega^o) \le \mathcal{E}(f; u(\widetilde{\Omega}), \widetilde{\Omega}) \quad \text{for all domain } \widetilde{\Omega}$$

under some restrictions, where $u(\widetilde{\Omega})$ is the solution of $\mathrm{P}(f, V(\widetilde{\Omega}, \widetilde{\Gamma_D}))$.
We now solve the optimization by Azegami's method as follows: Let $\mathcal{V}(\Omega)$ be the subspace of $W^{1,2}(\Omega, \mathbb{R}^d)$ and let $b_\Omega(V, \mu)$ be a bilinear defined on $\mathcal{V}(\Omega) \times \mathcal{V}(\Omega)$ satisfying the following conditions.

**[A1]** $b_\Omega(V, \mu) \le \alpha_8 \|V\|_{1,2,\Omega} \|\mu\|_{1,2,\Omega}$ for all $V, \mu \in \mathcal{V}(\Omega)$ with a constant $\alpha_8 > 0$.

**[A2]** $b_\Omega(V, V) \ge \alpha_9 \|V\|^2_{1,2,\Omega}$ for all $V \in \mathcal{V}(\Omega)$ with a constant $\alpha_9 > 0$.

Consider the variational problem $\Pi(u, f, \Omega)$: Under the condition of Theorem 1, find $V^o \in \mathcal{V}(\Omega)$ such that

$$b_\Omega(V^o, \mu) = R_\Omega(u, \mu) + \int_{\partial\Omega} f \cdot u(\mu \cdot n) ds \ \text{ for all } \mu \in \mathcal{V}(\Omega) \tag{5}$$

The mapping $\varphi_t(x) = x + tV^o(x)$ from $\Omega$ to $\mathbb{R}^d$ is 1-1 if $t$ is near $0$. Unfortunately, $[\mu \mapsto R_\Omega(u, \mu)]$ is linear functional on $W^{1,\infty}(\Omega, \mathbb{R}^d)$, and is not on $W^{1,2}(\Omega, \mathbb{R}^d)$. To extend it on $W^{1,2}(\Omega, \mathbb{R}^d)$, we need slightly smoothness of $u$.

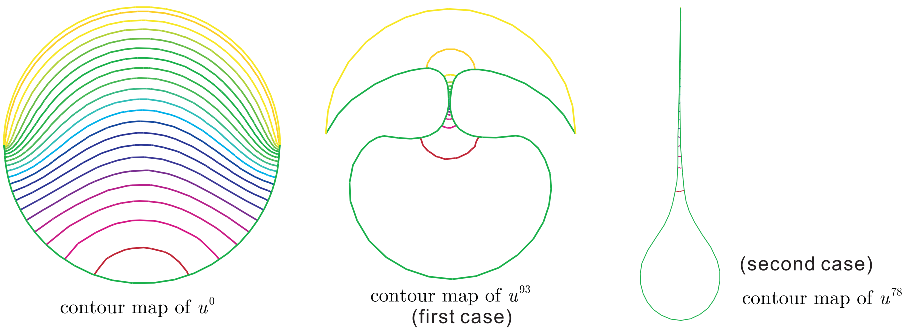
**Proposition 2** *If $d = 2$ or $3$, and the solution $u$ of $P(f, V(\Omega, \Gamma_D))$ is in $W^{1,2p}(\Omega, \mathbb{R}^m)$, then there is a constant $\alpha_{10} > 0$ such that*

$$R_\Omega(u, \mu) + \int_{\partial\Omega} f \cdot u(\mu \cdot n) ds \le \alpha_{10} \|\mu\|_{1,2,\Omega} \ \text{ for all } \mu \in W^{1,\infty}(\Omega, \mathbb{R}^d)$$

Using FreeFem++[2], we have numerical calculations that find the shapes $\Omega^i$, $i = 1, 2, \cdots$ such that $\mathcal{E}(0.5, V(\Omega^{i+1}, \Gamma_D^{i+1})) < \mathcal{E}(0.5, V(\Omega^i, \Gamma_D^i))$ starting from

$$\Omega^0 = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}, \Gamma_D^0 = \{(x_1, x_2) : x_1^2 + x_2^2 = 1, x_2 > 0\}$$

under the restriction that the areas of $\Omega^i$ are same $(= \pi)$. In first example, $\Gamma_D$ is fixed and $\Gamma_N^i = \partial\Omega^i \setminus \Gamma_D$ could change, and after 93 iteration the figure (first case) is obtained. In the second example, both boundaries (Neumann, Dirichlet) are changeable and after 78 iteration the figure (second case) is obtained.



contour map of $u^0$    contour map of $u^{93}$ (first case)    (second case) contour map of $u^{78}$

*Acknowledgement*

This work was supported by the Grant-in-Aid for Scientific Research(B)#1934007 and Research(C)#23540258, Japan. I would like to thank Prof. M.Kimura who made enormous contribution to prove Theorem 1 and to thank Prof. H.Azegami who provided great idea to solve shape optimization.

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