SNAP-BUCKLING INSTABILITY OF BEAMS AND SHELLS

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Summary Snap-buckling of geometric arches and thin spherical shells occur in a variety of different geometric situations. In this paper, we discuss the fundamental mechanics and dynamics of snap-buckling. We show that the stability can be determined geometrically, and that in the absence of geometric frustration the dynamics are controlled by the speed of sound within the material and its radius of curvature.

INTRODUCTION

Rapid elastic instabilities are observed in nature for pollen emplacement, seed dispersal, nutrition, and defense. These non-muscular, hydraulic plant movements use the storage of elastic energy as a way to respond to the environment around them. The carnivorous fungus *Dactylaria brochophaga* traps nematodes in less than 0:1s by swelling rapidly. The flower stamens of the bunchberry dogwood use the storage and release of elastic energy to catapult pollen into the air as the flower explosively fractures. The responsive mechanism used by the Venus flytrap to capture its prey has been attributed to a snap-through elastic instability[1].

Snap-buckling of curved beams, plates, and shells presents an interesting experimental paradigm due to the ability for one structure to rapidly change between two stable configurations. Instead of expending energy to force a structure with one stable configuration to continuously maintain an alternative configuration, one can design systems that expend a smaller amount of energy over a short timescale which can easily, quickly, and predictably change from one stable configuration to another. These principles have been recently used in the development of switches within MEMS devices, biomedical valves, switchable optical devices[2], adhesion of spherical caps, and responsive hydrogels.

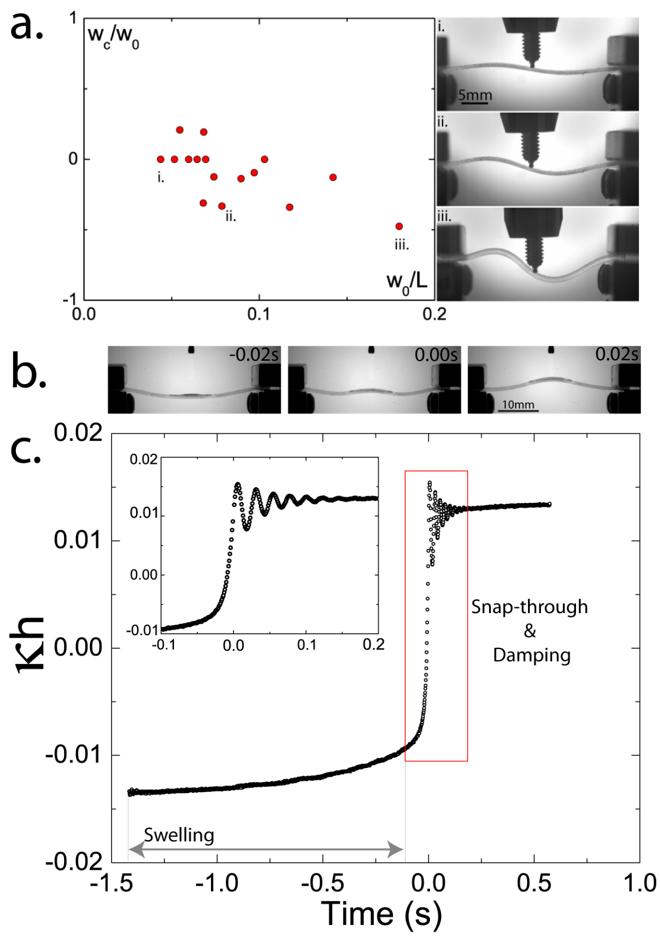

Figure 1: *a*. A clamped arch under a point load loses stability as the center of the arch reaches the height of the clamped edges. *b*. Images of an arch swollen by a favorable solvent, leading to a distributed stress that induces snap-buckling. *c*. A plot of the normalized curvature κh as a function of time, illustrating the dynamics of the snapping arch.

Snap-buckling of geometric arches and thin spherical shells occur in a variety of different geometric situations, exhibiting a highly nonlinear response dictated by the geometry and material properties of the system. As this elastic instability often precedes the catastrophic failure of a mechanical system, significant work has focused on the stability criteria for such structures[3,4]. What remains less understood are fundamental questions surrounding the dynamics of snap-buckling.

Experimental

To study the mechanics and dynamics of the snap-buckling instability, we investigated the snapping of an arch and an everted hemispherical shell. A beam of length $2L$ and thickness h made of polydimethylsiloxane (PDMS) is clamped at its edges and compressed to form an arch with curvature $\kappa_i = R^{-1}$. The arch is initially symmetric, with a profile described by a cosine shape, $w(x) = \frac{w(0)}{2}\left(1+\cos\frac{\pi x}{L}\right)$, where $w(0)$ is the displacement of the arch at its center. Upon loading, the beam generally adopts an asymmetric shape such that $w_l(x) = w(x) + \frac{w_2}{2}\left(1+\cos\frac{2\pi x}{L}\right)$, as seen in figure 1a. The beam is compressed at its center by a point load until structural stability is lost and the arch rapidly snaps to its stable, opposite configuration ($i.e.$ $\kappa = -\kappa_i$). To consider the effect of a distributed load on the mechanical stability of an arch, the

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experiments were repeated by swelling the crosslinked PDMS with a favorable solvent ($e.g.$ hexane, toluene, silicone oil). Swelling the beam causes the top surface to expand and generates a stress distributed along the length of the arch, similar to the thermal expansion of a heated material. At a critical stress, dictated by the geometry of the arch, the swollen beam snaps from one stable curvature to another (Figure 1a, b). Finally, we examined the dynamics of a snapping shell be everting a thick, hemispherical rubber shell into an unstable geometry. Being inherently unstable, the shell rapidly snaps to its opposite, and more stable configuration. The timescale of this snap-through transition was quantified using a high-speed camera and acoustic measurements.

Results

In the absence of any geometric frustration from the arch or shell's initial geometry, upon losing stability the structure will snap-though at a timescale dictated by the material's speed of sound and initial radius of curvature, such that $\tau_s \approx 2R\sqrt{\frac{\rho}{E}}$, where ρ is the density and E is Young's elastic modulus. The experimental results for τ_s for arches and shells are shown in figure 2 indicating good agreement with the inertial timescale of the structure.

Geometric frustration occurs due to the balance between deforming via bending versus stretching[5], such that the total energy of the snap-through transition is described by $U = U_b + \alpha U_s$, where $\alpha = L^4/R^2h^2$. When $\alpha > 1$, it is energetically favorable for the structure to rapid snap to a more stable geometry, while in the presence of geometric frustration, $i.e.$ $\alpha < 1$, the structure will undergo a smooth and slow transition between one shape to the next. Decreasing α leads to an increase in snap-through dynamics of over an order of magnitude greater than the inertial timescale.

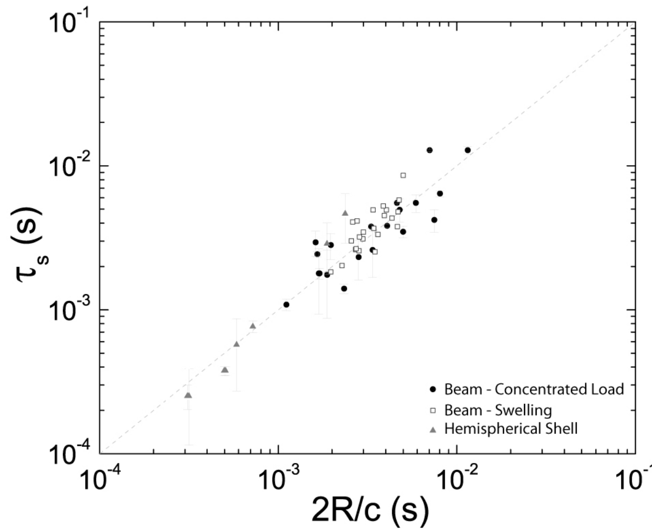

Figure 2: a. A plot of the snap-through time τ_s versus the inertial timescale $2R\sqrt{\rho/E}$

CONCLUSIONS

In conclusion, the mechanics and dynamics of a snapping structure are described by both material properties and geometry. An ideal structure will transition between stable states at a timescale described by the inertial time for the material, $i.e.$ $\tau_s \approx 2R/c$, where c is the speed of sound within the material. As the structure's initial curvature and length decreases, there is a far greater energetic barrier to undergo a snap-through transition, so a slow, smooth transition between stable geometries is observed. Understanding the impact of geometry and material properties on the structural snap-through instability is a necessary component in designing rapid, responsive advanced materials.

References

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